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MORE EXAMPLES OF INVARIANCE UNDER TWISTING

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Abstract. The so-called "invariance under twisting" for twisted tensor products of algebras is a result stating that, if we start with a twisted tensor product, under certain circumstances we can "deform" the twisting map and we obtain a new twisted tensor product, isomorphic to the given one. It was proved before that a number of independent and previously unrelated results from Hopf algebra theory are particular cases of this theorem. In this article we show that some more results from literature are particular cases of invariance under twisting, for instance a result of Beattie-Chen-Zhang that implies the Blattner-Montgomery duality theorem.

Keywords: twisted tensor product, invariance under twisting, duality theorem

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1. INTRODUCTION

If A and B are (associative unital) algebras and $R\colon B\otimes A \rightarrow A\otimes B$ is a linear map satisfying certain axioms (such an R is called a *twisting map*) then $A \otimes B$ becomes an associative unital algebra with a multiplication defined in terms of R and the multiplications of A and B; this algebra structure on $A \otimes B$ is denoted by $A \otimes_R B$ and called the *twisted tensor product* of A and B afforded by R (cf. [2], [11]).

A very general result about twisted tensor products of algebras was obtained in [8]. It states that, if $A \otimes_R B$ is a twisted tensor product of algebras and on the vector space A we have one more algebra structure denoted by A' and we have also two linear maps $\varrho, \lambda\colon A \rightarrow A \otimes B$ satisfying a set of conditions, then one can define a new map $R'\colon B \otimes A' \rightarrow A' \otimes B$ by a certain formula, this map turns out to be a twisting map and we have an algebra isomorphism $A' \otimes_{R'} B \simeq A \otimes_R B$.

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This result was directly inspired by the invariance under twisting of the Hopf smash product (and thus it was called *invariance under twisting* for twisted tensor products of algebras), but it also contains as particular cases a number of independent and previously unrelated results from Hopf algebra theory, for instance Majid's theorem stating that the Drinfeld double of a quasitriangular Hopf algebra is isomorphic to an ordinary smash product (cf. [9]), a result of Fiore-Steinacker-Wess from [5] concerning a situation where a braided tensor product can be "unbraided", and also a result of Fiore from [4] concerning a situation where a smash product can be "decoupled".

The aim of this paper is to show that some more results from literature can be regarded as particular cases of invariance under twisting. Among them is a result from [1] concerning twistings of comodule algebras (which implies the Blattner-Montgomery duality theorem) and a generalization (obtained in [3]) of Majid's theorem mentioned before, in which quasitriangularity is replaced by a weaker condition, called semiquasitriangularity (a concept introduced in [6]).

2. Preliminaries

We work over a commutative field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. By "algebra" we always mean an associative unital algebra. We will denote by $\Delta(h) = h_1 \otimes h_2$ the comultiplication of a Hopf algebra H.

We recall from [2], [11] that, given two algebras A, B and a k-linear map R: $B \otimes A \to A \otimes B$, with notation $R(b \otimes a) = a_R \otimes b_R$ for $a \in A$, $b \in B$, satisfying the conditions $a_R \otimes 1_R = a \otimes 1$, $1_R \otimes b_R = 1 \otimes b$, $(aa')_R \otimes b_R = a_R a'_r \otimes b_{R_r}$, $a_R \otimes (bb')_R = a_{R_r} \otimes b_r b'_R$ for all $a, a' \in A$ and $b, b' \in B$ (where r is another copy of R), if we define on $A \otimes B$ a new multiplication by $(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'$, then this multiplication is associative with unit $1 \otimes 1$. In this case, the map R is called a *twisting map* between A and B and the new algebra structure on $A \otimes B$ is denoted by $A \otimes_R B$ and called the *twisted tensor product* of A and B afforded by R.

Theorem 2.1 ([8]). *Let $A \otimes_R B$ be a twisted tensor product of algebras, and denote the multiplication on A by $a \otimes a' \mapsto aa'$. Assume that on the vector space A we have one more algebra structure, denoted by A', with the same unit as A and multiplication denoted by $a \otimes a' \mapsto a * a'$. Assume that we are given two linear maps ϱ, λ: $A \to A \otimes B$, with notation $\varrho(a) = a_{(0)} \otimes a_{(1)}$ and $\lambda(a) = a_{[0]} \otimes a_{[1]}$, such that ϱ is an algebra map from A' to $A \otimes_R B$, $\lambda(1) = 1 \otimes 1$ and the following relations hold* *for all $a, a' \in A$:*

$$\lambda(aa') = a_{[0]} * (a'_R)_{[0]} \otimes (a'_R)_{[1]}(a_{[1]})_R, \tag{2.1}$$

$$a_{(0)_{[0]}} \otimes a_{(0)_{[1]}} a_{(1)} = a \otimes 1, \tag{2.2}$$

$$a_{[0]_{(0)}} \otimes a_{[0]_{(1)}} a_{[1]} = a \otimes 1. \tag{2.3}$$

Then the map $R'\colon B \otimes A' \to A' \otimes B$, $R'(b \otimes a) = (a_{(0)_R})_{[0]} \otimes (a_{(0)_R})_{[1]} b_R a_{(1)}$, is a twisting map and we have an algebra isomorphism $A' \otimes_{R'} B \simeq A \otimes_R B$, $a \otimes b \mapsto a_{(0)} \otimes a_{(1)} b$.

Given an algebra A, another algebra structure A' on the vector space A (as in Theorem 2.1) may sometimes be obtained by using the following result:

Theorem 2.2 ([8])**.** *Let A, B be two algebras and $R\colon B \otimes A \to A \otimes B$ a linear map, with notation $R(b \otimes a) = a_R \otimes b_R$ for all $a \in A$ and $b \in B$. Assume that we are given two linear maps, $\mu\colon B \otimes A \to A$, $\mu(b \otimes a) = b \cdot a$, and $\varrho\colon A \to A \otimes B$, $\varrho(a) = a_{(0)} \otimes a_{(1)}$, and denote $a * a' := a_{(0)}(a_{(1)} \cdot a')$ for all $a, a' \in A$. Assume that the following conditions are satisfied:*

$$\varrho(1) = 1 \otimes 1, \quad 1 \cdot a = a, \quad a_{(0)}(a_{(1)} \cdot 1) = a, \tag{2.4}$$

$$b \cdot (a * a') = a_{(0)_R}(b_R a_{(1)} \cdot a'), \tag{2.5}$$

$$\varrho(a * a') = a_{(0)} a'_{(0)_R} \otimes a_{(1)_R} a'_{(1)} \tag{2.6}$$

for all $a, a' \in A$ and $b \in B$. Then $(A, *, 1)$ is an associative unital algebra.

3. Examples

3.1. Twisting comodule algebras. Let H be a finite dimensional Hopf algebra and A a right H-comodule algebra, with multiplication denoted by $a \otimes a' \mapsto aa'$ and comodule structure denoted by $A \to A \otimes H$, $a \mapsto a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle}$. Let $\nu\colon H \to \mathrm{End}(A)$ be a convolution invertible linear map, with convolution inverse denoted by ν^{-1}. For $h \in H$ and $a \in A$, we denote $\nu(h)(a) = a \cdot h \in A$. For $a, a' \in A$ we denote $a * a' = (a \cdot a'_{\langle 1 \rangle}) a'_{\langle 0 \rangle} \in A$. Assume that for all $a, a' \in A$ and $h \in H$, the following conditions are satisfied:

$$a \cdot 1_H = a, \quad 1_A \cdot h = \varepsilon(h) 1_A, \tag{3.1}$$

$$(a \cdot h_2)_{\langle 0 \rangle} \otimes (a \cdot h_2)_{\langle 1 \rangle} h_1 = a_{\langle 0 \rangle} \cdot h_1 \otimes a_{\langle 1 \rangle} h_2, \tag{3.2}$$

$$(a * a') \cdot h = (a \cdot a'_{\langle 1 \rangle} h_2)(a'_{\langle 0 \rangle} \cdot h_1). \tag{3.3}$$

Then, by [1], Proposition 2.1, $(A, *, 1_A)$ is also a right H-comodule algebra (with the same H-comodule structure as for A), denoted in what follows by A_ν, and moreover, ν^{-1} satisfies the relations (3.2) and (3.3) for A_ν, that is, for all $a, a' \in A$ and $h \in H$ we have

$$(\nu^{-1}(h_2)(a))_{\langle 0\rangle} \otimes (\nu^{-1}(h_2)(a))_{\langle 1\rangle} h_1 = \nu^{-1}(h_1)(a_{\langle 0\rangle}) \otimes a_{\langle 1\rangle} h_2, \tag{3.4}$$

$$\nu^{-1}(h)(aa') = \nu^{-1}(a'_{\langle 1\rangle} h_2)(a) * \nu^{-1}(h_1)(a'_{\langle 0\rangle}). \tag{3.5}$$

Theorem 3.1 ([1]). *There exists an algebra isomorphism $A_\nu \# H^* \simeq A \# H^*$.*

We will prove that Theorem 3.1 is a particular case of Theorem 2.1.

We take in Theorem 2.1 the algebra A to be the original H-comodule algebra A, the second algebra structure A' on A to be the comodule algebra A_ν, and $B = H^*$. We consider $A \# H^*$ as the twisted tensor product $A \otimes_R H^*$, where $R\colon H^* \otimes A \to A \otimes H^*$, $R(\varphi \otimes a) = \varphi_1 \cdot a \otimes \varphi_2 = a_{\langle 0\rangle} \otimes \varphi \leftharpoonup a_{\langle 1\rangle}$ for all $\varphi \in H^*$ and $a \in A$, where $\leftharpoonup$ is the right regular action of H on H^*. Define the map $\varrho\colon A_\nu \to A \# H^*$, $\varrho(a) = \sum_i a \cdot e_i \# e^i := a_{(0)} \otimes a_{(1)}$, where $\{e_i\}$ and $\{e^i\}$ are dual bases in H and H^*. We will prove that ϱ is an algebra map. First, by using (3.1), it is easy to see that $\varrho(1_A) = 1_A \# \varepsilon$. We prove that ϱ is multiplicative. For $a, a' \in A$, we have

$$\varrho(a * a') = \sum_i (a * a') \cdot e_i \otimes e^i \overset{(3.3)}{=} \sum_i (a \cdot a'_{\langle 1\rangle}(e_i)_2)(a'_{\langle 0\rangle} \cdot (e_i)_1) \otimes e^i,$$

which applied to some $h \in H$ on the second component gives $(a \cdot a'_{\langle 1\rangle} h_2)(a'_{\langle 0\rangle} \cdot h_1)$. On the other hand, we have

$$\varrho(a)\varrho(a') = \sum_{i,j} (a \cdot e_i \# e^i)(a' \cdot e_j \# e^j) = \sum_{i,j} (a \cdot e_i)((e^i)_1 \cdot (a' \cdot e_j) \# (e^i)_2 e^j),$$

which applied to some $h \in H$ on the second component gives

$$\begin{aligned}
&\sum_i (a \cdot e_i)((e^i)_1 (e^i)_2(h_1) \cdot (a' \cdot h_2)) \\
&\qquad = \sum_i (a \cdot e_i)(e^i((a' \cdot h_2)_{\langle 1\rangle} h_1)(a' \cdot h_2)_{\langle 0\rangle}) \\
&\qquad = (a \cdot (a' \cdot h_2)_{\langle 1\rangle} h_1)(a' \cdot h_2)_{\langle 0\rangle} \overset{(3.2)}{=} (a \cdot a'_{\langle 1\rangle} h_2)(a'_{\langle 0\rangle} \cdot h_1),
\end{aligned}$$

showing that ϱ is indeed multiplicative.

Define now the map $\lambda\colon A \to A \otimes H^*$, $\lambda(a) = \sum_i \nu^{-1}(e_i)(a) \otimes e^i := a_{[0]} \otimes a_{[1]}$. First, it is obvious that $\lambda(1_A) = 1_A \otimes \varepsilon$, because ν^{-1} satisfies also the condition (3.1). We need to prove now that the relations (2.1), (2.2) and (2.3) are satisfied. It is easy to prove (2.2) and (2.3), because ν^{-1} is the convolution inverse of ν. We prove now (2.1). We have $\lambda(aa') = \sum_i \nu^{-1}(e_i)(aa') \otimes e^i$, which applied to some $h \in H$ on the second component gives $\nu^{-1}(h)(aa')$. On the other hand, we have

$$\begin{aligned} a_{[0]} * (a'_R)_{[0]} \otimes (a'_R)_{[1]}(a_{[1]})_R &= a_{[0]} * (a'_{\langle 0\rangle})_{[0]} \otimes (a'_{\langle 0\rangle})_{[1]}(a_{[1]} \leftharpoonup a'_{\langle 1\rangle}) \\ &= \sum_{i,j} \nu^{-1}(e_i)(a) * \nu^{-1}(e_j)(a'_{\langle 0\rangle}) \otimes e^j(e^i \leftharpoonup a'_{\langle 1\rangle}), \end{aligned}$$

which applied to some $h \in H$ on the second component gives $\nu^{-1}(a'_{\langle 1\rangle}h_2)(a) * \nu^{-1}(h_1)(a'_{\langle 0\rangle})$, and this is equal to $\nu^{-1}(h)(aa')$ because of the relation (3.5). Thus, all hypotheses of Theorem 2.1 are fulfilled, so we obtain the twisting map $R'\colon\ H^* \otimes A_\nu \to A_\nu \otimes H^*$, which looks as follows:

$$\begin{aligned} R'(\varphi \otimes a) &= (a_{(0)_R})_{[0]} \otimes (a_{(0)_R})_{[1]} \varphi_R a_{(1)} \\ &= a_{(0)_{\langle 0\rangle_{[0]}}} \otimes a_{(0)_{\langle 0\rangle_{[1]}}}(\varphi \leftharpoonup a_{(0)_{\langle 1\rangle}}) a_{(1)} \\ &= \sum_i (a \cdot e_i)_{\langle 0\rangle_{[0]}} \otimes (a \cdot e_i)_{\langle 0\rangle_{[1]}}(\varphi \leftharpoonup (a \cdot e_i)_{\langle 1\rangle}) e^i \\ &= \sum_{i,j} \nu^{-1}(e_j)((a \cdot e_i)_{\langle 0\rangle}) \otimes e^j(\varphi \leftharpoonup (a \cdot e_i)_{\langle 1\rangle}) e^i, \end{aligned}$$

which applied to some $h \in H$ on the second component gives

$$\begin{aligned} &\sum_i \nu^{-1}(h_1)((a \cdot e_i)_{\langle 0\rangle}) \varphi((a \cdot e_i)_{\langle 1\rangle} h_2) e^i(h_3) \\ &\qquad = \nu^{-1}(h_1)((a \cdot h_3)_{\langle 0\rangle}) \varphi((a \cdot h_3)_{\langle 1\rangle} h_2) \\ &\qquad \overset{(3.2)}{=} \nu^{-1}(h_1)(a_{\langle 0\rangle} \cdot h_2) \varphi(a_{\langle 1\rangle} h_3) \\ &\qquad = \nu^{-1}(h_1)(\nu(h_2)(a_{\langle 0\rangle})) \varphi(a_{\langle 1\rangle} h_3) \\ &\qquad = a_{\langle 0\rangle} \varphi(a_{\langle 1\rangle} h). \end{aligned}$$

Thus, we obtained $R'(\varphi \otimes a) = a_{\langle 0\rangle} \otimes \varphi \leftharpoonup a_{\langle 1\rangle}$ for all $\varphi \in H^*$ and $a \in A$, that is $R' = R$ and $A_\nu \otimes_{R'} H^* = A_\nu \# H^*$, and so Theorem 2.1 provides the algebra isomorphism $A_\nu \# H^* \simeq A \# H^*$, $a \otimes \varphi \mapsto a_{(0)} \otimes a_{(1)}\varphi = \sum_i a \cdot e_i \otimes e^i \varphi$, which is exactly Theorem 3.1.

3.2. External homogenization. Let H be a Hopf algebra and A a right H-comodule algebra, with comodule structure denoted by $a \mapsto a_{(0)} \otimes a_{(1)}$. We also denote $a_{(0)} \otimes a_{(1)} \otimes a_{(2)} = a_{(0)_{(0)}} \otimes a_{(0)_{(1)}} \otimes a_{(1)} = a_{(0)} \otimes a_{(1)_1} \otimes a_{(1)_2}$. The external homogenization of A, introduced in [10] and denoted by $A[H]$, is an H-comodule algebra structure on $A\otimes H$, with multiplication $(a\otimes h)(a'\otimes h') = aa'_{(0)}\otimes S(a'_{(1)})ha'_{(2)}h'$. By [10], $A[H]$ is isomorphic as an algebra to the ordinary tensor product $A\otimes H$.

We want to obtain this as a consequence of Theorem 2.1, actually, we will see that the data in Theorem 2.1 lead naturally to the multiplication of $A[H]$. Indeed, we will apply Theorem 2.1 to the following data: A is the original comodule algebra we started with, $B = H$, R is the usual flip between A and H, $A' = A$ as an algebra, ϱ is the comodule structure of A and $\lambda\colon\ A\to A\otimes H$ is given by $\lambda(a) = a_{(0)}\otimes S(a_{(1)}) := a_{[0]}\otimes a_{[1]}$. It is very easy to see that the hypotheses of Theorem 2.1 are fulfilled, so we obtain the twisting map $R'\colon\ H\otimes A\to A\otimes H$ given by

$$\begin{aligned} R'(h\otimes a) &= (a_{(0)})_{[0]}\otimes (a_{(0)})_{[1]}ha_{(1)}\\ &= a_{(0)_{(0)}}\otimes S(a_{(0)_{(1)}})ha_{(1)}\\ &= a_{(0)}\otimes S(a_{(1)})ha_{(2)}, \end{aligned}$$

and obviously $A\otimes_{R'} H = A[H]$. Thus, as a consequence of Theorem 2.1, we obtain the algebra isomorphism from [10]: $A[H]\simeq A\otimes H$, $a\otimes h\mapsto a_{(0)}\otimes a_{(1)}h$.

3.3. Doubles of semiquasitriangular Hopf algebras. Let H be a finite dimensional Hopf algebra and $r\in H\otimes H$ an invertible element, denoted by $r = r^1\otimes r^2$, with inverse $r^{-1} = u^1\otimes u^2$. Consider the Drinfeld double $D(H)$, which is the tensor product $H^*\otimes H$ endowed with the multiplication $(\varphi\otimes h)(\varphi'\otimes h') = \varphi(h_1\rightharpoonup\varphi'\leftharpoonup S^{-1}(h_3))\otimes h_2h'$ for all $h, h'\in H$ and $\varphi,\varphi'\in H^*$, where $\rightharpoonup$ and $\leftharpoonup$ are the regular actions of H on H^*.

Define maps

$$\begin{aligned} &f\colon\ D(H)\to H^*\otimes H,\ f(\varphi\otimes h) = \varphi\leftharpoonup S^{-1}(u^1)\otimes u^2h,\\ &g\colon\ H^*\otimes H\to D(H),\ g(\varphi\otimes h) = \varphi\leftharpoonup S^{-1}(r^1)\otimes r^2h. \end{aligned}$$

It is obvious that f and g are linear isomorphisms, inverse to each other, so we can transfer the algebra structure of $D(H)$ to $H^*\otimes H$ via these maps. It is natural to ask under what conditions on r this algebra structure on $H^*\otimes H$ is a twisted tensor product between H and a certain algebra structure on H^*.

We claim that this is the case if r satisfies the following conditions:

$$\Delta(r^1)\otimes r^2 = \mathcal{R}^1\otimes r^1\otimes\mathcal{R}^2r^2, \tag{3.6}$$

$$r^1\otimes\Delta(r^2) = \mathcal{R}^1r^1\otimes r^2\otimes\mathcal{R}^2, \tag{3.7}$$

$$\mathcal{R}^1\otimes\mathcal{R}^2_2r^1\otimes\mathcal{R}^2_1r^2 = \mathcal{R}^1\otimes r^1\mathcal{R}^2_1\otimes r^2\mathcal{R}^2_2, \tag{3.8}$$

where $\mathcal{R}^1 \otimes \mathcal{R}^2$ is another copy of r. We will obtain this result as a consequence of Theorem 2.1, combined with Theorem 2.2. Note that the above conditions are part of the axioms of a so-called semiquasitriangular structure (cf. [6]), and that if r satisfies also the other axioms in [6] then it was proved in [3] that $D(H)$ is isomorphic as a Hopf algebra to a Hopf crossed product in the sense of [7].

We take $A = H^*$, with its ordinary algebra structure, $B = H$, and $R\colon H \otimes H^* \to H^* \otimes H$, $R(h \otimes \varphi) = h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_3) \otimes h_2$, hence $A \otimes_R B = D(H)$. Then we define maps

$$
\begin{aligned}
&\mu\colon\ H \otimes H^* \to H^*,\ \ \mu(h \otimes \varphi) = h \cdot \varphi := h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_2),\\
&\varrho\colon\ H^* \to H^* \otimes H,\ \ \varrho(\varphi) = \varphi_{(0)} \otimes \varphi_{(1)} := \varphi \leftharpoonup S^{-1}(r^1) \otimes r^2,\\
&\lambda\colon\ H^* \to H^* \otimes H,\ \ \lambda(\varphi) = \varphi_{[0]} \otimes \varphi_{[1]} := \varphi \leftharpoonup S^{-1}(u^1) \otimes u^2.
\end{aligned}
$$

The corresponding product $*$ on H^* provided by Theorem 2.2 is given by

$$
\begin{aligned}
\varphi * \varphi' &= \varphi_{(0)}(\varphi_{(1)} \cdot \varphi')\\
&= (\varphi \leftharpoonup S^{-1}(r^1))(r^2 \cdot \varphi')\\
&= (\varphi \leftharpoonup S^{-1}(r^1))(r^2_1 \rightharpoonup \varphi' \leftharpoonup S^{-1}(r^2_2)).
\end{aligned}
$$

We need to prove that the relations (2.4)–(2.6) hold. We note first that as consequences of (3.6) and (3.7) we obtain $\varepsilon(r^1)r^2 = r^1\varepsilon(r^2) = 1 = \varepsilon(u^1)u^2 = u^1\varepsilon(u^2)$, hence we have $\varrho(\varepsilon) = \lambda(\varepsilon) = \varepsilon \otimes 1$ and also we obtain immediately $1 \cdot \varphi = \varphi$ and $\varphi_{(0)}(\varphi_{(1)} \cdot \varepsilon) = \varphi$ for all $\varphi \in H^*$, thus (2.4) holds. We prove now (2.5). We compute:

$$
\begin{aligned}
h \cdot (\varphi * \varphi') &= h_1 \rightharpoonup (\varphi * \varphi') \leftharpoonup S^{-1}(h_2)\\
&= (h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_4 r^1))(h_2 r^2_1 \rightharpoonup \varphi' \leftharpoonup S^{-1}(h_3 r^2_2)),\\
{\varphi_{(0)}}_R(h_R \varphi_{(1)} \cdot \varphi') &= (\varphi \leftharpoonup S^{-1}(r^1))_R(h_R r^2 \cdot \varphi')\\
&= (h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_3 r^1))(h_2 r^2 \cdot \varphi')\\
&= (h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_4 r^1))(h_2 r^2_1 \rightharpoonup \varphi' \leftharpoonup S^{-1}(h_3 r^2_2)),\ \text{q.e.d.}
\end{aligned}
$$

In order to prove (2.6), we prove first the following relation:

$$
r^1 \otimes r^2_1 \otimes r^2_3 \mathcal{R}^1 \otimes r^2_2 \mathcal{R}^2 = \mathcal{R}^1_2 r^1 \otimes r^2_1 \otimes \mathcal{R}^1_1 r^2_2 \otimes \mathcal{R}^2. \tag{3.9}
$$

We compute (denoting by $r = \mathfrak{R}^1 \otimes \mathfrak{R}^2 = \varrho^1 \otimes \varrho^2$ two more copies of r):

$$\begin{aligned} r^1 \otimes r_1^2 \otimes r_3^2\mathcal{R}^1 \otimes r_2^2\mathcal{R}^2 &\overset{(3.7)}{=} \mathfrak{R}^1 r^1 \otimes r^2 \otimes \mathfrak{R}_2^2\mathcal{R}^1 \otimes \mathfrak{R}_1^2\mathcal{R}^2 \\ &\overset{(3.8)}{=} \mathfrak{R}^1 r^1 \otimes r^2 \otimes \mathcal{R}^1\mathfrak{R}_1^2 \otimes \mathcal{R}^2\mathfrak{R}_2^2 \\ &\overset{(3.7)}{=} \mathfrak{R}^1\varrho^1 r^1 \otimes r^2 \otimes \mathcal{R}^1\varrho^2 \otimes \mathcal{R}^2\mathfrak{R}^2, \\ \mathcal{R}_2^1 r^1 \otimes r_1^2 \otimes \mathcal{R}_1^1 r_2^2 \otimes \mathcal{R}^2 &\overset{(3.6)}{=} \mathfrak{R}^1 r^1 \otimes r_1^2 \otimes \mathcal{R}^1 r_2^2 \otimes \mathcal{R}^2\mathfrak{R}^2 \\ &\overset{(3.7)}{=} \mathfrak{R}^1\varrho^1 r^1 \otimes r^2 \otimes \mathcal{R}^1\varrho^2 \otimes \mathcal{R}^2\mathfrak{R}^2, \end{aligned}$$

and we see that the two terms coincide. Now we prove (2.6); we compute:

$$\begin{aligned} \varrho(\varphi * \varphi') &= (\varphi * \varphi') \leftharpoonup S^{-1}(\mathcal{R}^1) \otimes \mathcal{R}^2 \\ &= (\varphi \leftharpoonup S^{-1}(\mathcal{R}_2^1 r^1))(r_1^2 \rightharpoonup \varphi' \leftharpoonup S^{-1}(\mathcal{R}_1^1 r_2^2)) \otimes \mathcal{R}^2, \\ \varphi_{(0)}\varphi'_{(0)_R} \otimes \varphi_{(1)_R}\varphi'_{(1)} &= (\varphi \leftharpoonup S^{-1}(r^1))(\varphi' \leftharpoonup S^{-1}(\mathcal{R}^1))_R \otimes r_R^2\mathcal{R}^2 \\ &= (\varphi \leftharpoonup S^{-1}(r^1))(r_1^2 \rightharpoonup \varphi' \leftharpoonup S^{-1}(r_3^2\mathcal{R}^1)) \otimes r_2^2\mathcal{R}^2, \end{aligned}$$

and the two terms are equal because of (3.9).

Thus, we can apply Theorem 2.2 and we obtain that $(H^*, *, \varepsilon)$ is an associative algebra, which will be denoted in what follows by $\underline{H}^*$.

We will prove now that the hypotheses of Theorem 2.1 are fulfilled for $A' = \underline{H}^*$. Note first that the relations (2.4) and (2.6) proved before imply that ϱ is an algebra map from $\underline{H}^*$ to $H^* \otimes_R H$. We have already seen that $\lambda(\varepsilon) = \varepsilon \otimes 1$, so we only have to check the relations (2.1)–(2.3). To prove (2.1), we compute (we denote by $r^{-1} = U^1 \otimes U^2 = \mathfrak{U}^1 \otimes \mathfrak{U}^2$ some more copies of r^{-1}):

$$\begin{aligned} \lambda(\varphi\varphi') &= (\varphi\varphi') \leftharpoonup S^{-1}(u^1) \otimes u^2 \\ &= (\varphi \leftharpoonup S^{-1}(u_2^1))(\varphi' \leftharpoonup S^{-1}(u_1^1)) \otimes u^2 \\ &\overset{(3.6)}{=} (\varphi \leftharpoonup S^{-1}(u^1))(\varphi' \leftharpoonup S^{-1}(U^1)) \otimes u^2U^2, \end{aligned}$$

$$\begin{aligned} &\varphi_{[0]} * (\varphi'_R)_{[0]} \otimes (\varphi'_R)_{[1]}(\varphi_{[1]})_R \\ &= (\varphi \leftharpoonup S^{-1}(u^1)) * (\varphi'_R \leftharpoonup S^{-1}(U^1)) \otimes U^2u_R^2 \\ &= (\varphi \leftharpoonup S^{-1}(u^1)) * (u_1^2 \rightharpoonup \varphi' \leftharpoonup S^{-1}(U^1u_3^2)) \otimes U^2u_2^2 \\ &= (\varphi \leftharpoonup S^{-1}(r^1u^1))(r_1^2u_1^2 \rightharpoonup \varphi' \leftharpoonup S^{-1}(r_2^2U^1u_3^2)) \otimes U^2u_2^2 \\ &\overset{(3.7)}{=} (\varphi \leftharpoonup S^{-1}(r^1u^1\mathfrak{U}^1))(r_1^2u^2 \rightharpoonup \varphi' \leftharpoonup S^{-1}(r_2^2U^1\mathfrak{U}_2^2)) \otimes U^2\mathfrak{U}_1^2 \\ &\overset{(3.8)}{=} (\varphi \leftharpoonup S^{-1}(r^1u^1\mathfrak{U}^1))(r_1^2u^2 \rightharpoonup \varphi' \leftharpoonup S^{-1}(r_2^2\mathfrak{U}_1^2U^1)) \otimes \mathfrak{U}_2^2U^2 \\ &\overset{(3.7)}{=} (\varphi \leftharpoonup S^{-1}(r^1\mathfrak{U}^1))(\varphi' \leftharpoonup S^{-1}(r^2\mathfrak{U}_1^2U^1)) \otimes \mathfrak{U}_2^2U^2 \\ &\overset{(3.7)}{=} (\varphi \leftharpoonup S^{-1}(u^1))(\varphi' \leftharpoonup S^{-1}(U^1)) \otimes u^2U^2, \end{aligned}$$

and we see that the two terms are equal. The remaining relations (2.2) and (2.3) are very easy to prove and are left to the reader. Thus, we can apply Theorem 2.1 and we obtain the twisting map $R'\colon\ H \otimes \underline{H}^* \to \underline{H}^* \otimes H$,

$$R'(h \otimes \varphi) = ({\varphi_{(0)}}_R)_{[0]} \otimes ({\varphi_{(0)}}_R)_{[1]} h_R \varphi_{(1)} = h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(u^1 h_3 r^1) \otimes u^2 h_2 r^2,$$

and the algebra isomorphism $\underline{H}^* \otimes_{R'} H \simeq H^* \otimes_R H = D(H)$, given by

$$\varphi \otimes h \mapsto \varphi_{(0)} \otimes \varphi_{(1)} h = \varphi \leftharpoonup S^{-1}(r^1) \otimes r^2 h,$$

which is exactly the linear isomorphism g defined before. Thus, we have proved that if r satisfies the conditions (3.6)–(3.8) then $D(H)$ is isomorphic as an algebra to a twisted tensor product between $\underline{H}^*$ and H.

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