Czechoslovak Mathematical Journal

Ján Brajerčík

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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 4, 1063–1076

Persistent URL: `http://dml.cz/dmlcz/141807`

Czechoslovak Mathematical Journal, 61 (136) (2011), 1063–1076

ORDER REDUCTION OF THE EULER-LAGRANGE EQUATIONS OF HIGHER ORDER INVARIANT VARIATIONAL PROBLEMS ON FRAME BUNDLES

Ján Brajerčík, Prešov

(Received September 14, 2010)

Abstract. Let $\mu\colon FX \rightarrow X$ be a principal bundle of frames with the structure group $\mathrm{Gl}_n(\mathbb{R})$. It is shown that the variational problem, defined by $\mathrm{Gl}_n(\mathbb{R})$-invariant Lagrangian on J^rFX, can be equivalently studied on the associated space of connections with some compatibility condition, which gives us order reduction of the corresponding Euler-Lagrange equations.

Keywords: Frame bundle, Euler-Lagrange equations, invariant Lagrangian, Euler-Poincaré reduction

MSC 2010: 53C05, 53C10, 58A20, 58E30

1. Introduction

In the Lagrange theory of the calculus of variations on fibred manifolds, it is well known that the Euler-Lagrange equations are employed for finding the extremal values of the corresponding variational function. In general, for variational problems defined by Lagrangians of order r, the order of the Euler-Lagrange equations is $2r$. In the case when an underlying manifold has the structure of a principal fibre bundle with the structure group G and G-invariant Lagrangians, the original system of the Euler-Lagrange equations can be simplified.

The idea of reduction of the Euler-Lagrange equations originally comes from mechanics, where G-invariant Lagrangian defined on the tangent bundle of a Lie group G is considered. The equations of new kind for the reduced Lagrangian are called the

The author is grateful to the Ministry of Education of the Slovak Republic (Grant VEGA 1/0577/10), to the Slovak Research and Development Agency (Grant MVTS SK-CZ-0006-09), and to the Czech Grant Agency (Grant 201/09/0981).

Euler-Poincaré equations (see, e.g., [14]). The extension of such mechanism to field theory was done in [3], [4], [5], and is called the Euler-Poincaré reduction. In this case, we consider a Lagrangian defined on the first jet prolongation J^1P of a principal fibre bundle P, invariant under the natural action induced by the structure group G on J^1P. The reduced variational problem, in the sense of Euler-Poincaré reduction, is defined on the bundle of connections $C(P)$, which can be identified with $(J^1P)/G$. It was shown that locally there is an equivalence between the solutions of the Euler-Lagrange equations for G-invariant Lagrangian on the one side, and the solutions of the Euler-Poincaré equations for reduced Lagrangian plus the condition of flat connection on the other side.

In this paper, we consider the same type of invariant variational problems defined on frame bundles FX over an n-dimensional manifold X. Our aim is to generalize the reduction problem to higher order cases, i.e., to variational problems on J^rFX defined by $\mathrm{Gl}_n(\mathbb{R})$-invariant Lagrangians.

In Sections 2 and 3 we recall the basic notions from the variational theory on fibred manifolds and frame bundles, respectively. Section 4 deals with the correspondence between the sections of J^1FX and C^1X, where C^1X is the bundle of linear connections of X. Using this correspondence it is shown that the $\mathrm{Gl}_n(\mathbb{R})$-invariant variational problem on $J^{r+1}FX$ can be alternatively studied as the reduced problem on J^rC^1X, which gives us the order reduction of the variational equations for extremals. Again, the necessity of additional compatibility conditions for connections, in the form of zero curvature, occurs. Having a solution of the reduced equations, the compatibility conditions ensure the existence of the associated extremal frame field, because they correspond to the conditions of complete integrability of the arising system of first order equations. In Section 5 we give explicit expressions of the reduced, first order equations for connections, which replace the second order Euler-Lagrange equations for the associated frame field.

For reduction of the solution of the Euler-Lagrange equations defined by higher order invariant Lagrangians it is also possible to use the corresponding bundles of higher order connections $C^rX = J^rFX/\mathrm{Gl}_n(\mathbb{R})$ (see [1]).

2. Preliminaries

Y is a *fibred manifold* with an oriented *base manifold* X and a *projection* π. We denote $n = \dim X$, $m = \dim Y - n$. J^rY is the *r-jet prolongation* of Y. The *r-jet* of a section γ of Y at a point $x \in X$ is denoted by $J^r_x\gamma$; and $x \mapsto J^r\gamma(x) = J^r_x\gamma$ is the *r-jet prolongation* of γ. If $J^r_x\gamma \in J^rY$, the *canonical jet projections* $\pi^{r,0}\colon J^rY \to Y$ (the target projection), $\pi^r\colon J^rY \to X$ (the source projection), are defined by $\pi^{r,0}(J^r_x\gamma) = \gamma(x)$, and $\pi^r(J^r_x\gamma) = x$, respectively.

Any *fibred chart* (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, where $1 \leqslant i \leqslant n$, $1 \leqslant \sigma \leqslant m$, induces the *associated charts* on X and on J^rY, (U, φ), $\varphi = (x^i)$, and (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, respectively; here $V^r = (\pi^{r,0})^{-1}(V)$, and $U = \pi(V)$. For functions on V^r, we define the *formal derivative operator* (with respect to the fibred chart (V, ψ)) by

$$d_p = \frac{\partial}{\partial x^p} + y^\sigma_p \frac{\partial}{\partial y^\sigma} + y^\sigma_{i_1 p} \frac{\partial}{\partial y^\sigma_{i_1}} + \ldots + y^\sigma_{i_1 i_2 \ldots i_r p} \frac{\partial}{\partial y^\sigma_{i_1 i_2 \ldots i_r}}.$$

A *Lagrangian* (of *order* r) for Y is any π^r-horizontal n-form on some $W^r \subset J^rY$. If, in a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, we denote $\omega_0 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n$, then a Lagrangian, defined on $V^r = (\pi^{r,0})^{-1}(V)$, has in this chart an expression

$$\lambda = \mathcal{L}\omega_0,$$

where $\mathcal{L}\colon\ V^r \to \mathbb{R}$ is a function (the *Lagrange function* associated with λ and (V, ψ)).

The *Euler-Lagrange form* E_λ, associated with a Lagrangian λ of order r, is a $\pi^{2r,0}$-horizontal $(n+1)$-form, in fibred chart defined by

$$E_\lambda = E_\sigma(\mathcal{L})\omega^\sigma \wedge \omega_0,$$

where

$$E_\sigma(\mathcal{L}) = \sum_{k=0}^{r} (-1)^k d_{i_1} d_{i_2} \ldots d_{i_k} \frac{\partial \mathcal{L}}{\partial y^\sigma_{i_1 i_2 \ldots i_k}}$$

are the *Euler-Lagrange expressions* associated with λ.

Let Ω be a piece of X (a compact, n-dimensional submanifold of X with boundary $\partial\Omega$), let $\Gamma_{\Omega,W}(\pi)$ be the set of smooth sections γ over Ω such that $\gamma(\Omega) \subset W$. If we have a Lagrangian λ of order r defined on W^r, the *variational function* associated with λ over Ω is the mapping $\Gamma_{\Omega,W}(\pi) \ni \gamma \to \lambda_\Omega(\gamma) \in \mathbb{R}$ defined by

$$\lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\lambda.$$

Let ξ be a π-projectable vector field on Y, and let $\partial_{J^r\xi}\lambda$ denote the Lie derivative of λ by the r-jet prolongation $J^r\xi$ of ξ. The number

$$(\partial_{J^r\xi}\lambda)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\xi}\lambda$$

is the *variation* of the variational function λ_Ω at γ, *induced* by the vector field ξ. We say that a section $\gamma \in \Gamma_{\Omega,W}(\pi)$ is *stable* with respect to a variation ξ of γ, if

$(\partial_{J^r\xi}\lambda)_\Omega(\gamma) = 0$. If γ is stable with respect to all ξ with support contained in $\pi^{-1}(\Omega)$, we say that γ is an *extremal* of λ_Ω. A section γ which is an extremal of every λ_Ω is called an *extremal* of λ. A section γ is an extremal of a Lagrangian λ of order r if and only if, for every fibred chart on Y, γ satisfies the system of partial differential equations

$$E_\sigma(\mathcal{L}) \circ J^{2r}\gamma = 0.$$

For more detailed exposition of the variational theory on fibred spaces, including local variational principles, see [2], [11].

3. Frame bundles and their prolongations

In this section we recall the basic notions from the theory of frame bundles; the reader can also consult [1], [9], [13].

Henceforth, instead of Y, let us consider the *frame bundle* $\mu\colon FX \to X$ over an n-dimensional manifold X, which has the structure of a principal fiber bundle with the structure group $\mathrm{Gl}_n(\mathbb{R})$. A *frame* at a point $x \in X$ is a pair $\Xi = (x, \zeta)$, where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ is an ordered basis of the tangent space T_xX. For every chart (U, φ), $\varphi = (x^i)$, on X, the *associated chart* (V, ψ), $\psi = (x^i, x^i_j)$, on FX, is defined by $V = \mu^{-1}(U)$, and

$$x^i(\Xi) = x^i(\mu(\Xi)), \quad \zeta_j = x^i_j(\Xi)\Big(\frac{\partial}{\partial x^i}\Big)_x,$$

where $\Xi \in V$. We denote by y^j_k the *inverse matrix* of x^i_j.

Let us denote by J^rFX an *r-jet prolongation* of FX. With any chart (U, φ), $\varphi = (x^i)$, on X, we also associate a chart (V^r, ψ^r) on J^rFX, where $V^r = (\mu^r)^{-1}(U)$ and $\psi^r = (x^i, x^i_j, x^i_{j,k_1}, x^i_{j,k_1k_2}, \dots, x^i_{j,k_1k_2\dots k_r})$.

The *right action* $(\Xi, A) \mapsto \Xi \cdot A \equiv R_A(\Xi)$ of $\mathrm{Gl}_n(\mathbb{R})$ on FX is expressed, in a chart (U, φ), $\varphi = (x^i)$, on X, by the equations

$$\bar{x}^i = x^i \circ R_A = x^i, \quad \bar{x}^i_j = x^i_j \circ R_A = x^i_k a^k_j,$$

where a^k_j are the canonical coordinates on $\mathrm{Gl}_n(\mathbb{R})$. This action can be canonically prolonged to the action of $\mathrm{Gl}_n(\mathbb{R})$ on J^rFX. If $A \in \mathrm{Gl}_n(\mathbb{R})$, $J^r_x\gamma \in J^rFX$, we define

$$(J^r_x\gamma, A) \mapsto J^r_x\gamma \cdot A \equiv J^rR_A(J^r_x\gamma) = J^r_x(R_A \circ \gamma).$$

In the associated charts on FX and J^rFX, we have

$$\bar{x}^i = x^i \circ J^rR_A = x^i,$$
$$\bar{x}^i_{j,k_1k_2\dots k_m} = x^i_{j,k_1k_2\dots k_m} \circ J^rR_A = x^i_{l,k_1k_2\dots k_m} a^l_j, \quad 0 \leqslant m \leqslant r.$$

A form η on J^rFX is *invariant* with respect to the action of $\mathrm{Gl}_n(\mathbb{R})$ (or, $\mathrm{Gl}_n(\mathbb{R})$-*invariant*), if

$$(J^rR_A)^*\eta = \eta$$

holds for all $A \in \mathrm{Gl}_n(\mathbb{R})$. In particular, a function f on J^rFX is $\mathrm{Gl}_n(\mathbb{R})$-invariant if $(J^rR_A)^*f = f \circ J^rR_A = f$ for all $A \in \mathrm{Gl}_n(\mathbb{R})$.

The functions

$$\Gamma^i_{k_1k_2\ldots k_mp} = -y^l_p x^i_{l,k_1k_2\ldots k_m}, \tag{3.1}$$

defined for $1 \leqslant m \leqslant r$, are examples of $\mathrm{Gl}_n(\mathbb{R})$-invariant functions on J^rFX. Note that the functions $\Gamma^i_{k_1k_2\ldots k_mp}$ are *symmetric* in $k_1, k_2, \ldots, k_m$.

Let (V^r, ψ^r) be any fibred chart on J^rFX. Setting $\Psi^r = (x^i, x^i_p, \Gamma^i_{k_1p}, \Gamma^i_{k_1k_2p}, \ldots, \Gamma^i_{k_1k_2\ldots k_rp})$, we obtain the associated chart (V^r, Ψ^r) on J^rFX, which is said to be $\mathrm{Gl}_n(\mathbb{R})$-*adapted*.

For $r \geqslant 1$, let us denote

$$C^rX = J^rFX/\mathrm{Gl}_n(\mathbb{R}).$$

The quotient projection $\mu^{(r)}\colon J^rFX \to C^rX$ is in $\mathrm{Gl}_n(\mathbb{R})$-adapted coordinates expressed as

$$\mu^{(r)}\colon (x^i, x^i_j, \Gamma^i_{k_1j}, \Gamma^i_{k_1k_2j}, \ldots, \Gamma^i_{k_1k_2\ldots k_rj}) \to (x^i, \Gamma^i_{k_1j}, \Gamma^i_{k_1k_2j}, \ldots, \Gamma^i_{k_1k_2\ldots k_rj}).$$

Let $\{V^r_\alpha, \Psi^r_\alpha\}_{\alpha\in\mathcal{A}}$ be a covering of J^rFX by $\mathrm{Gl}_n(\mathbb{R})$-adapted charts. For each $\alpha \in \mathcal{A}$, we set $W^r_\alpha = \mu^{(r)}(V^r_\alpha)$, and define the system of coordinate functions χ^r_α by $\chi^r_\alpha \circ \mu^{(r)} = \Psi^r_\alpha$. Then $\{W^r_\alpha, \chi^r_\alpha\}_{\alpha\in\mathcal{A}}$ is a covering of C^rX consisting of the fibred charts and C^rX is the fibred manifold over X with the projection $\kappa^r\colon C^rX \to X$, in the corresponding coordinates defined by $\kappa^r\colon (x^i, \Gamma^i_{k_1j}, \Gamma^i_{k_1k_2j}, \ldots, \Gamma^i_{k_1k_2\ldots k_rj}) \to (x^i)$.

Lemma 3.1. *J^rFX has the structure of a right principal $\mathrm{Gl}_n(\mathbb{R})$-bundle over C^rX such that the diagram*

$$\begin{tikzcd} J^rFX \arrow[r, "\mu^{(r)}"] \arrow[d, "\mu^{r,0}"] & C^rX \arrow[d, "\kappa^r"] \\ FX \arrow[r, "\mu"] & X \end{tikzcd}$$

commutes.

For $r = 1$, the space $C^1X = J^1FX/\mathrm{Gl}_n(\mathbb{R})$ can be identified with the bundle of linear connections (García Pérez [7], Krupka [12]). In the case $r > 1$, the space C^rX is said to be the bundle of an rth order connections (Kolář [10]).

Lemma 3.2.

(a) *A function on J^rFX is $\mathrm{Gl}_n(\mathbb{R})$-invariant if and only if it depends, locally, on the coordinates $(x^i, \Gamma^i_{k_1p}, \Gamma^i_{k_1k_2p}, \dots, \Gamma^i_{k_1k_2\dots k_rp})$ on C^rX only.*

(b) *A k-form η on J^rFX is $\mathrm{Gl}_n(\mathbb{R})$-invariant if and only if in any chart on X,*

$$\begin{aligned}\eta = \Delta_0 &+ y^{q_1}_{s_1} dx^{p_1}_{q_1} \wedge \Delta^{s_1}_{p_1} + y^{q_1}_{s_1} y^{q_2}_{s_2} dx^{p_1}_{q_1} \wedge dx^{p_2}_{q_2} \wedge \Delta^{s_1s_2}_{p_1p_2} \\ &+ \dots + y^{q_1}_{s_1} y^{q_2}_{s_2} \dots y^{q_k}_{s_k} dx^{p_1}_{q_1} \wedge dx^{p_2}_{q_2} \dots \wedge dx^{p_k}_{q_k} \wedge \Delta^{s_1s_2\dots s_k}_{p_1p_2\dots p_k},\end{aligned}$$

where Δ_0, $\Delta^{s_1}_{p_1}$, $\Delta^{s_1s_2}_{p_1p_2}$, $\dots$, $\Delta^{s_1s_2\dots s_k}_{p_1p_2\dots p_k}$ are arbitrary forms defined on C^rX.

(c) *Let f be a $\mathrm{Gl}_n(\mathbb{R})$-invariant function on J^rFX. Then the pth formal derivative d_pf is again a $\mathrm{Gl}_n(\mathbb{R})$-invariant function on $J^{r+1}FX$, and in the corresponding adapted chart it can be expressed as*

$$d_pf = \frac{\partial f}{\partial x^p} + \sum_{s=1}^{r} \sum_{k_1 \leqslant k_2 \leqslant \dots \leqslant k_s} \frac{\partial f}{\partial \Gamma^i_{k_1k_2\dots k_sj}} (\Gamma^i_{k_1k_2\dots k_spj} + \Gamma^i_{k_1k_2\dots k_sq}\Gamma^q_{pj}). \tag{3.2}$$

Note that the operator d_p introduced in (3.2), restricted to the invariant functions on J^rFX, corresponds to the operator of the formal derivative of any function on C^rX.

The functions (3.1) can be rewritten in another form. By (3.2),

$$\Gamma^i_{klj} = d_l\Gamma^i_{kj} - \Gamma^i_{ks}\Gamma^s_{lj} = d_k\Gamma^i_{lj} - \Gamma^i_{ls}\Gamma^s_{kj},$$

where the functions Γ^i_{kj} and their formal derivatives $d_l\Gamma^i_{kj}$ satisfy the compatibility condition

$$d_l\Gamma^i_{kj} - d_k\Gamma^i_{lj} + \Gamma^i_{ls}\Gamma^s_{kj} - \Gamma^i_{ks}\Gamma^s_{lj} = 0. \tag{3.3}$$

Analogously,

$$\begin{aligned}\Gamma^i_{k_1k_2k_3j} &= d_{k_3}\Gamma^i_{k_1k_2j} - \Gamma^i_{k_1k_2s}\Gamma^s_{k_3j} \\ &= d_{k_2}\Gamma^i_{k_1k_3j} - \Gamma^i_{k_1k_3s}\Gamma^s_{k_2j} = d_{k_1}\Gamma^i_{k_2k_3j} - \Gamma^i_{k_2k_3s}\Gamma^s_{k_1j}, \\ &\vdots \\ \Gamma^i_{k_1k_2\dots k_rj} &= d_{k_r}\Gamma^i_{k_1k_2\dots k_{r-1}j} - \Gamma^i_{k_1k_2\dots k_{r-1}s}\Gamma^s_{k_rj} \\ &= d_{k_{r-1}}\Gamma^i_{k_1k_2\dots k_{r-2}k_rj} - \Gamma^i_{k_1k_2\dots k_{r-2}k_rs}\Gamma^s_{k_{r-1}j} \\ &= \dots \\ &= d_{k_1}\Gamma^i_{k_2k_3\dots k_rj} - \Gamma^i_{k_2k_3\dots k_rs}\Gamma^s_{k_1j},\end{aligned}$$

and we have the corresponding compatibility conditions

$$\begin{aligned}
d_{k_1}\Gamma^i_{k_2k_3p} - \Gamma^i_{k_2k_3s}\Gamma^s_{k_1p} &= d_{k_2}\Gamma^i_{k_1k_3p} - \Gamma^i_{k_1k_3s}\Gamma^s_{k_2p}\\
&= d_{k_3}\Gamma^i_{k_1k_2p} - \Gamma^i_{k_1k_2s}\Gamma^s_{k_3p}\\
&\;\;\vdots\\
d_{k_1}\Gamma^i_{k_2k_3\ldots k_rp} - \Gamma^i_{k_2k_3\ldots k_rs}\Gamma^s_{k_1p} &= d_{k_2}\Gamma^i_{k_1k_3\ldots k_rp} - \Gamma^i_{k_1k_3\ldots k_rs}\Gamma^s_{k_2p}\\
&= \ldots\\
&= d_{k_{r-1}}\Gamma^i_{k_1k_2\ldots k_{r-2}k_rp} - \Gamma^i_{k_1k_2\ldots k_{r-2}k_rs}\Gamma^s_{k_{r-1}p}\\
&= d_{k_r}\Gamma^i_{k_1k_2\ldots k_{r-1}p} - \Gamma^i_{k_1k_2\ldots k_{r-1}s}\Gamma^s_{k_rp}.
\end{aligned}$$

Let λ be a Lagrangian on J^rFX. Recall that the coordinate expression of λ in any $\mathrm{Gl}_n(\mathbb{R})$-adapted chart is $\lambda = \mathcal{L}\omega_0$, where $\mathcal{L}\colon V^r \to \mathbb{R}$ is the associated Lagrange function.

Lemma 3.3. *The Euler-Lagrange form E_λ, associated with the $\mathrm{Gl}_n(\mathbb{R})$-invariant Lagrangian λ on J^rFX, is $\mathrm{Gl}_n(\mathbb{R})$-invariant.*

Corollary 3.1.

(a) *A Lagrangian λ on J^rFX is $\mathrm{Gl}_n(\mathbb{R})$-invariant if and only if in any $\mathrm{Gl}_n(\mathbb{R})$-adapted chart, the associated Lagrange function $\mathcal{L}$ is $\mathrm{Gl}_n(\mathbb{R})$-invariant.*

(b) *The coordinate expression of the Euler-Lagrange form E_λ, associated with $\mathrm{Gl}_n(\mathbb{R})$-invariant Lagrangian λ on J^rFX, is*

$$E_\lambda = F^l_i(\mathcal{L})y^j_l\,\mathrm{d}x^i_j \wedge \omega_0,$$

where $F^l_i(\mathcal{L})$ are $\mathrm{Gl}_n(\mathbb{R})$-invariant functions.

Corollary 3.2. *A Lagrangian λ on J^rFX is $\mathrm{Gl}_n(\mathbb{R})$-invariant if and only if there exists a Lagrangian $\tilde{\lambda}$ on C^rX such that $\lambda = (\mu^{(r)})^*\tilde{\lambda}$.*

Finally, we prove an auxiliary assertion.

Lemma 3.4. *In a $\mathrm{Gl}_n(\mathbb{R})$-adapted chart (V^r, Ψ^r), $\Psi^r = (x^i, x^i_j, \Gamma^i_{p_1j}, \Gamma^i_{p_1p_2j}, \ldots, \Gamma^i_{p_1p_2\ldots p_rj})$, on J^rFX, $r > 2$,*

$$\Gamma^i_{p_1p_2\ldots p_mklj} - \Gamma^i_{p_1p_2\ldots p_mlkj} = -\Gamma^i_{p_1p_2\ldots p_mq}(\Gamma^q_{klj} - \Gamma^q_{lkj})$$

holds for all $1 \leqslant m \leqslant r-2$.

Proof. According to (3.2), for adapted coordinates $\Gamma^i_{p_1p_2\dots p_mj}$, $m \geqslant 1$ we have

$$\Gamma^i_{p_1p_2\dots p_mkj} = d_k\Gamma^i_{p_1p_2\dots p_mj} - \Gamma^i_{p_1p_2\dots p_mq}\Gamma^q_{kj} \tag{3.4}$$

and

$$d_l\Gamma^i_{p_1p_2\dots p_mkj} = d_ld_k\Gamma^i_{p_1p_2\dots p_mj} - d_l\Gamma^i_{p_1p_2\dots p_mq}\Gamma^q_{kj} - \Gamma^i_{p_1p_2\dots p_mq}d_l\Gamma^q_{kj}. \tag{3.5}$$

Then, using (3.4) and (3.5), we obtain

$$\begin{aligned}
&\Gamma^i_{p_1p_2\dots p_mklj} - \Gamma^i_{p_1p_2\dots p_mlkj} \\
&\qquad = d_l\Gamma^i_{p_1p_2\dots p_mkj} - \Gamma^i_{p_1p_2\dots p_mkt}\Gamma^t_{lj} - d_k\Gamma^i_{p_1p_2\dots p_mlj} + \Gamma^i_{p_1p_2\dots p_mlt}\Gamma^t_{kj} \\
&\qquad = d_ld_k\Gamma^i_{p_1p_2\dots p_mj} - d_l\Gamma^i_{p_1p_2\dots p_mq}\Gamma^q_{kj} - \Gamma^i_{p_1p_2\dots p_mq}d_l\Gamma^q_{kj} \\
&\qquad\quad - d_k\Gamma^i_{p_1p_2\dots p_mt}\Gamma^t_{lj} + \Gamma^i_{p_1p_2\dots p_mq}\Gamma^q_{kt}\Gamma^t_{lj} \\
&\qquad\quad - d_kd_l\Gamma^i_{p_1p_2\dots p_mj} + d_k\Gamma^i_{p_1p_2\dots p_mq}\Gamma^q_{lj} + \Gamma^i_{p_1p_2\dots p_mq}d_k\Gamma^q_{lj} \\
&\qquad\quad + d_l\Gamma^i_{p_1p_2\dots p_mt}\Gamma^t_{kj} - \Gamma^i_{p_1p_2\dots p_mq}\Gamma^q_{lt}\Gamma^t_{kj} \\
&\qquad = -\Gamma^i_{p_1p_2\dots p_mq}(d_l\Gamma^q_{kj} - d_k\Gamma^q_{lj} + \Gamma^q_{lt}\Gamma^t_{kj} - \Gamma^q_{kt}\Gamma^t_{lj}) \\
&\qquad = -\Gamma^i_{p_1p_2\dots p_mq}(\Gamma^q_{klj} - \Gamma^q_{lkj}).
\end{aligned}$$

□

4. Variational problems defined by higher order invariant Lagrangians

First we discuss relations between linear connections on X and sections of the fibred manifold J^1FX (see [15]).

Lemma 4.1. *Equations*

$$\mu^{1,0} \circ w = \mathrm{id}_{\mu^{-1}(U)}, \quad \mu^{(1)} \circ w = \Gamma \circ \mu \tag{4.1}$$

define a bijective correspondence between the connections Γ, *defined on open sets* $U \subset X$, *and the* $\mathrm{Gl}_n(\mathbb{R})$*-equivariant sections* w *of* J^1FX, *defined on open,* μ*-saturated sets* $V = \mu^{-1}(U) \subset FX$.

If $w\colon V \to J^1FX$ is the $\mathrm{Gl}_n(\mathbb{R})$-equivariant section, then the connection Γ, defined by (4.1), is said to be *associated* with w. The diagram

$$\begin{tikzcd}
J^1FX \arrow[r, "\mu^{(1)}"] & C^1X \\
V \arrow[u, "w"] \arrow[r, shift left, "\mu"] & U \arrow[u, "\Gamma"'] \arrow[ul, "\delta"'] \arrow[l, shift left, "\gamma"]
\end{tikzcd}$$

also induces the natural correspondence between the sections of J^1FX over X, and the sections of C^1X: if δ is a section of J^1FX, then $\Gamma = \mu^{(1)} \circ \delta$ is a section of C^1X over the same domain of definition. Conversely, given Γ, one can construct, for every chart (U, φ), $\varphi = (x^i)$, on X, a section $\delta\colon U \to J^1FX$. We choose *any* frame field $\gamma\colon U \to FX$, and then define $\delta\colon U \to J^1FX$ by setting $x^i_j \circ \delta = x^i_j \circ \gamma$, $x^i_{j,k} \circ \delta = -(x^l_j \circ \gamma) \cdot \Gamma^i_{kl}$. If

$$\Gamma = \mu^{(1)} \circ \delta$$

over an open set U, we say that δ *generates* Γ on U.

Consider the restriction of the correspondence $\delta \to \mu^{(1)} \circ \delta$ to *holonomic* sections, i.e., to sections δ of the form $\delta = J^1\gamma$, where γ is a section of FX. If

$$\Gamma = \mu^{(1)} \circ J^1\gamma$$

for some frame field γ, we say that γ *generates* Γ on U. If every point of X has a neighborhood U such that $\Gamma = \mu^{(1)} \circ J^1\gamma$ for some frame field γ, we say that Γ is *locally generated* by frame fields.

Lemma 4.2. *The following conditions are equivalent.*

(a) Γ *is locally generated by frame fields.*

(b) Γ *is flat.*

Proof. Let Γ be a linear connection on X. Denote by

$$R^i_{lkj} = \frac{\partial \Gamma^i_{kj}}{\partial x^l} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} + \Gamma^i_{ls}\Gamma^s_{kj} - \Gamma^i_{ks}\Gamma^s_{lj} \tag{4.2}$$

the components of the *curvature tensor* of Γ in a chart (U, φ), $\varphi = (x^i)$, on X (see [8]).

(a) $\Rightarrow$ (b): Assume that the first condition holds. Then there exists a system of functions $x^i_j\colon U \to \mathbb{R}$ such that

$$\frac{\partial x^i_j}{\partial x^k} = -x^l_j \Gamma^i_{kl}. \tag{4.3}$$

This system satisfies

$$\begin{aligned} -\frac{\partial x^s_j}{\partial x^l}\Gamma^i_{ks} - x^s_j \frac{\partial \Gamma^i_{ks}}{\partial x^l} &= \frac{\partial}{\partial x^l}(-x^s_j\Gamma^i_{ks}) = \frac{\partial^2 x^i_j}{\partial x^l \partial x^k} \\ &= \frac{\partial^2 x^i_j}{\partial x^k \partial x^l} = \frac{\partial}{\partial x^k}(-x^s_j\Gamma^i_{ls}) = -\frac{\partial x^s_j}{\partial x^k}\Gamma^i_{ls} - x^s_j \frac{\partial \Gamma^i_{ls}}{\partial x^k}, \end{aligned}$$

i.e.,

$$-x^t_j\Gamma^s_{lt}\Gamma^i_{ks} + x^s_j\frac{\partial\Gamma^i_{ks}}{\partial x^l} + x^t_j\Gamma^s_{kt}\Gamma^i_{ls} - x^s_j\frac{\partial\Gamma^i_{ls}}{\partial x^k} = 0.$$

Thus,

$$-\Gamma^s_{lj}\Gamma^i_{ks} + \frac{\partial\Gamma^i_{kj}}{\partial x^l} + \Gamma^s_{kj}\Gamma^i_{ls} - \frac{\partial\Gamma^i_{lj}}{\partial x^k} = 0, \tag{4.4}$$

or,

$$R^i_{lkj} = 0,$$

which means that Γ is flat.

(b) $\Rightarrow$ (a): If Γ is a flat connection, it satisfies the system of equations (4.4). This system represents the conditions of complete integrability for the system (4.3), which means that Γ is locally generated by the frame fields (see [6]). □

Note that the conditions of zero curvature (4.4) for a connection Γ to be locally generated by frame fields are equivalent to the compatibility conditions (3.3) (symmetry of Γ^i_{klj} in the indices k, l) for the associated section γ.

The correspondence between sections, as described above, allows us to consider a variational problem for a frame field $\gamma\colon X \to FX$ as a problem to find the corresponding linear connection $\Gamma\colon X \to C^1X$. For higher order variational problems on frame bundles we consider the commutative diagram

$$\begin{tikzcd} J^rJ^1FX \arrow[r, "J^r\mu^{(1)}"] \arrow[d] & J^rC^1X \arrow[d] \\ J^1FX \arrow[r, "\mu^{(1)}"] & C^1X \end{tikzcd}$$

where $J^r\mu^{(1)}$ denotes the r-jet prolongation of the quotient projection $\mu^{(1)}$ (over X), and the canonical inclusions $C^{r+1}X \hookrightarrow J^{r+1}FX$, and $J^{r+1}FX \hookrightarrow J^rJ^1FX$. The composed mapping

$$\nu\colon\ C^{r+1}X \to J^rC^1X \tag{4.5}$$

has the expression

$$\begin{aligned} x^i \circ \nu &= x^i, \\ \Gamma^i_{kj} \circ \nu &= \Gamma^i_{kj}, \\ \Gamma^i_{kj,l} \circ \nu &= d_l\Gamma^i_{kj} = \Gamma^i_{klj} + \Gamma^i_{ks}\Gamma^s_{lj}, \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
\Gamma^i_{kj,l_1l_2} \circ \nu &= d_{l_2} d_{l_1} \Gamma^i_{kj} \\
&= \Gamma^i_{kl_1l_2j} + \Gamma^i_{kl_1s}\Gamma^s_{l_2j} + \Gamma^i_{kl_2s}\Gamma^s_{l_1j} + \Gamma^i_{ks}(\Gamma^s_{l_1l_2j} + \Gamma^s_{l_1q}\Gamma^q_{l_2j} + \Gamma^s_{l_2q}\Gamma^q_{l_1j}), \\
&\dots \\
\Gamma^i_{kj,l_1l_2\dots l_r} \circ \nu &= d_{l_r} \dots d_{l_2} d_{l_1} \Gamma^i_{kj}.
\end{aligned}
$$

To describe the image of ν in J^rC^1X, we denote by τ^1_3X the bundle of tensors of type $(1,3)$ over X, with induced coordinates S^i_{lkj}. We introduce the *formal curvature tensor* on J^rC^1X (see [12]) as a morphism $\varrho\colon\ J^rC^1X \to \tau^1_3X$ expressed by

$$S^i_{lkj} \circ \varrho = R^i_{lkj} = \Gamma^i_{kj,l} - \Gamma^i_{lj,k} + \Gamma^i_{ls}\Gamma^s_{kj} - \Gamma^i_{ks}\Gamma^s_{lj}$$

(compare with (4.2)).

Consider the sequence

$$0 \longrightarrow C^{r+1}X \xrightarrow{\ \nu\ } J^rC^1X \xrightarrow{\ \varrho\ } \tau^1_3X \longrightarrow 0$$

of bundles over X. We have the following result.

Lemma 4.3. *Let $J^r_x\Gamma \in J^rC^1X$. Then $\varrho(J^r_x\Gamma) = 0$ if and only if there exists an element $\Delta \in C^{r+1}X$ such that $\nu(\Delta) = J^r_x\Gamma$.*

Proof. $\Rightarrow$: Let $J^r_x\Gamma = (x^i, \Gamma^i_{kj}, \Gamma^i_{kj,l}, \Gamma^i_{kj,l_1l_2}, \dots, \Gamma^i_{kj,l_1l_2\dots l_r})$ be the coordinates of $J^r_x\Gamma \in J^rC^1X$ with respect to a fibred coordinate system (W^r, χ^r) on J^rC^1X, associated with the coordinate system (W, χ), $\chi = (x^i, \Gamma^i_{kj})$ on C^1X, where Γ is a section of C^1X defined in the neighborhood of $x \in X$. Let $\varrho(J^r_x\Gamma) = 0$, i.e., the coordinates of $J^r_x\Gamma$ satisfy $\Gamma^i_{kj,l} - \Gamma^i_{lj,k} + \Gamma^i_{ls}\Gamma^s_{kj} - \Gamma^i_{ks}\Gamma^s_{lj} = 0$. It means that $\Gamma^i_{klj} = (\Gamma^i_{kj,l} - \Gamma^i_{ks}\Gamma^s_{lj}) \circ \nu = (\Gamma^i_{lj,k} - \Gamma^i_{ls}\Gamma^s_{kj}) \circ \nu = \Gamma^i_{lkj}$, and Γ^i_{klj} are symmetric in k, l. Moreover, by (3.2), $\Gamma^i_{klmj} = d_m\Gamma^i_{klj} - \Gamma^i_{kls}\Gamma^s_{mj}$, which gives us symmetry of Γ^i_{klmj} in k, l. According to Lemma 3.4, $\Gamma^i_{klmj} - \Gamma^i_{kmlj} = -\Gamma^i_{kq}(\Gamma^q_{lmj} - \Gamma^q_{mlj}) = 0$, which gives us symmetry of Γ^i_{klmj} in l, m. It means that Γ^i_{klmj} is symmetric in the indices k, l, m and these coordinates of Δ are well defined. Continuing in this way, suppose that the coordinates $\Gamma^i_{kl_1l_2\dots l_{s-1}j}$ of Δ are defined. Again, by (3.2), $\Gamma^i_{kl_1l_2\dots l_{s-1}l_sj} = d_{l_s}\Gamma^i_{kl_1l_2\dots l_{s-1}j} - \Gamma^i_{kl_1l_2\dots l_{s-1}q}\Gamma^q_{l_sj}$, and symmetry of $\Gamma^i_{kl_1l_2\dots l_{s-1}j}$ in the indices $k, l_1, l_2, \dots, l_{s-1}$ ensures symmetry of $\Gamma^i_{kl_1l_2\dots l_sj}$ in $k, l_1, l_2, \dots, l_{s-1}$. According to Lemma 3.4, $\Gamma^i_{kl_1l_2\dots l_{s-1}l_sj} - \Gamma^i_{kl_1l_2\dots l_sl_{s-1}j} = -\Gamma^i_{kl_1l_2\dots l_{s-1}q}(\Gamma^q_{l_{s-1}l_sj} - \Gamma^q_{l_sl_{s-1}j}) = 0$, which gives us symmetry of $\Gamma^i_{kl_1l_2\dots l_sj}$ in l_{s-1}, l_s. It means that $\Gamma^i_{kl_1l_2\dots l_sj}$ is symmetric in the indices $k, l_1, l_2, \dots, l_{s-1}, l_s$ and $\Delta = (x^i, \Gamma^i_{kj}, \Gamma^i_{klj}, \Gamma^i_{kl_1l_2j}, \dots, \Gamma^i_{kl_1l_2\dots l_rj})$ is well defined. By (4.6), we have $\nu(\Delta) = J^r_x\Gamma$.

$\Leftarrow$: Let there exist $\Delta = J^{r+1}_x\gamma \in C^{r+1}X$ such that $\nu(J^{r+1}_x\gamma) = J^r_x\Gamma$. Then for the adapted coordinates $(x^i, \Gamma^i_{kj}, \Gamma^i_{klj}, \Gamma^i_{kl_1l_2j}, \dots, \Gamma^i_{kl_1l_2\dots l_rj})$ of $J^{r+1}_x\gamma$, $\Gamma^i_{klj} = \Gamma^i_{lkj}$ holds, which is equivalent to the relation $\Gamma^i_{kj,l} - \Gamma^i_{lj,k} + \Gamma^i_{ls}\Gamma^s_{kj} - \Gamma^i_{ks}\Gamma^s_{lj} = 0$ for the coordinates of $J^r_x\Gamma$ and $\varrho(J^r_x\Gamma) = 0$. $\square$

Let us denote by $\mathcal{S}$ the submanifold of J^rC^1X determined by $\mathcal{S} = \{\Lambda \in J^rC^1X\colon \varrho(\Lambda) = 0\}$. Lemma 4.3 shows that for $\Lambda \in \mathcal{S}$ the inverse

$$\sigma\colon\ \mathcal{S} \to C^{r+1}X \tag{4.7}$$

of ν (4.5) is defined. In the corresponding coordinates,

$$\begin{aligned}
x^i \circ \sigma &= x^i, \\
\Gamma^i_{kj} \circ \sigma &= \Gamma^i_{kj}, \\
\Gamma^i_{klj} \circ \sigma &= \Gamma^i_{kj,l} - \Gamma^i_{ks}\Gamma^s_{lj}, \\
\Gamma^i_{kl_1l_2j} \circ \sigma &= \tilde{d}_{l_2}(\Gamma^i_{kl_1j} \circ \sigma) - (\Gamma^i_{kl_1s} \circ \sigma)\Gamma^s_{l_2j} \\
&= \Gamma^i_{kj,l_1l_2} - \Gamma^i_{ks,l_1}\Gamma^s_{l_2j} - \Gamma^i_{ks,l_2}\Gamma^s_{l_1j} - \Gamma^i_{ks}\Gamma^s_{l_1j,l_2} + \Gamma^i_{kq}\Gamma^q_{l_2s}\Gamma^s_{pj}, \\
&\ \vdots \\
\Gamma^i_{kl_1l_2\dots l_rj} \circ \sigma &= \tilde{d}_{l_r}(\Gamma^i_{kl_1l_2\dots l_{r-1}j} \circ \sigma) - (\Gamma^i_{kl_1l_2\dots l_{r-1}s} \circ \sigma)\Gamma^s_{l_rj}.
\end{aligned} \tag{4.8}$$

The operator $\tilde{d}_m$ of formal derivative on J^rC^1X corresponds to the operator of formal derivative d_m on $C^{r+1}X$, given by Lemma 3.2. More exactly, $\tilde{d}_m(f \circ \sigma) = d_mf \circ \sigma$ for any function f on $C^{r+1}X$.

Using the mapping σ (4.7), it is possible to rewrite the Euler-Lagrange equations of the invariant variational problem, defined on $C^{r+1}X$ by a Lagrangian λ, as associated equations on J^rC^1X for connections with compatibility conditions as constraints.

Theorem 4.1. *Let $\lambda = \mathcal{L}\omega_0$ be a $\mathrm{Gl}_n(\mathbb{R})$-invariant Lagrangian on J^qFX and let $E_\lambda = F^l_i(\mathcal{L})y^j_l\,\mathrm{d}x^i_j \wedge \omega_0$ be its associated Euler-Lagrange form. The system of n^2 equations of order $r+1$ for extremals of the given variational problem,*

$$F^l_i(\mathcal{L}) \circ J^{r+1}\gamma = 0, \tag{4.9}$$

is equivalent to the system of n^2 rth order equations

$$G^l_i(\mathcal{L}) \circ J^r\Gamma = 0, \tag{4.10}$$

where

$$G^l_i(\mathcal{L}) = F^l_i(\mathcal{L}) \circ \sigma,$$

and σ is defined by (4.8), *together with the system of $\binom{n}{2} \cdot n^2$ first order equations in the form*

$$(\Gamma^i_{kj,l} - \Gamma^i_{lj,k} + \Gamma^i_{ls}\Gamma^s_{kj} - \Gamma^i_{ks}\Gamma^s_{lj}) \circ J^1\Gamma = 0. \tag{4.11}$$

Finding a solution $\Gamma\colon\ X \to C^1X$ of (4.10), the corresponding solution $\gamma\colon\ X \to FX$ of (4.9) is then obtained by solving the system (4.3).

5. Example

Let λ be a $\mathrm{Gl}_n(\mathbb{R})$-invariant Lagrangian on J^1FX. The coordinate expression of λ, in the coordinates $(x^i, x^i_j, \Gamma^i_{jk})$ adapted to the action of $\mathrm{Gl}_n(\mathbb{R})$ on J^1FX, is given by

$$\lambda = \mathcal{L}\omega_0, \quad \mathcal{L} = \mathcal{L}(x^i, \Gamma^i_{jk}).$$

The Euler-Lagrange form E_λ of λ is expressed as

$$E_\lambda = F^l_i(\mathcal{L}) y^j_l \, \mathrm{d}x^i_j \wedge \omega_0,$$

where

$$F^l_i(\mathcal{L}) = -\Gamma^p_{qi}\frac{\partial \mathcal{L}}{\partial \Gamma^p_{ql}} + \Gamma^l_{pq}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pq}} + \frac{\partial^2 \mathcal{L}}{\partial x^p \partial \Gamma^i_{pl}} + (\Gamma^k_{mpq} + \Gamma^k_{ms}\Gamma^s_{pq})\frac{\partial^2 \mathcal{L}}{\partial \Gamma^k_{mq}\partial \Gamma^i_{pl}}.$$

A section $\gamma\colon X \to FX$ is an extremal of the first order variational problem defined by λ if and only if the system of second order partial differential equations

$$F^l_i(\mathcal{L}) \circ J^2\gamma = 0$$

is satisfied.

According to Theorem 4.1, the problem of finding the extremal γ can be also represented as to find the corresponding section $\Gamma\colon X \to C^1X$ which is the solution of the system

$$G^l_i(\mathcal{L}) \circ J^1\Gamma = 0$$

of first order equations, where

$$G^l_i(\mathcal{L}) = -\Gamma^p_{qi}\frac{\partial \mathcal{L}}{\partial \Gamma^p_{ql}} + \Gamma^l_{pq}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pq}} + \frac{\partial^2 \mathcal{L}}{\partial x^p \partial \Gamma^i_{pl}} + \Gamma^k_{mq,p}\frac{\partial^2 \mathcal{L}}{\partial \Gamma^k_{mq}\partial \Gamma^i_{pl}},$$

satisfying the additional conditions (4.11) for Γ.

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Author's address: Ján Brajerčík, Department of Physics, Mathematics and Technology, University of Prešov, ul. 17. novembra 1, 081 16 Prešov, Slovak Republic, e-mail: `jan.brajercik@unipo.sk`.