Tomáš Kepka

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T. KEPKA
Department of Mathematics, Charles University, Prague

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1° We shall usually write the binary operation of a groupoid multiplicatively. When using the other symbol of the binary operation, we put this symbol into brackets, e. g.: $G(\cdot)$, $H(o)$ e.t.c.

Let G be a groupoid and $x \in G$. By the symbol L_x (R_x) we shall denote the mapping of the set G into G such that for every $y \in G$, $L_x(y) = xy$ $(R_x(y) = yx)$.

A groupoid G is called a groupoid with left (right) cancellation, if for every $x \in G$ the mapping L_x (R_x) is one — to — one.

A groupoid G is called a groupoid with left (right) division, if for every $x \in G$ the mapping L_x (R_x) is onto G.

A groupoid G is called a left (right) quasigroup, if for every $x \in G$ the mapping L_x (R_x) is a permutation of the set G (permutation is a mapping of a set into itself, which is one — to — one and onto the set).

A groupoid, which is simultaneously with left and right cancellation (division), is called a groupoid with cancellation (division).

A groupoid, which is simultaneously a left and right quasigroup, is called a quasigroup.

2° **Definition 1:** Let G be a groupoid. A mapping λ (ϱ) of the set G into G is called left (right) regular, if there is a mapping λ^* (ϱ^*) such that for every $x, y \in G$, $\lambda(xy) = \lambda^*(x) \, . \, y$ $(\varrho(xy) = x \, . \, \varrho^*(y))$.

A mapping φ is called central regular, if there is a mapping φ^* such that for every $x, y \in G$, $\varphi(x) \, . \, y = x \, . \, \varphi^*(y)$. By the symbol Λ_G we shall denote the set of all left regular mappings of the groupoid G and by Λ_G^* the set of all possible mappings λ^* corresponding to the left regular mappings. Similarly introduce the symbols R_G, R_G^*, Φ_G, Φ_G^*.

Lemma 1: Let G be a groupoid. Then the sets Λ_G, Λ_G^*, R_G, R_G^*, Φ_G, Φ_G^* are semigroups with unit under the binary operation of composition of mappings.

Proof: We shall prove the Lemma for Λ_G, Λ_G^* only. For the other cases the proof is similar.

Let $\lambda_1, \lambda_2 \in \Lambda_G$. Let $\lambda_1^*, \lambda_2^* \in \Lambda_G^*$ be arbitrary mappings corresponding to the mappings λ_1, λ_2. For every $x, y \in G$, $\lambda_1\lambda_2(xy) = \lambda_1(\lambda_2^*(x) \, . \, y) = \lambda_1^*\lambda_2^*(x) \, . \, y$.

Hence $\lambda_1\lambda_2 \in \Lambda_G$ and $\lambda_1^*\lambda_2^* \in \Lambda_G^*$ Evidently $1_G \in \Lambda_G$, $1_G \in \Lambda_G^*$.

Lemma 2: Let G be a groupoid and $\lambda \in \Lambda_G$, $\varrho \in R_G$, $\varphi \in \Phi_G$. Then $\lambda L_x = L_{\lambda^*(x)}$, $\lambda R_x = R_x\lambda^*$, $\varrho R_x = R_{\varrho^*(x)}$, $\varrho L_x = L_x\varrho^*$, $R_x\varphi = R_{\varphi^*(x)}$, $L_{\varphi(x)} = L_x\varphi^*$ for every $x \in G$.

Proof: By Definition 1.

Corollary: 1. Let G be a groupoid with left (right) division. Then all left (right) regular mappings of the groupoid G are onto G.

2. Let G be a groupoid with cancellation. Then all central regular mappings of G are one — to — one.

3. Let G be a quasigroup. Let $\lambda \in \Lambda_G$, $\varrho \in R_G$, $\varphi \in \Phi_G$. Then the mappings $\lambda,^* \varrho^*, \varphi^*$. are uniquely determined and λ, λ^*, ϱ, ϱ^*, φ, φ^* are permutations.

Lemma 3: Let G be a groupoid and $\lambda \in \Lambda_G$, $\varrho \in R_G$, $\varphi \in \Phi_G$ such that λ^*, ϱ^*, φ^* are mappings onto G. Let α, β, γ be arbitrary mappings such that $\alpha\lambda = \beta\varrho = \varphi\gamma = 1_G$. Then also $\alpha \in \Lambda_G$, $\beta \in R_G$, $\gamma \in \Phi_G$.

Proof: 1) Since λ^* is a mapping onto, there is a mapping δ so that $\lambda^*\delta = 1_G$. For every $x, y \in G$ we have $\lambda(xy) = \lambda^*(x) . y$. Hence $\alpha\lambda(\delta(x) . y) = \delta(x) . y = \alpha(\lambda^*\delta(x) . . y) = \alpha(xy)$. Thus $\alpha \in \Lambda_G$. For β similarly.

2) There is a mapping ε such that $\varphi^*\varepsilon = 1_G$. For every x, $y \in G$, $\varphi(x) . y = = x . \varphi^*(y)$. Hence $x . \varepsilon(y) = \gamma(x) . y$. Thus $\gamma \in \Phi_G$.

Theorem 1: Let G be a quasigroup. Then the semigroups Λ_G, R_G, Φ_G are groups.

Proof: By Corollary and Lemma 1, 3.

Lemma 4: Let G be a groupoid with right (left) unit e. Let $\lambda \in \Lambda_G$ ($\varrho \in R_G$) Then the mapping λ^* (ϱ^*) is uniquely determined and $\lambda = \lambda^*$ ($\varrho = \varrho^*$).

Proof.: Let $x \in G$. Then $\lambda(x) = \lambda(xe) = \lambda^*(x) . e = \lambda^*(x)$. Hence $\lambda = \lambda^*$. Similarly for ϱ.

Lemma 5: Let G be a groupoid with unit e. Let $\lambda \in \Lambda_G$, $\varrho \in R_G$, $\varphi \in \Phi_G$. Then $\lambda = L_{\lambda(e)}$, $\varrho = R_{\varrho(e)}$, $\varphi = R_{\varphi(e)}$, $\varphi^* = L_{\varphi(e)}$. Let x, $y \in G$. Then $\lambda(e)(xy) = = (\lambda(e)x)y$, $(xy)\varrho(e) = x(y . \varrho(e))$, $(x\varphi(e))y = x(\varphi(e) . y)$.

Proof: By Lemma 4, $\lambda = \lambda^*$. Let $x \in G$. Then $\lambda(x) = \lambda(ex) = \lambda(e)x = L_{\lambda(e)}(x)$. Hence $L_{\lambda(e)} = \lambda$. For every x, $y \in G$, $\lambda(e)(xy) = \lambda(e . xy) = \lambda(xy) = \lambda(x) . y = = (\lambda(e)x) . y$.

For ϱ similarly.

2) Let $x, y \in G$. We have $\varphi(x) = \varphi(x) . e = \varphi^*(e)$. Hence $\varphi^*(e) = e . \varphi^*(e) = \varphi(e)$, hence, $\varphi = R_{\varphi(e)}$. Further, $x(\varphi(e) . y) = x(e . \varphi^*(y)) = x . \varphi^*(y) = \varphi(x) . y = = (x . \varphi(e)) . y$.

Theorem 2: Let G be a groupoid (quasigroup) with unit e.

Put $A_G = E(x \in G/\exists\ \lambda \in \Lambda_G, x = \lambda(e))$,

$B_G = E(x/\exists\varrho \in R_G, x = \varrho(e))$, $C_G = E(x/\exists\varphi \in \Phi_G, x = \varphi(e))$.

Then the sets A_G, B_G, C_G are subsemigroups (subgroups) with unit of the groupoid (quasigroup) G.

Proof: We shall prove the theorem for A_G only.
1) Let $x, y \in A_G$. Then there are $\lambda_1, \lambda_2 \in \Lambda_G$ so that $x = \lambda_1(e)$, $y = \lambda_2(e)$. Hence $xy = \lambda_1(e) . \lambda_2(e) = \lambda_1(e . \lambda_2(e)) = \lambda_1\lambda_2(e)$. But $\lambda_1\lambda_2 \in \Lambda_G$ by Lemma 1. Hence $xy \in A_G$. We have proved that A_G is a subgroupoid of G. By Lemma 5, A_G is a semigroup. Evidently $e \in A_G$.
2) Let G be a quasigroup. By 1), A_G is a semigroup with unit e.
Let $x \in A_G$. Then there is $\lambda \in \Lambda_G$ such that $x = \lambda(e)$.
By Theorem 1, λ is a permutation and $\lambda^{-1} \in \Lambda_G$. Hence $\lambda^{-1}(e) \in A_G$.
But $x . \lambda^{-1}(e) = \lambda(e) . \lambda^{-1}(e) = \lambda(e . \lambda^{-1}(e)) = \lambda\lambda^{-1}(e) = e$.
The element $\lambda^{-1}(e)$ is a right inverse element to x. Therefore A_G is a group.

Definition 2: A groupoid G is called Λ — transitive if for every $x, y \in G$ there is $\lambda \in \Lambda_G$ such that $\lambda(x) = y$. Similarly for Λ^*, R, R^*, Φ, Φ^* — transitivity. A groupoid G is called transitive, if at least one of the cases defined is valid.

Lemma 6: Every group is transitive in all possible cases.
Proof: As a group G is a groupoid with unit, we have $\Lambda_G = \Lambda_G^*$, $R_G = R_G^*$. Further for all $x \in G$, $L_x \in \Lambda_G$, $R_x \in R_G$, $R_x \in \Phi_G$, $L_x \in \Phi_G^*$. Hence G is Λ, Λ^*, R, R^*, Φ, Φ^* — transitive.

Lemma 7: Let G be a Λ or Φ^* — transitive groupoid with left unit. Then G is a groupoid with right division. Let G be a R or Φ — transitive groupoid with right unit. Then G is a groupoid with left division.
Proof: 1) Let G be Λ — transitive and e be a left unit of G.
Let $x, y \in G$. There is $\lambda \in \Lambda_G$ such that $\lambda(x) = y$. But $\lambda(x) = \lambda(ex) = \lambda^*(e) . x = y$. Hence R_x is a mapping onto G. Hence G is with right division.
2) Let G be Φ^* — transitive and e be a left unit of G.
Let $x, y \in G$. There is $\varphi^* \in \Phi_G^*$ such that $\varphi^*(x) = y$. We have, $y = e . \varphi^*(x) =$
$= \varphi(e) . x$. Hence G is with right division.
Similarly for the other cases.

Theorem 3: A groupoid with unit is transitive if and only if it is a group.
Proof: 1) Let G be a transitive groupoid with unit. Hence $\Lambda_G = \Lambda_G^*$, $R_G = R_G^*$ by Lemma 4. Now we can use Lemma 7. Hence G is with left or right division. Since G is transitive, we have, by Definition 2, $G = A_G$ or $G = B_G$ or $G = C_G$. Thus by Theorem 2 G is a semigroup with unit. But every semigroup with unit, which is with left or right division, is a group.
2) Let G be a group. By Lemma 6 G is transitive in all possible cases.

3° **Definition 3:** Let G be a groupoid. A groupoid $G(\bullet)$ is called a homotope of the groupoid G, if there are mappings α, β of the set G into G and a permutation γ of the set G so that for every $x, y \in G$, $\gamma(x \bullet y) = \alpha(x) . \beta(y)$. We shall write $G(\bullet) =$
$= G^{(\alpha,\beta,\gamma)}$. The groupoid $G(\bullet)$ is called a μ — homotope of the groupoid G, if α, β are onto G. The groupoid $G(\bullet)$ is called an isotope of G, if α, β are permutations. The groupoid $G(\bullet)$ is called a principal homotope, if $\gamma = 1_G$.
The following Lemma is evident.

Lemma 8: 1) Let $G(\bullet) = G(o)^{(\alpha,\beta,\gamma)}$ and $G(o) = G^{(\delta,\varepsilon,\varkappa)}$. Then $G(\bullet) = G^{(\delta\alpha,\varepsilon\beta,\varkappa\gamma)}$.
2) For every G is $G = G^{(1_G,1_G,1_G)}$.
3) A mapping $\gamma : G(\bullet) \to G$ is an isomorphism if and only if $G(\bullet) = G^{(\gamma,\gamma,\gamma)}$.
4) Let $G(\bullet) = G^{(\alpha,\beta,\gamma)}$ and δ, ε be arbitrary mappings such that $\alpha\delta = \beta\varepsilon = 1_G$. Then G is a homotope of $G(\bullet)$ and $G = G(\bullet)^{(\delta,\varepsilon,\gamma^{-1})}$.
5) Let $G(\bullet) = G^{(\alpha,\beta,\gamma)}$ be an isotope of G. Then G is an isotope of $G(\bullet)$ and $G = G(\bullet)^{(\alpha^{-1},\beta^{-1},\gamma^{-1})}$.
6) Let $G(\bullet) = G^{(\alpha,\beta,\gamma)}$. Put $G(o) = G(\bullet)^{(\gamma^{-1},\gamma^{-1},\gamma^{-1})}$. Then $\gamma : G(\bullet) \to G(o)$ is an isomorphism and $G(o) = G^{(\gamma^{-1}\alpha,\gamma^{-1}\beta,1_G)}$.

Lemma 9: Every μ – homotope of a groupoid with division is a groupoid with division.
Proof: Let G be a groupoid with division and $G(o)$ be a μ – homotope of G; $G(o) = G^{(\alpha,\beta,\gamma)}$. Denote R_x^*, L_y^* translations of the groupoid $G(o)$. Let $x, y \in G$. We have $\gamma(x \, o \, y) = \alpha(x) \, . \, \beta(y)$, hence $\gamma R_y^* = R_{\beta(y)} \alpha$, and hence, $R_y^* = \gamma^{-1} R_{\beta(y)} \alpha$. But γ^{-1}, $R_{\beta(y)}$, α are mappings onto G, hence R_y^* is a mapping onto G.
Similarly for L_x^*.

Lemma 10: Let G be a groupoid with cancellation and $G(o) = G^{(\alpha,\beta,\gamma)}$. Let α, β be one — to — one mappings. Then $G(o)$ is also a groupoid with cancellation.
Proof: Similarly as for Lemma 9.

Theorem 4: Every groupoid which is an isotope of a quasigroup, is a quasigroup.
Proof: By Lemma 9, 10.

Lemma 11: Let $G(o) = G^{(\alpha,\beta,\gamma)}$. Let $\lambda \in \Lambda_{G(o)}$, $\varrho \in R_{G(o)}$, $\varphi \in \Phi_{G(o)}$ and δ, ε be arbitrary mappings such that $\alpha\delta = \beta\varepsilon = 1_G$. Then $\gamma\lambda\gamma^{-1} \in \Lambda_G$, $\alpha\lambda^*\delta \in \Lambda_G^*$, $\gamma\varrho\gamma^{-1} \in R_G$, $\beta\varrho^*\varepsilon \in R_G^*$, $\alpha\varphi\delta \in \Phi_G$, $\beta\varphi^*\varepsilon \in \Phi_G^*$.
Proof: 1) For every $x, y \in G$, $\gamma\lambda\gamma^{-1}(\alpha(x) \, . \, \beta(y)) = \gamma \, \lambda(x \, o \, y) = \gamma(\lambda^*(x) \, o \, y) = \alpha\lambda^*(x) \, . \, \beta(y)$. Hence for every $u, v \in G$, $\gamma\lambda\gamma^{-1}(uv) = \gamma\lambda\gamma^{-1}(\alpha\delta(u) \, . \, \beta\varepsilon(v)) = \alpha\lambda^*\delta(u) \, . \, \beta\varepsilon(v) = \alpha\lambda^*\delta(u) \, . \, v$. Thus $\gamma\lambda\gamma^{-1} \in \Lambda_G$, $\alpha\lambda^*\delta \in \Lambda_G^*$.
Similarly for ϱ.
2) For every $x, y \in G$, $\alpha\varphi(x) \, . \, \beta(y) = \gamma(\varphi(x) o y) = \gamma(x \, o \, \varphi^*(y)) = \alpha(x) \, o \, \beta\varphi^*(y)$. Hence for every $u, v \in G$, $\alpha\varphi\delta(u) \, . \, v = \alpha\varphi\delta(u) \, . \, \beta\varepsilon(v) = \alpha\delta(u) \, . \, \beta\varphi^*\varepsilon(v) = u \, . \, \beta\varphi^*\varepsilon(v)$. Thus $\alpha\varphi\delta \in \Phi_G$, $\beta\varphi^*\varepsilon \in \Phi_G^*$.

Theorem 5: Let $G(o)$ be an isotope of a groupoid G. Then the following isomorphisms are valid: $\Lambda_{G(o)} \cong \Lambda_G$, $\Lambda_{G(o)}^* \cong \Lambda_G^*$, $R_{G(o)} \cong R_G$, $R_{G(o)}^* \cong R_G^*$, $\Phi_{G(o)} \cong \Phi_G$, $\Phi_{G(o)}^* \cong \Phi_G^*$.
Proof: Let $G(o) = G^{(\alpha,\beta,\gamma)}$. Then, by Lemma 8, $G = G(o)^{(\alpha^{-1},\beta^{-1},\gamma^{-1})}$. By Lemma 11, $\lambda \in \Lambda_{G(o)} \Leftrightarrow \gamma\lambda\gamma^{-1} \in \Lambda_G$, $\lambda^* \in \Lambda_{G(o)}^* \Leftrightarrow \alpha\lambda^*\alpha^{-1} \in \Lambda_G^*$. The mappings $A : \Lambda_{G(o)} \to \Lambda_G$, $B : \Lambda_{G(o)}^* \to \Lambda_G^*$ such that $A(\lambda) = \gamma\lambda\gamma^{-1}$, $B(\lambda^*) = \alpha\lambda^*\alpha^{-1}$ for all $\lambda \in \Lambda_{G(o)}$, $\lambda_* \in \Lambda_{G(o)}^*$, are evidently isomorphisms.
Similarly for the other cases.

Theorem 6: Let $G(o)$ be a μ – homotope of G. Let $G(o)$ be Λ – transitive

($\Lambda^{\bullet}$, R, $R^{\bullet}$, Φ, $\Phi^{\bullet}$ — transitive). Then G is Λ — transitive ($\Lambda^{\bullet}$, R, $R^{\bullet}$, Φ, $\Phi^{\bullet}$ — transitive).

Proof: Let $G(o) = G^{(\alpha,\beta,\gamma)}$. Since the mappings α, β are onto G, there are mappings δ, ε such that $\alpha\delta = \beta\varepsilon = 1_G$. Let $G(o)$ be Λ — transitive and $x, y \in G$. There is $\lambda \in \Lambda_{G(o)}$ such that $\lambda\gamma^{-1}(x) = \gamma^{-1}(y)$. Hence $\gamma\lambda\gamma^{-1}(x) = y$. By Lemma 11, $\gamma\lambda\gamma^{-1} \in \Lambda_G$. Hence G is Λ — transitive.

Similarly for the other cases.

Theorem 7: Let a transitive groupoid $G(o)$ be a μ — homotope of a groupoid G, which has a unit. Then G is a group.

Proof: By Theorem 6 and Theorem 3.

Theorem 8: Let a commutative groupoid $G(o)$ be a μ — homotope of a group G. Then G is an Abelian group.

Proof: Let $G(o) = G^{(\alpha,\beta,\gamma)}$. Since the mappings α, β are onto G, there are mappings δ, ε such that $\alpha\delta = \beta\varepsilon = 1_G$. Let e be a unit of the group G and $x, y \in G$. We have $\alpha(x) . \beta(y) = \gamma(xoy) = \gamma(yox) = \alpha(y) . \beta(x)$. Let $u \in G$ be such that $\beta(u) = e$. Then $\alpha(x) = \alpha(u) . \beta(x)$. Hence $\alpha(u) . \beta(x) . \beta(y) = \alpha(x) . \beta(y) = \alpha(y) . \beta(x) = \alpha(u) . \beta(y) .$ $. \beta(x)$, hence $\beta(x) . \beta(y) = \beta(y) . \beta(x)$. Thus for every v, $z \in G$ we have $vz =$ $= \beta\varepsilon(v) . \beta\varepsilon(z) = \beta\varepsilon(z) . \beta\varepsilon(v) = zv$.

Lemma 12: Let $G(o) = G^{(\alpha,\beta,\gamma)}$ and let $G(o)$ be a groupoid with unit e. Then the translations $L_{\alpha(e)}$, $R_{\beta(e)}$ of the groupoid G are mappings onto G.

Proof: For every $x \in G$ we have $\gamma(x) = \gamma(x\ o\ e) = \alpha(x) . \beta(e)$. Hence $\gamma = R_{\beta(e)}\alpha$. Similarly $\gamma = L_{\alpha(e)} . \beta$. Thus $L_{\alpha(e)}$, $R_{\beta(e)}$ are mappings onto G.

Lemma 13: Let G be a groupoid and $x, y \in G$. Let α, β be arbitrary mappings such that $L_x\beta = R_y\alpha = 1_G$ and $\alpha(xy) = x$, $\beta(xy) = y$ (the mappings α, β exist if and only if the mappings L_x, R_y are onto G). Put $G(o) = G^{(\alpha,\beta,1_G)}$. Then $G(o)$ is a groupoid with unit e, where $e = xy$.

Proof: Let $u \in G$. Then $u\ o\ e = u\ o\ (xy) = \alpha(u) . \beta(xy) = \alpha(u) . y = R_y\alpha(u) = u$, $e\ o\ u = (xy)\ o\ u = \alpha(xy) . \beta(u) = L_x\beta(u) = u$.

Definition 4: Let G be a groupoid and $x, y \in G$. We say that two elements x, y satisfy the μ — condition if:

1) The mappings L_x, R_y are onto G.
2) For every $u, v, z \in G$,
   $R_y(u) = R_y(v)$ implies $R_z(u) = R_z(v)$
3) For every $u, v, z \in G$,
   $L_x(u) = L_x(v)$ implies $L_z(u) = L_z(v)$.

Lemma 14: Let G be a groupoid and $x, y \in G$. Then the following conditions are equivalent:

1) The elements x, y satisfy the μ — condition.
2) There are mappings α, β such that $R_y\alpha = L_x\beta = 1_G$ and $uv = \alpha R_y(u) . \beta L_x(v)$ for every $u, v \in G$.
3) There are mappings α, β such that $R_y\alpha = L_x\beta = 1_G$. For all possible mappings δ, ε such that $R_y\delta = L_x\varepsilon = 1_G$ and for all $u, v \in G$, $uv = \delta R_y(u) . \varepsilon\, L_x(v)$.

Proof: 1) Implies 3). Since R_y, L_x are onto G, there are mappings α, β such that $R_y\alpha = L_x\beta = 1_G$. Let δ,ε be arbitrary mappings such that $R_y\delta = L_x\varepsilon = 1_G$. Let $u, v \in G$. Set $z = \delta R_y(u)$, $t = \varepsilon L_x(v)$. We have $R_y(z) = R_y\delta R_y(u) = R_y(u)$. Hence $zt = ut$ (by μ — condition). Further, $L_x(t) = L_x\varepsilon L_x(v) = L_x(v)$. Hence $ut = uv$. Thus $zt = uv$.
Evidently 3) implies 2).
2) implies 1). Since $R_y\alpha = L_x\beta = 1_G$, the mappings R_y, L_x are onto G. Let $u, v \in G$ and $R_y(u) = R_y(v)$. Let $z \in G$ be arbitrary element. Then $R_z(u) = uz = \alpha\, R_y(u)\,.\,\beta L_x(z) = \alpha\, R_y(v)\,.\,\beta L_x(z) = vz = R_z(v)$. Hence we have proved that:
$R_y(u) = R_y(v)$ implies $R_z(u) = R_z(v)$.
Similarly we can prove the last part of the μ — condition.

Lemma 15: Let G be a groupoid and $x, y \in G$. Then the following conditions are equivalent:
1) The elements x, y satisfy the μ — condition.
2) There are mappings α, β such that $R_y\alpha = L_x\beta = 1_G$ and $\alpha(xy) = x$, $\beta(xy) = y$. Let α_1, β_1 be arbitrary such mappings. Put $G(o) = G^{(\alpha_1,\beta_1,1)}$. Then $G(o)$ is a groupoid with unit xy and $G = G(o)^{(R_y,L_x,1)}$. (R_y, L_x are taken in G).

Proof: 1) implies 2). The mappings R_y, L_x are onto. Hence there are mappings α, β such that $R_y\alpha = L_x\beta = 1_G$, $\alpha(xy) = x$, $\beta(xy) = y$. Let α_1, β_1 be arbitrary such mappings. For every $u, v \in G$ by Lemma 14, we have $uv = \alpha_1 R_y(u)\,.\,\beta_1 L_x(v)$. Hence $R_y(u)\; o\; L_x(v) = \alpha_1 R_y(u)\,.\,\beta_1 L_x(v) = uv$. Thus $G = G(o)^{(R_y,L_x,1)}$.
2) implies 1). The mappings R_y, L_x are evidently onto G. Let be $u, v \in G$ such that $R_y(u) = R_y(v)$. Let $z \in G$ be an arbitrary element. We have $R_z(u) = uz = R_y(u)\; o\; L_x(z) = R_y(v)\; o\; L_x(z) = vz = R_z(v)$.
Similarly we can prove the last part of the μ — condition

Definition 5: A groupoid G is called μ — groupoid if there is a groupoid with unit, $G(o)$, such that the groupoid G is a μ — homotope of the groupoid $G(o)$.

Lemma 16: Let G be a μ — groupoid. Then there is a groupoid with unit, $G(o)$, such that G is a principal μ — homotope of $G(o)$.

Proof: By Lemma 8.

Theorem 9: Every groupoid G is a μ — groupoid if and only if there are two elements $x, y \in G$ such that x, y satisfy the μ — condition.

Proof: 1) Let G be a μ - groupoid. By Lemma 16 there is a groupoid $G(o)$, which has a unit e, such that $G = G(o)^{(\delta,\varepsilon,1)}$. Moreover, the mappings δ, ε are onto G. Hence there are mappings α, β such that $\delta\alpha = \varepsilon\,\beta = 1_G$. Set $x = \alpha(e), y = \beta(e)$. For every $u \in G$, $R_y(u) = uy = \delta(u)\; o\; \varepsilon(y) = \delta(u)\; o\; \varepsilon\beta(e) = \delta(u)\; o\; e = \delta(u)$, $L_x(u) = \delta\alpha(e)\; o\; \varepsilon(u) = \varepsilon(u)$. Thus $\delta = R_y$, $\varepsilon = L_x$. Further, $\alpha(xy) = \alpha(\alpha(e)\,.\,\beta(e)) = \alpha(\delta\alpha(e)\; o\; \varepsilon\beta(e)) = \alpha(e) = x$. Similarly $\beta(xy) = y$. Finally, $\alpha(u)\,.\,\beta(v) = \delta\alpha(u)\; o\; \varepsilon\beta(v) = u\; o\; v$. Now we can use Lemma 15. Therefore x, y satisfy the μ - condition.
2) Let $x, y \in G$ be two elements satisfying the μ - condition. By Lemma 15 there is

a groupoid with unit, $G(o)$, such that $G = G(o)^{(R_y, L_x, 1)}$. Since R_y, L_x are mappings onto G, G is a μ – groupoid.

Theorem 10: Let G be a transitive μ – groupoid. Then G is a principal μ – homotope of a group. Hence G is with division.

Proof: There is a groupoid with unit, $G(o)$, and there are mappings α, β, which are onto G, such that $G = G(o)^{(\alpha, \beta, 1)}$. By Theorem 6, $G(o)$ is transitive and hence, by Theorem 3, $G(o)$ is a group. By Lemma 9, G is a groupoid with division.

Theorem 11: Let G be a transitive groupoid. Let there be two elements of G which satisfy the μ – condition. Then arbitrary two elements of G satisfy the μ – condition.

Proof: The groupoid G is a μ – groupoid. Then, by Theorems 9, 10, there is a group $G(o)$ and there are mappings α, β (which are onto G) such that $G = G(o)^{(\alpha, \beta, 1)}$. Let x, y be arbitrary elements of G. By Theorem 10, G is a groupoid with division, hence L_x, R_y are mappings onto G. Let $u, v \in G$ be such that $R_y(u) = R_y(v)$ and $z \in G$ be arbitrary element. We have $R_y(u) = uy = \alpha(u) \circ \beta(y) = vy = \alpha(v) \circ \beta(y)$. Hence $\alpha(u) = \alpha(v)$, and hence, $R_z(u) = uz = \alpha(u) \circ \beta(z) = \alpha(v) \circ \beta(z) = vz = R_z(v)$. Similarly we can prove the last part of the μ – condition. Thus x, y satisfy the μ – condition.

Lemma 17: Let G be a group and α, β, γ be three mappings of G into G such that for every $x, y \in G$ is $\gamma(xy) = \alpha(x) . \beta(y)$. Then there are elements a, b, c of the group G such that the mappings $L_a\alpha$, αR_a, $L_b\beta$, βL_b, $L_c\gamma$, γR_c are endomorphisms of the group G.

Proof: Let 1 be the unit of G. For every $x \in G, \gamma(x) = \alpha(1) . \beta(x), \gamma(x) = \alpha(x) . \beta(1)$. Therefore $\alpha(x) . \beta(1) = \alpha(1) . \beta(x)$. Hence $\alpha(x) = \alpha(1) . \beta(x) . (\beta 1)^{-1}$. Further, for every x, $y \in G, \gamma(xy) = \alpha(x) . \beta(y) = \alpha(1) . \beta(xy) = \alpha(1) . \beta(x) . (\beta 1)^{-1} . \beta(y)$.
Hence $\beta(xy) = \beta(x) b \beta(y)$, where $b = (\beta(1))^{-1}$. Thus the mappings $L_b\beta$, $R_b\beta$ are endomorphisms of G.
Similarly, there exist $a \in G$ such that $L_a\alpha$, $R_a\alpha$ are endomorphisms of G.
Now for γ. We have $\beta(x) = (\alpha 1)^{-1} . \gamma(x)$, $\alpha(x) = \gamma(x) . (\beta 1)^{-1}$ for every x, $y \in G$. Since $\gamma(xy) = \alpha(x) . \beta(y)$, we have $\gamma(xy) = \gamma(x) . (\beta 1)^{-1} . (\alpha 1)^{-1} . \gamma(y) = \gamma(x) . c . \gamma(y)$, where $c = (\beta 1)^{-1} . (\alpha 1)^{-1}$. Thus $L_c\gamma$, γR_c are endomorphisms of the group G.

4° **Definition 6:** A groupoid G is called B_1 (B_2) – groupoid if $x(yz) = y(xz)$ ($xy . z = xz . y$) for all $x, y, z \in G$.

Lemma 18: Let G be a B_1 – groupoid. Let $x \in G$ be such that R_x is onto G. Then G has a left unit e. Moreover, the elements e, x satisfy the μ – condition.

Proof: Let $y \in G$. There are $e, z \in G$ such that $zx = y$ and $ex = x$. We have $y = zx = z(ex) = e(zx) = ey$. Therefore, e is a left unit of G. The mappings L_e, R_x are onto G. Further, let u, v be elements of G such that $R_x(u) = R_x(v)$. Let $z \in G$. There is $t \in G$ such that $tx = z$. Then $uz = u(tx) = t(ux) = t(vx) = v(tx) = vz$. The last part of the μ – condition (for e) is evident (as $L_e = 1_G$).

Lemma 19: Every B_1 – groupoid with right division is R – transitive.

Proof: For all $x, y, z \in G$, $x . yz = y . xz$. Hence $L_x(yz) = y . L_x(z)$. Thus $L_x \in R_G$.

Let $u, v \in G$ be arbitrary elements. There is $z \in G$ such that $zu = L_z(u) = v$. Hence G is R – transitive.

Theorem 12: Let G be a B_1 – groupoid. Then the following conditions are equivalent:

1) There exists $x \in G$ such that R_x is onto G.

2) There is a commutative semigroup with unit, $G(o)$, and a mapping α which is onto G such that $uv = \alpha(u) \, o \, v$ for every $u, v \in G$.

Proof: 1) implies 2). By Lemma 18, G has a left unit e. Since R_x is onto G, there is a mapping β such that $R_x\beta = 1_G$ and $\beta(x) = \beta(ex) = e$. Put $G(o) = G^{(\beta,1,1)}$. Since e, x satisfy the μ – condition, hence, by Lemma 15, $G(o)$ is a groupoid with unit x and $G = G(o)^{(R_x,1,1)}$. Let $u, v, z \in G$. We have $u(vz) = R_x(u) \, o \, (R_x(v) \, o \, z) = v(uz) = R_x(v) \, o \, (R_x(u) \, o \, z)$. From this we deduce that $G(o)$ is B_1 – groupoid. But every B_1 – groupoid with unit is a commutative semigroup.

2) implies 1). This part of the proof is evident

Theorem 13: Let G be a groupoid. Then the following conditions are equivalent:

1) G is a B_1 – groupoid with right division.

2) G is a B_1 – groupoid with division and simultaneously a left quasigroup.

3) There is an Abelian group $G(+)$ and a mapping α which is onto G such that $xy = \alpha(x) + y$ for every $x, y \in G$.

Proof: 3) implies 2) and 2) implies 1) evidently.

1) implies 3). By Theorem 12, there is a commutative semigroup with unit, $G(+)$, and a mapping α which is onto G such that $G = G(+)^{(\alpha,1,1)}$. Therefore, G is a μ – homotope of $G(+)$. Since, by Lemma 19, G is transitive, the semigroup $G(+)$ is, by Theorem 7, a (Abelian) group.

Theorem 14: Let G be a B_1–groupoid with left cencellation. Let there be $x \in G$ such that R_x is onto G (a permutation). Then the groupoid G can be imbedded in a B_1 – groupoid G_1 which is with division (which is a quasigroup).

Proof: By Theorem 12, there is a commutative semigroup $G(o)$ and a mapping α which is onto G such that $G = G(o)^{(\alpha,1,1)}$. Let β be a mapping such that $\alpha \, \beta = 1_G$. Then, by Lemma 18, $G^{(\beta,1,1)} = G(o)$. Therefore, for every $u, v \in G$ we have,

(1) $u \, o \, v = \beta(u) \, . \, v.$

Since G is with left cancellation, we get, applying (1), that $G(o)$ is with left cancellation, too. As $G(o)$ is commutative, $G(o)$ is with cancellation. It is well known that every commutative semigroup with cancellation can be imbedded in an Abelian group. Let $G_1(+)$ be any such Abelian group and $\varphi : G(o) \to G_1(+)$ be a monomorphism. Define the mapping $\varkappa$ of G_1 into G_1 as follows: $\varkappa(y) = \varphi R_x \varphi^{-1}(y)$ for $y \in \varphi(G)$, $\varkappa(y) = y$ for $y \in G_1, y \notin \varphi(G)$. The mapping $\varkappa$ is, evidently, onto G_1. When R_x is moreover one – to – one, then $\varkappa$ is a permutation. Put $G_1(\cdot) = G_1{}^{(\varkappa,1,1)}$. $G_1(\cdot)$ is a B_1 – groupoid with division. If $\varkappa$ is one – to – one, $G_1(\cdot)$ is a B_1 – quasigroup. The mapping φ is also monomorphism of G into $G_1(\cdot)$. This completes the proof.

For B_2 – groupoids we can prove Theorems dual to Theorems 12—14

Definition 7: A groupoid G is called an A_1 – groupoid (A_2 – groupoid) if $xy \,.\, uv = xu \,.\, yv$ ($xy \,.\, uv = vy \,.\, ux$), for all $x, y, u, v \in G$.

Lemma 19: Let G be a groupoid with left (right) division. Let there be $x \in G$ such that the mapping R_x (L_x) is onto G and for $y, u, v \in G$, $yx \,.\, uv = yu \,.\, xv$. Then the groupoid G is Φ (Φ^*) – transitive.

Proof: For every $y, u, v \in G$ we have $R_u(y) \,.\, L_x(v) = R_x(y) \,.\, L_u(v)$. Let α, β be any mappings such that $R_x\alpha = L_x\beta = 1_G$. Then $R_u\alpha(y) \,.\, v = y \,.\, L_u\beta(v)$. Hence $R_u\alpha \in \Phi_G$ for every $u \in G$. From this we see that G is a Φ – transitive groupoid. Similarly for the remaining case.

Corollary: Every A_1 – groupoid with division is Φ and Φ^* – transitive.

Theorem 15: Let G be a groupoid. Then the following conditions are equivalent:

1) G is a μ – groupoid with division and there is $x \in G$ such that for every $u, y, v \in G$, $yx \,.\, uv = yu \,.\, xv$.

2) There is a group $G(o)$, its endomorphisms φ, ψ which are onto $G(o)$ and $g, h \in G(o)$ such that for every $u, v \in G$, $uv = \varphi(u) \, o \, g \, o \, \psi(v)$, $\varphi\psi(u) \, o \, h = h \, o \, \psi\varphi(u)$.

Proof: 1) implies 2). Since G, by Lemma 19, is Φ – transitive, there is a group $G(o)$ and mappings α, β such that $G = G(o)^{(\alpha,\beta,1)}$. The mappings α, β are onto G. For all $y, u, v \in G$ we have $ux \,.\, yv = \alpha\,(\alpha(u) \, o \, \beta(x)) \, o \, \beta(\alpha(y) \, o \, \beta(v)) = uy \,.\, xv =$
$= \alpha(\alpha(u) \, o \, \beta(y)) \, o \, (\beta(\alpha(x) \, o \, \beta(v))$. Hence $\alpha(u \, o \, y) = \alpha_1(u) \, o \, \beta_1(y)$, $\beta(y \, o \, v) =$
$= \alpha_2(y) \, o \, \beta_2(v)$, where α_i, β_i are convenient mappings. Thus, by Lemma 17, there exist endomorphisms φ, ψ of the group $G(o)$ and elements a, b in G such that $\alpha(u) =$
$= \varphi(u) \, o \, a$, $\beta(u) = b \, o \, \psi(u)$ for every $u \in G$. Therefore, $uv = \varphi(u) \, o \, g \, o \, \psi(v)$, where $g = a \, o \, b$. Now we can write,

$ux \,.\, yv = \varphi^2(u) \, o \, \varphi(g) \, o \, \varphi\psi(x) \, o \, g \, o \, \psi\varphi(y) \, o \, \psi(g) \, o \, \psi^2(v) =$
$= uy \,.\, xv = \varphi^2(u) \, o \, \varphi(g) \, o \, \varphi\psi(y) \, o \, g \, o \, \psi\varphi(x) \, o \, \psi(g) \, o \, \psi^2(v)$.

From this we get $\varphi\psi(y) \, o \, g \, o \, \psi\varphi(x) = \varphi\psi(x) \, o \, g \, o \, \psi\varphi(y)$.

Put $y = 1$, where 1 is the unit of the group $G(o)$. Then $g \, o \, \psi\varphi(x) = \varphi\psi(x) \, o \, g = h$.
Hence $\varphi\psi(y) \, o \, h = h \, o \, \psi\varphi\,(y)$ for every $y \in G$.

2) implies 1). The groupoid G is, evidently, a μ – homotope of the group $G(o)$. Hence G is a μ – groupoid with division. Put $x = \psi^{-1}\varphi^{1}\,(h \, o \, g^{-1})$. Then
$h = \varphi\psi(x) \, o \, g = \varphi\psi(x) \, o \, h \, o \, h^{-1} \, o \, g = h \, o \, \psi\varphi(x) \, o \, h^{-1} \, o \, g$.
Hence $g^{-1} = \psi\varphi(x) \, o \, h^{-1}$, and hence, $h = g \, o \, \psi\varphi(x)$.
For every $u, y, v \in G$ we have,

$yx \,.\, uv = \varphi^2(y) \, o \, \varphi(g) \, o \, \varphi\psi(x) \, o \, g \, o \, \psi\varphi(u) \, o \, \psi(g) \, o \, \psi^2(v) =$
$= \varphi^2(y) \, o \, \varphi(g) \, o \, h \, o \, \psi\varphi(u) \, o \, \psi(g) \, o \, \psi^2(v) =$
$= \varphi^2(y) \, o \, \varphi(g) \, o \, \varphi\psi(u) \, o \, h \, o \, \psi(g) \, o \, \psi^2(v) =$
$= \varphi^2(y) \, o \, \varphi(g) \, o \, \varphi\psi(u) \, o \, g \, o \, \psi\varphi(x) \, o \, \psi(g) \, o \, \psi^2(v) = yu \,.\, xv$.

This completes the proof.

Theorem 16: Let G be a groupoid. Then the following conditions are equivalent:

1) G is a μ – groupoid with division and there exist elements x, a, b of G such that $ux \,.\, vt = uv \,.\, xt$, $au \,.\, vb = av \,.\, ub$ for all $u, v, t \in G$.
2) G is a μ – groupoid with division and G is an A_1 – groupoid.
3) There is an Abelian group $G(+)$, its endomorphisms φ, ψ which are onto G and $g \in G$ such that $uv = \varphi(u) + \psi(v) + g$ for all $u, v \in G$ and $\varphi\psi = \psi\varphi$.

Proof: 1) implies 3). By Theorem 15, there is a group $G(+)$, its endomorphisms φ, ψ which are onto G and $g, h \in G$ such that $uv = \varphi(u) + g + \psi(v)$, $h + \psi\varphi(u) = \varphi\psi(u) + h$ for all $u, v \in G$.
Put $G(\cdot) = G^{(L_a, R_b, 1)}$. For every $u, v \in G$ we have $u \cdot v = au \,.\, vb = av \,.\, ub = v \cdot u$. Thus $G(\cdot)$ is a commutative μ – homotope of G. Hence $G(\cdot)$ is a commutative μ – homotope of the group $G(+)$. Therefore, by Theorem 8, $G(+)$ is an Abelian group. Hence $h + \psi\varphi(u) = \psi\varphi(u) + h = \varphi\psi(u) + h$, and hence, $\varphi\psi = \psi\varphi$.
3) implies 2) and 2) implies 1) evidently.

5° **Definition 8:** Let G be a non-empty set, $n \geq 2$ be a positive integer and f be an n – ary operation completely defined on G. The algebra (G,f) is called n – groupoid. Instead of (G,f) and $f(x_1, \ldots, x_n)$ we shall usually write G and $(x_1 \ldots, x_n)$ only.

Definition 9: Let G be a n – groupoid. A mapping λ of the set G into G is called i – regular, where $1 \leq i \leq n$ if there exists a mapping λ^* such that for every $x_1, \ldots, x_n \in G$, $\lambda(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \lambda^*(x_i), x_{i+1}, \ldots, x_n)$.
Denote by symbol Λ_G^i the set of all i – regular mappings of the n – groupoid G.

Lemma 20: Let G be a n – groupoid. Then for every i, $1 \leq i \leq n$, the set Λ_G^i is a semigroup with unit under the operation of composition of mappings.

Proof: Proof is the same as for Lemma 1.

Definition 10: Let G be a n – groupoid. Let i be a positive integer, $1 \leq i \leq n$. An element e of G is called an i – unit if for every $x \in G$,
$(\overset{1}{e}, \ldots, \overset{i-1}{e}, \overset{i}{x}, \overset{i+1}{e}, \ldots, \overset{n}{e}) = x$. An element e is called a unit if e is a j – unit for every, j, $1 \leq j \leq n$.

Lemma 21: Let G be a n – groupoid with i – unit e, $1 \leq i \leq n$. Let $\lambda \in \Lambda_G^i$. Then $\lambda = \lambda^*$.

Proof: For every $x \in G$ we have $\lambda(x) = \lambda\,(\overset{1}{e}, \ldots, \overset{i-1}{e}, \overset{i}{x}, \overset{i+1}{e}, \ldots, \overset{n}{e}) = (\overset{1}{e}, \ldots, \overset{i-1}{e}, \overset{i}{\lambda^*(x)}, \overset{i+1}{e}, \ldots, \overset{n}{e}) = \lambda^*(x)$. Thus $\lambda = \lambda^*$.

Definition 11: Let G be a n – groupoid and a be an element of G. We say that a satisfies the ν – condition if for every j, $1 \leq j \leq n$, and for every $x_1, \ldots, x_n \in G$,
$(\overset{1}{x_1}, \ldots, \overset{j-1}{x_{j-1}}, \overset{j}{a}, \overset{j+1}{x_j}, \ldots, \overset{n}{x_{n-1}}) = (\overset{1}{x_1}, \ldots, \overset{j-1}{x_{j-1}}, \overset{j}{x_j}, \overset{j+1}{a}, \overset{j+2}{x_{j+1}}, \ldots, \overset{n}{x_{n-1}})$

Lemma 22: Let G be a n – groupoid with i – unit e, $1 \leq i \leq n$. Let e satisfy the ν – condition. Then e is a unit of G.

Proof: This Lemma follows directly from Definition 11.

Definition 12: Let G be a n – groupoid and i be a positive integer, $1 \leq i \leq n$. The n – groupoid G is called Λ^i – transitive if for every $x, y \in G$ there is $\lambda \in \Lambda^i_G$ such that $\lambda(x) = y$.

Definition 13: Let G be a n – groupoid with i, j – unit e, where $1 \leq i, j \leq n$, $i \neq j$. Define the binary operation $f_{i,j}$ on G as follows:

For every $x, y \in G$, $f_{i,j}(x, y) = (e, \ldots, e, \overset{i}{x}, e, \ldots, e, \overset{j}{y}, e, \ldots, e)$ if $i < j$

and $f_{i,j}(x, y) = (e, \ldots, e, \overset{j}{y}, e, \ldots, e, \overset{i}{x}\, e, \ldots, e)$ if $j < i$.

Just defined groupoid $(G, f_{i,j})$ we shall denote by symbol $G(o)^{i,j}$.

Theorem 17: Let G be a Λ^i – transitive n – groupoid with i, j – unit e, where $1 \leq i, j \leq n$, $i \neq j$. Then $G(o)^{i,j}$ is a group.

Proof: Suppose $i < j$. The element e is a unit of the groupoid $G(o)^{i,j}$. Indeed, for every $x \in G$ we have $x \, o \, e = (e, \ldots, e, \overset{i}{x}, e, \ldots, \overset{j}{e}, \ldots e) = (e, \ldots, \overset{i}{e}, \ldots e, \overset{j}{x}, e, \ldots, e) = e \, o \, x$. Further, let $\lambda \in \Lambda^i_G$. For every $x, y \in G$ we have,

$\lambda(x \, o \, y) = \lambda(e, \ldots, e, x, e, \ldots e, \overset{j}{y}, e, \ldots, e) = (e, \ldots, e, \overset{i}{\lambda(x)}, e, \ldots, e, \overset{j}{y}, e, \ldots, e) =$
$= \lambda(x) \, o \, y$.

Hence λ is a left regular mapping of $G(o)^{i,j}$. Since G is Λ^i – transitive, $G(o)^{i,j}$ is Λ – transitive. Therefore, by Theorem 3, $G(o)^{i,j}$ is a group.

If $j < i$ the proof is similar.

Definition 13: Let G be a n – groupoid and σ be a permutation of elements $1, 2, \ldots, n$. The n – groupoid G is called a $\sigma - n -$ groupoid if there exists a group $G(o)$ such that for every $x_1, \ldots, x_n \in G$,

$$(x_1, x_2, \ldots, x_n) = x_{\sigma(1)} \, o \, x_{\sigma(2)} \, o \ldots o \, x_{\sigma(n)}.$$

Lemma 23: Let G be a $\sigma - n -$ groupoid. Then G is $\Lambda^{\sigma(1)}$ – transitive and $\Lambda^{\sigma(n)}$ – transitive.

Proof: There exists a group $G(o)$ such that for every $x_1, \ldots, x_n \in G$,

$$(x_1, \ldots, x_n) = x_{\sigma(1)} \, o \ldots o \, x_{\sigma(n)}.$$

Let $u \in G$. The translation R_u of the group $G(o)$ is a $\sigma(n)$ – regular mapping of the n – groupoid G. Indeed,

$R_u(x_1, \ldots, x_n) = x_{\sigma(1)} \, o \, x_{\sigma(2)} \, o \ldots o \, x_{\sigma(n)} \, o \, u =$
$= x_{\sigma(1)} \, o \ldots o \, x_{\sigma(n-1)} \, o \, (x_{\sigma(n)} \, o \, u) = (x_1, \ldots, x_{\sigma(n)-1}, R_u(x_{\sigma(n)}), x_{\sigma(n)+1}, \ldots, x_n)$.

Since the group $G(o)$ is a groupoid with division, G is $\Lambda^{\sigma(n)}$ – transitive. Similarly, G is $\Lambda^{\sigma(1)}$ – transitive.

Lemma 24: Let G be a Λ^i – transitive n – groupoid with i – unit e, $1 \leq i \leq n$. Let e satisfy the ν – condition. Then G is a $\sigma - n -$ groupoid for

$\sigma = (\overset{1}{i}, \overset{2}{i + 1}, \ldots, n, i - 1, i - 2, \ldots, \overset{n}{1})$.

Proof: There is j, $1 \leq j \leq n$, such that $i \neq j$. Suppose $i < j$.

By Lemma 22, the element e is an unit of G. Therefore, by Theorem 17, the groupoid $G(o)^{i,j}$ is a group.
Let $x_1, \ldots, x_n \in G$ be arbitrary elements. Since G is Λ^i – transitive, there are mappings $\lambda_1, \ldots, \lambda_n \in \Lambda^i_G$ such that $x_1 = \lambda_1(e)$, $x_2 = \lambda_2(e), \ldots, x_n = \lambda_n(e)$. Since e satisfies the ν – condition and λ_k are i – regular, we have $(x_1, \ldots, x_n) =$
$= (\lambda_1(e), \ldots, \lambda_n(e)) = \lambda_i\,(\lambda_1(e), \ldots, \lambda_{i-1}(e), e, \lambda_{i+1}(e), \ldots, \lambda_n(e)) =$
$= \lambda_i\,(\lambda_1(e), \ldots, \lambda_{i-1}(e), \lambda_{i+1}(e)\, e, \lambda_{i+2}(e), \ldots, \lambda_n(e)) = \ldots\ldots =$
$= \lambda_i\lambda_{i+1} \ldots \lambda_n\lambda_{i-1}\, \lambda_{i-2} \ldots \lambda_1(e, \ldots, e) = \lambda_i \ldots \lambda_n\, \lambda_{i-1} \ldots \lambda_1(e).$
Conversely, $x_i\, o \ldots o\, x_n\, o\, x_{i-1}\, o \ldots o\, x_1 = \lambda_i(e)\, o \ldots o\, \lambda_n(e)\, o\, \lambda_{i-1}(e)\, o \ldots o\, \lambda_1(e) =$

$$\overset{1, \ldots, i-1,\; i,\; i+1 \ldots, j-1,\; j \qquad\qquad j+1, \ldots, n}{= (e, \ldots, e, \lambda_i(e), e, \ldots, e, (e, \ldots, e, \lambda_{i+1}(e), e, \ldots\ldots), e, \ldots, e) =}$$

$= \lambda_i(e, \ldots, e, \lambda_{i+1}(e), e, \ldots, e, (e, \ldots, e, \lambda_{i+2}(e), e, \ldots), e, \ldots, e) =$
$= \lambda_i\lambda_{i+1} \ldots \lambda_n\, \lambda_{i-1}\, \lambda_{i-2} \ldots \lambda_1(e).$

Thus G is a $\sigma - n$ groupoid for $\sigma = \begin{pmatrix} 1, & \ldots\ldots\ldots & , n \\ i, \ldots, n, & i-1, \ldots, & 1 \end{pmatrix}$
If $j < i$ the proof is similar.

Theorem 18: Let G be a n – groupoid. Then the following conditions are equivalent:
1) There exists i, $1 \leq i \leq n$, that G is Λ^i – transitive and G has an i – unit e which satisfies the ν – condition.
2) G is Λ^1 and Λ^n – transitive and G has a unit g which satisfies the ν – condition.
3) There is a group $G(o)$ such that for every $x_1, \ldots, x_n \in G$, $(x_1, \ldots, x_n) =$
$= x_1\, o\, x_2\, o \ldots o\, x_n$.

Proof: 1) implies 3). By Lemma 24, G is $\sigma - n$ – groupoid for $\sigma = \begin{pmatrix} 1, & \ldots\ldots\ldots & , n \\ i, \ldots, n, & i-1, \ldots, & 1 \end{pmatrix}$.
Hence, by Lemma 23, G is $\Lambda^{\sigma(n)}$ – transitive. But $\sigma(n) = 1$. Hence G is Λ^1 – transitive. The element e is, by Lemma 22, a unit of G. Hence, by Lemma 24, G is $\varepsilon - n$ – groupoid for $\varepsilon = \begin{pmatrix} 1, & 2, & \ldots, & n \\ 1, & 2, & \ldots, & n \end{pmatrix}$.
Since ε is the identity permutation, there is a group $G(o)$ such that for every $x_1, \ldots, x_n \in G$,

$$(x_1, \ldots, x_n) = x_1\, o\, x_2\, o \ldots o\, x_n$$

3) implies 2) and 2) implies 1) evidently.

Theorem 19: Let G be a n – groupoid. Then the following conditions are equivalent:
1) There exists i, $1 < i < n$, such that G is Λ^i – transitive and G has an i – unit e, which satisfies the ν – condition.
2) G is Λ^j – transitive for all j, $1 \leq j \leq n$. G has a unit g and an arbitrary element of G satisfies the ν – condition.
3) There is an Abelian group $G(+)$ such that for every $x_1, \ldots, x_n \in G$,

$$(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n.$$

Proof: 1) implies 3). By Theorem 18, there is a group $G(+)$ such that for every $x_1, \ldots, x_n \in G$,

$$(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n.$$

Let $\lambda \in \Lambda_G^i$. Then $\lambda(x) = \lambda(x, e, \ldots, e) = \overset{1}{(x}, \overset{\ldots}{e}, \ldots, \overset{i-1}{e}, \overset{i}{\lambda(e)}, \overset{i+1}{e}, \ldots, \overset{n}{e)} =$
$= x + e + \ldots + \lambda(e) + e + \ldots + e = x + \lambda(e)$. Hence for every $x_1, \ldots, x_n \in G$ we have $\lambda(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n + \lambda(e) =$
$= (x_1, \ldots, x_{i-1}, \lambda(x_i), x_{i+1}, \ldots, x_n) = x_1 + \ldots + x_{i-1} + x_i + \lambda(e) + x_{i+1} +$
$+ \ldots + x_n$.
Since $1 < i < n$, $i + 1 \leq n$. Hence $x_{i+1} + \ldots + x_n + \lambda(e) = \lambda(e) + x_{i+1} + \ldots + x_n$, and hence, $\lambda(e) + x = x + \lambda(e)$ for all $x \in G$. Using the Λ^i - transitivity, we get that $G(+)$ is commutative.
3) implies 2) and 2) implies 1) evidently.