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# Multiplication Groups of Quasigroups and Loops III

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Groups possessing connected transversals to Abelian subgroups are studied in more detail.

Podrobněji se studují grupy se spojenými transversálami k abelovským podgrupám.

#### 1. Preliminaries

**1.1 Lemma.** Let H be a non-normal finitely generated subgroup of a finitely generated group G. Then they exists a subgroup N of G maximal with respect to  $N \subseteq G$  and  $NH \npreceq G$ .

**Proof.** Let  $\mathfrak{M}$  be the set of subgroups M such that  $M \leq G$  and  $MH \not \supseteq G$ . Then the unit subgroup is in  $\mathfrak{M}$  and  $\mathfrak{M}$  is ordered by inclusion. Moreover, if  $M_i$ ,  $i \in I$ , is a chain of subgroups from  $\mathfrak{M}$ , then certainly  $M = \bigcup_I M_i \leq G$ . Now, suppose for a moment that  $MH \leq G$ . If A and B are finite subsets of G such that  $H = \langle A \rangle$  and  $G = \langle B \rangle$ , then  $b^{-1}ab \in MH$  for all  $a \in A$  and  $b \in B \cup B^{-1}$ . Consequently there is  $i \in I$  such that all these elements  $b^{-1}ab$  are in  $M_i$ . But then  $M_iH \leq G$ , a contradiction.

We have proved that the ordered set  $\mathfrak{M}$  is upwords-inductive.

- **1.2 Remark.** Consider the situation from 1.1 and put  $\overline{G} = G/N$ . Then  $\overline{H} = HN/N$  is not normal in  $\overline{G}$  and, if  $\overline{M} \neq 1$  is normal subgroup of  $\overline{G}$ , then  $\overline{M} \cdot \overline{H} \leq \overline{G}$ .
- **1.3 Lemma.** Let H be a subgroup of group G such that H is nilpotent and  $\mathbb{N}_G(K) \subseteq H$  for every subgroup  $K \neq 1$  of H. Then  $H \cap H^x = 1$  for every  $x \in G \setminus H$ .

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- **Proof.** Suppose that  $L = H \cap H^x \neq 1$  for some  $x \in G$ . Now,  $\mathbb{N}_G(L) \subseteq H \cap H^x = L$  and since H and  $H^x$  are nilpotent, we conclude that  $L = H = H^x$ . Then  $x \in \mathbb{N}_G(H) = H$ .
- **1.4 Lemma.** Let A, B, C be subsets of a group G such that G = BC, C is finite  $(\operatorname{card}(C) = n \ge 1)$  and  $[a, b] \in C$  for all  $a \in A$  and  $b \in B$ . Then  $[G : \mathbb{C}_G(a)] \le n^2$  for every  $a \in A$ .
- **Proof.** Let  $a \in A$ . For  $c \in C$ , put  $B_c = \{b \in B, [a, b] = c\}$ . Now,  $B = \bigcup_{c \in C} B_c$  and this union is disjoint. If  $b_1, b_2 \in B_c$ , then  $b_1^{-1}ab_1 = b_2^{-1}ab_2$  i.e.,  $b_2b_1^{-1} \in \mathbb{C}_G(a)$  and  $\mathbb{C}_G(a)b_1 = \mathbb{C}_G(a)b_2$ .

For every  $c \in C$  such that  $B_c \neq \emptyset$ , choose  $b_c \in B_c$  and put  $D = \{b_c; c \in C\}$ . Ten  $G = \mathbb{C}_G(a)DC$ . Indeed, if  $x \in G$ , then x = bd for some  $b \in B$  and  $d \in C$ . Of course,  $b \in B_c$  for some  $c \in C$  and we have  $x(b_cd)^{-1} = bdd^{-1}b_c = bb_c^{-1} \in \mathbb{C}_G(a)$ . Clearly,  $\operatorname{card}(DC) \leq n^2$ .

- **1.5 Lemma.** Let H be a finite subgroup of a group G and let A, B be H-connected pseudotransversals to H in G such that  $G = \langle A, B \rangle$ . Then the index  $[G : \mathbb{C}_G(H)]$  is finite. Moreover, if  $\mathbb{L}_G(H) = 1$ , then the index  $[G : \mathbb{Z}(G)]$  is also finite.
- **Proof.** Since H is finite,  $H \subseteq \langle C \rangle$  for a finite subset  $C \subseteq A \cup B$ . Now, put  $K = \bigcap_{c \in C} \mathbb{C}_G(c)$ . By 1.4, each of the subgroups  $\mathbb{C}_G(c)$  is of finite index in G. Consequently, the index [G:K] is also finite. Clearly,  $K \subseteq \mathbb{C}_G(H)$ .

Now, assume that  $\mathbb{L}_G(H) = 1$ . By the preceding observation and [1,3.18],  $\mathbb{N}_G(H) = H\mathbb{Z}(G)$  is of finite index in G. But H is finite,  $H \cap \mathbb{Z}(G) = 1$  and  $[G : \mathbb{Z}(G)] = [G : H\mathbb{Z}(G)][H\mathbb{Z}(G) : \mathbb{Z}(G)] = [G : \mathbb{N}_G(H)] \operatorname{card}(H)$  is finite.

- **1.6 Remark.** Let H be a finite subgroup of a group G such that there exist H-connected pseudotransversals A, B to H in G.
- (i) If  $G = \langle A, B \rangle$ , then the index [G:K] is finite, where  $K/\mathbb{L}_G(H) = \mathbb{Z}(G/\mathbb{L}_G(H))$ .
- (ii) Put  $G_1 = \langle A, B \rangle$  and  $H_1 = H \cap G_1$ . Then A, B are  $H_1$ -connected pseudotransversals to  $H_1$  in  $G_1$  and by 1.5, the index  $[G_1 : \mathbb{C}_{G_1}(H_1)]$  is finite. On the other hand, since H is finite also the index  $[G:G_1]$  is finite. Consequently  $[G:\mathbb{C}_{G_1}(H_1)]$  is finite. In particular, there is a normal subgroup  $K \subseteq G$  such that  $K \subseteq \mathbb{C}_{G_1}(H_1)$  and [G:K] is finite.
- **1.7 Lemma.** Let H be an Abelian subgroup of a group G such that  $\mathbb{N}_G(H) = H$ . If  $x \in G$  and  $\mathbb{N}_G(T) \subseteq H$ , where  $T = H \cap H^x$ , then  $x \in H$  and T = H.
- **Proof.** We have  $H \cup H^x \subseteq \mathbb{N}_G(T) \subseteq H$ . Hence  $H^x \subseteq H$  and  $T = H^x$ . Further,  $H \subseteq H^{x^{-1}}$  is an Abelian group,  $H^{x^{-1}} \subseteq \mathbb{N}_G(H) = H$  and  $H \subseteq H^x$ . Thus  $H = H^x = T$  and  $x \in \mathbb{N}_G(H) = H$ .

- **1.8 Lemma.** Let  $H \subseteq R \subseteq G$  be subgroups of a group G such that H is Abelian,  $H \not \supseteq R$  and  $R \subseteq K$ , whenever K is a subgroup of G properly containing H. Then:
  - (i)  $\mathbb{N}_G(H) = H$  and  $\mathbb{Z}(R) \subseteq H$ .
  - (ii) If T is a subgroup of H such that  $\mathbb{N}_G(T) \not\subseteq H$ , then  $T \subseteq \mathbb{Z}(R)$ .
  - (iii)  $\mathbb{L}_G(H) \subseteq \mathbb{Z}(R) \subseteq \mathbb{L}_R(H)$ .
  - **Proof.** (i) If  $\mathbb{N}_G(H) \neq H$ , then  $H \subseteq R \subseteq \mathbb{N}_G(H)$  and  $H \subseteq R$ , a contradiction.
- (ii) We have  $H \subseteq \mathbb{C}_G(T) \preceq \mathbb{N}_G(T)$ . Since  $\mathbb{N}_G(H) = H$  and  $\mathbb{N}_G(T) \neq H$ , H is not normal in  $\mathbb{N}_G(T)$ , and hence  $H \neq \mathbb{C}_G(T)$ . Consequently  $R \subseteq \mathbb{C}_G(T)$  and  $T \subseteq \mathbb{Z}(R)$ .
- (iii) By (i) and (ii),  $\mathbb{L}_G(H) \subseteq \mathbb{Z}(R) \subseteq H$ . Clearly  $\mathbb{Z}(R) \subseteq \mathbb{L}_R(H)$ . On the other hand,  $\mathbb{N}_R(\mathbb{L}_R(H)) = R \not\subseteq H$ , and so  $\mathbb{L}_R(H) \subseteq \mathbb{Z}(R)$  again by (ii) (where we take G = R).

## 2. Connected transversals to cyclic subgroups

- **2.1** In this section, let H be a cyclic subgroup of a group G and let A, B be H-connected pseudotransversals to H in G.
- **2.2 Theorem.** Suppose that H is a cyclic p-group for a prime p and that  $\mathbb{L}_G(H) = 1$ . Then A = B is an Abelian subgroup of G.
- **Proof.** First, A and B are transversals [1, 3.9]. Now, for every  $a \in A$  there exists a (uniquely determined)  $f(a) \in B$  with  $f(a)^{-1} a \in H$  and we have  $f(a)^{-1} a^2 H = f(a)^{-1} a f(a) H = f(a)^{-1} a f(a) f(a)^{-1} a^{-1} f(a) a H = aH$  (since  $f(a)^{-1} a^{-1} f(a) a \in H$ ). Let  $d \in A$  and  $c = f(d)^{-1} d \in H$ . Further, let  $b \in A$  and let K denote the subgroup  $\langle c, f(b)^{-1} b \rangle$ . Then  $K \subseteq H$ , K is a cyclic p-group, and hence either  $K = \langle c \rangle$  or  $K = \langle f(b)^{-1} b \rangle$ . In the latter case,  $cbH = (f(b)^{-1} b)^n bH = (f(b)^{-1} b)^{n-1} f(b)^{-1} b^2 H = (f(b)^{-1} b)^{n-1} bH = ... = bH$  (see the above observation) for some  $n \ge 1$  and we have  $b^{-1}cb \in H$ .

Now, assume that  $K = \langle c \rangle$ . Then  $f(b)^{-1}bdH = c^ndH = dH = f(d)H$ . Consequently,  $d^{-1}f(b)^{-1}bd \in H$  and thus  $b^{-1}d^{-1}bd = b^{-1}f(b) \cdot f(b)^{-1}d^{-1}f(b) \cdot d \cdot d^{-1}f(b)^{-1}bd \in H$ . Moreover,  $f(d)^{-1}f(b)^{-1}bd \in H$ , and hence  $f(b)^{-1}f(d)^{-1}bd = f(b)^{-1}b \cdot b^{-1}f(d)^{-1}bf(d)f(d)^{-1}f(b)^{-1}bd \cdot d^{-1}b^{-1}f(b)d \in H$ . Finally  $f(b)^{-1}cb = f(b)^{-1}f(d)^{-1}db = f(b)^{-1}f(d)^{-1}bd \cdot d^{-1}b^{-1}db \in H$ , and therefore  $b^{-1}cb = b^{-1}f(b) \cdot f(b)^{-1}cb \in H$ .

We have shown that (in both cases)  $b^{-1}cb \in H$  for every  $b \in A$ . Since G = AH, we conclude that  $c \in \mathbb{L}_G(H) = 1$ . Thus c = 1 and f(d) = d. But this means that A = B. Now A is an H-selfconnected transversal and, in particular, abH = baH for all  $a, b \in A$ .

In the remaining part of the proof, we are going to show that A is a subgroup of G. For every ordered pair  $(a, b) \in A \times A$  there exists a (unique)

 $g(a, b) \in A$  such that  $h(a, b) = g(a, b)^{-1} ab \in H$ . Now, g(a, b) = g(b, a) and  $h(a, b) b^{-1} a^{-1} ba = h(b, a)$ . Moreover,  $h(a, b) aH = g(a, b)^{-1} abaH = g(a, b)^{-1} a^2 bH = g(a, b)^{-1} a g(a, b) H = aH$  (since  $a^{-1}g(a, b)^{-1} ag(a, b) \in H$ ) and  $a^{-1}h(a, b) a \in H$ .

Let  $a, b, c \in A$  and  $R = \langle h(a, b), h(c, b), h(c, c) \rangle$ . Again, either  $R = \langle h(a, b) \rangle$  or  $R = \langle h(c, b) \rangle$  or  $R = \langle h(c, c) \rangle$ . In the latter two cases we have  $c^{-1}h(a, b) c \in H$ , since  $c^{-1}h(c, b) c$ ,  $c^{-1}h(c, c) c \in H$  (see the above observation).

Next assume that  $R = \langle h(a,b) \rangle$ . Then  $u = a^{-1}h(c,b) \ a \in H$ ,  $v = g(c,b)^{-1} \ a^{-1}cba = g(c,b)^{-1} \ a^{-1}g(c,b) \ a.u \in H$ ,  $w = b^{-1}c^{-1}a^{-1}cba = h(c,b)^{-1} \ v \in H$ ,  $z = c^{-1}b^{-1}a^{-1}cba = c^{-1}b^{-1}cbw \in H$ ,  $r = c^{-1}b^{-1}a^{-1}cg(a,b) = za^{-1}b^{-1}g(a,b) = zh(b,a)^{-1} \in H$ ,  $s = c^{-1}h(a,b)^{-1} \ c = rg(a,b)^{-1} \ c^{-1}g(a,b) \ c \in H$  and, finally,  $t = s^{-1} = c^{-1}h(a,b) \ c \in H$ . We have shown that  $c^{-1}h(a,b) \ c \in H$  for every  $c \in A$  and it follows that  $h(a,b) \in H$ .

We have shown that c = h(a, b)  $c \in H$  for every  $c \in A$  and it follows that  $h(a, b) \in \mathbb{L}_G(H) = 1$  and  $g(a, b)^{-1} ab = h(a, b) = 1$ , i.e.,  $ab = g(a, b) \in A$ . We conclude that A is a subsemigroup of G. On the other hand, if  $a \in A$ , then  $b^{-1}a^{-1} \in H$  for some  $b \in A$  and we have  $ab \in H \cap A = 1$  [1, 3.12(i)] and  $a^{-1} = b \in A$ . This shows that A is a subgroup of G. Since  $a^{-1}b^{-1}ab \in H \cap A$  for all  $a, b \in A$ , we get ab = ba. The proof is complete.

- **2.3 Corollary.** Suppose that H is a cyclic p-group. Then  $A\mathbb{L}_G(H) = B\mathbb{L}_G(H) = K$  is a subgroup of G and  $K/\mathbb{L}_G(H)$  is an Abelian group (and consequently  $[A, B] \subseteq \mathbb{L}_G(H)$ ).
- **2.4 Remark.** The preceding results remain true for H being the Prüfer quasicyclic p-group (the same proof).
- **2.5 Lemma.** Suppose that  $\mathbb{L}_G(H) = 1$ , [A, B] = 1 and  $G = \langle A, B \rangle$ . Then G = A = B is an Abelian group.
- **Proof.** Clearly,  $C = \langle A \rangle \cap H$  is a normal in G, hence C = 1 and  $\langle A \rangle = A$  is a normal subgroup of G. Quite similarly,  $\langle B \rangle = B$  is normal in G. And we have G = AB. In particular,  $H = \langle ab \rangle$  for some  $a \in A$ ,  $b \in B$  and  $G = AH = A\langle b \rangle$ . If  $c \in B$ , then  $c = db^n$ ,  $d \in A$ , and since  $d \in A \cap B$  it follows that  $d \in \mathbb{Z}(G)$ . We conclude that B is a Abelian, H = 1 and G = A = B.
- **2.6 Colorally.** Suppose that  $[A, B] \subseteq \mathbb{L}_G(H)$  and  $G = \langle A, B \rangle$ . Then  $G' \subseteq H$  and  $H \subseteq G$ .
  - **2.7 Theorem.** If  $G = \langle A, B \rangle$ , then  $G' \subseteq H$  and  $H \subseteq G$ .
- **Proof.** Clearly,  $G' \subseteq H$  iff  $H \subseteq G$ . Now assume that  $H \not = G$  and let  $H = \langle u \rangle$ . Then  $G' \not = H$  and there are  $x, y \in G$  such that  $[x, y] \not \in H$ . Further, since  $G = \langle A, B \rangle$ , there is a finite subset E of  $A \cup B$  such that  $x, y, u \in G_1 = \langle E \rangle$ . Now, put  $C = G_1 \cap A$  and  $D = G_1 \cap B$ . Then C, D are H-connected transversals to H in  $G_1$  and  $G_1 = \langle C, D \rangle$ . Moreover  $G'_1 \not = H$ . From now on we assume that G is finitely generated. With respect to 1.1 and 1.2, we can also assume that  $MH \subseteq G$  whenever  $1 \not = M \subseteq G$ . The rest of the proof is divided into four parts:

- (i) If  $\mathbb{L}_G(H) \neq 1$ , then  $H = \mathbb{L}_G(H) \leq G$ , a contradiction. Thus  $\mathbb{L}_G(H) = 1$ .
- (ii) Next we show that  $\mathbb{Z}(G) = 1$ . Assume this be not true and put  $L = \cap \langle H, z \rangle$ , where z goes trough non-identical elements of  $\mathbb{Z}(G)$ . We have  $\langle H, z \rangle = H \langle z \rangle \trianglelefteq G$ , and henceforth  $L \trianglelefteq G$ . Put also  $V = \cap \langle z \rangle$ , so that  $V \trianglelefteq G$ . Since  $H \cap \mathbb{Z}(G) = 1$ , we have L = VH and, since  $L \neq H$ , it follows that  $V \neq 1$ . We conclude that  $\mathbb{Z}(G)$  is either a cyclic p-group or quasicyclic p-group and, anyway, V is a (cyclic) group of prime order P. Moreover, P0 and P1 for every P2. Now, we see that P3 and P4 and P5 and P5 and P6 are the cyclic p-group of prime order P6.

Let T be a left transversal to L in G with  $1 \in T$ . Put  $f = \prod f_t$ ,  $t \in T$ , where  $f_t : L = H' \times V \to V$  is the natural projection, so that f is a homomorphism of L into the cartesian product W of  $\operatorname{card}(T) = [G:L]$  copies of V. If  $v \in \operatorname{Ker}(f)$ , then  $v \in H'$  for every  $t \in T$  and thus  $v \in \mathbb{L}_G(H) = 1$ . It follows that f is injective and then L (and, in particular, H) can be imbedded into L. But W is an Abelian elementary p-group, the same is true for H and H is a (cyclic) group of order P. Now, by P0.2, P1 and P2 is an Abelian group and P3 is not true. We have proved that P3 is an Abelian group and P4.

(iii) Let K be a subgroup of H such that  $K \neq 1$  and  $H \neq \mathbb{N}_G(K)$ . Let  $a \in (A \cap \mathbb{N}_G(K)) \setminus H$  and  $T = \langle H, a \rangle$ . We have  $\mathbb{N}_T(H) \subseteq \mathbb{N}_G(H) = H$ , so that  $\mathbb{N}_T(H) = H$ . Now,  $H \subseteq \mathbb{C}_T(K) \supseteq \mathbb{N}_T(K) \neq H$  and consequently  $H \neq \mathbb{C}_T(K)$ . Let  $b \in (A \cap \mathbb{C}_T(K)) \setminus H$  and  $S = \langle H, b \rangle$ . By [1, 3.11(i)],  $b \in \mathbb{L}_G(S) \neq 1$  and hence  $S = \mathbb{L}_G(S) H \supseteq G$ . Further,  $\mathbb{C}_S(K) = S$ , and so  $K \subseteq \mathbb{Z}(S)$ . Clearly,  $\mathbb{Z}(S) \subseteq H$  (since  $\mathbb{N}_G(H) = H$ ) and since  $S \supseteq G$ , we have also  $\mathbb{Z}(S) \supseteq G$ . Thus  $\mathbb{Z}(S) \subseteq \mathbb{L}_G(H) = 1$  and K = 1, a contradiction.

We have proved that  $\mathbb{N}_G(K) = H$  for every non-trivial subgroup K of H.

- (iv) By (iii) and 1.3, we have  $H \cap H^x = 1$  for every  $x \in G \setminus H$ . Now, by [1, 3.20],  $A = B = \langle A, B \rangle = G$  is an Abelian group, and hence G' = 1, which is the final contradiction.
- **2.8 Theorem.** Suppose that  $\mathbb{L}_G(H) = 1$ . Then G'' = 1 and A = B is an Abelian subgroup of G.
- **Proof.** Put  $K = \langle A, B \rangle$  and  $E = K \cap H$ . Now, A and B are E-connected transversals to E in K and  $\mathbb{L}_K(E) = 1$ . By 2.7,  $K' \subseteq E$ , and hence K' = 1. Then K is an Abelian group, E = 1, A = B = K and G'' = 1 by [3].
- **2.9 Corollary.**  $G'' \subseteq \mathbb{L}_G(H)$ , G''' = 1,  $K = A\mathbb{L}_G(H) = B\mathbb{L}_G(H)$  is a subgroup of G and  $K' \subseteq \mathbb{L}_G(H)$  (and consequently  $[A, B] \subseteq \mathbb{L}_G(H)$ ).

#### 3. Products of Abelian groups

**3.1** In this section, let G be a group such that G = KH, where both K and H are Abelian subgroups of G,  $H \neq G$ ,  $K \neq 1$  and  $K \leq G$ .

The following four lemmas are obvious:

#### 3.2 Lemma.

- (i)  $H \cap K \subseteq H \cap \mathbb{C}_G(K) = H \cap \mathbb{Z}(G) \subseteq \mathbb{L}_G(H)$ .
- (ii)  $\mathbb{Z}(G) = (K \cap \mathbb{Z}(G))(H \cap \mathbb{Z}(G)).$
- (iii) If  $\mathbb{L}_G(H) = 1$ , then  $H \cap K = 1 = H \cap \mathbb{C}_G(K)$  and  $\mathbb{Z}(G) \subseteq K$ .
- (iv) If  $\mathbb{Z}(G) = 1$  then  $H \cap K = 1 = H \cap \mathbb{C}_G(K)$ .
- (v) If  $H \cap K = 1$ , then  $\mathbb{L}_G(H) = H \cap \mathbb{C}_G(K) = H \cap \mathbb{Z}(G)$ .

#### 3.3 Lemma.

- (i) If E is a subgroup of G such that  $H \subseteq E \subseteq G$ , then  $E = (E \cap K) H$  and  $E \cap K \subseteq G$ .
- (ii) If no non-trivial proper subgroup of K is normal in G, then  $H \cap K = 1$  and H is maximal in G.
  - **3.4 Lemma.** Suppose that H is a maximal subgroup of G.
- (i) If L is a subgroup of K and  $L \subseteq G$ , then either  $L \subseteq H \cap K$  or  $K = (H \cap K)L$ .
  - (ii) If  $H \cap K = 1$ , then no non-trivial proper subgroup of K is normal in G.
  - (iii) If  $H \not \triangleq G$ , then  $\mathbb{Z}(G) \subseteq \mathbb{L}_G(H)$ .
  - **3.5 Lemma.** The following conditions are equivalent:
    - (i) H is maximal in G and  $H \cap K = 1$ .
  - (ii) No non-trivial proper subgroup of K is normal in G.
- **3.6** In the remaining part of this section, we shall assume that the equivalent conditions of 3.5 are satisfied. Now, by 3.2(v),  $\mathbb{L}_G(H) = H \cap \mathbb{C}_G(K) = H \cap \mathbb{Z}(G)$ . If  $H \not \supseteq G$ , then  $\mathbb{Z}(G) \subseteq H$  and  $\mathbb{L}_G(H) = \mathbb{Z}(G)$ .

If  $H \subseteq G$ , then  $G \cong K \times H$  is Abelian and K is (cyclic) of prime order.

For every  $u \in H$ , the map  $q_u : a \to a^u = u^{-1}au$  is an automorphism of K and we denote by F the subring generated by all these automorphisms  $q_u$  in the endomorphism ring of K. Moreover, we put  $q = -1_F \in F$ ; then  $q(a) = a^{-1}$  for every  $a \in K$  and  $q^2 = 1_F (= id_K)$ .

#### 3.7 Lemma.

- (i) F is a field and K, as a vector space over F, is of dimension 1. In particular, the groups K and F(+) are isomorphic.
- (ii) If H is finitely generated, then both F and K are finite. If, moreover,  $\mathbb{L}_G(H) = 1$ , then H is finite cyclic and G is finite.
- **Proof.** (i) Since H is abelian, F is a commutative ring. If  $f \in F$ ,  $f \neq 0_F$ , then both f(K) and Ker(f) are subgroups of K, they are normal in G, and hence f(K) = K and Ker(f) = 1, i.e., f is an automorphism of K.

Now, let  $a \in K$ ,  $a \neq 1$ . Then F(a) is a subgroup of K (use the fact that  $q \in F$ ) and  $F(a) \subseteq G$ . Since  $a \in F(a)$ , we have F(a) = K. If  $f \in F$ ,  $f \neq 0_F$ , then  $f^{-1}(a) = g(a)$  for some  $g \in F$ , a = fg(a) and the equality F(a) = K implies  $fg = id_K = 1$ . Consequently,  $f^{-1} = g \in F$ .

We have proved that F is a field.

- (ii) Any field, finitely generated as a ring, is finite. Now, if  $\mathbb{L}_G(H) = 1$ , then the mapping  $u \to q_u^{-1}$  is an injective homomorphism of H into the multiplicative group  $F^*$  of non-zero elements from F; the group  $F^*$  is cyclic.
  - **3.8** Let A be a left pseudotransversal to H in G such that [A, A] = 1.

#### 3.9 Lemma.

- (i)  $A \subseteq K \mathbb{L}_G(H)$ .
- (ii) If  $\mathbb{L}_G(H) = 1$ , then A = K.

**Proof.** There is a uniquely determined subset S of  $K \times H$  such that  $A = \{au, (a, u) \in S\}$ . Now, fix an element  $r \in K$ ,  $r \neq 1$ . For every  $a \in K$ , there is a unique  $p_a \in F$  with  $a = p_a(r)$ ; we have  $p_a \neq 0_F$  iff  $a \neq 1$ .

Assume that  $b \neq 1$  and  $u \notin L = \mathbb{L}_G(H)$  for some  $(b, u) \in S$  and put  $p = (q + q_u^{-1}) p_b^{-1} \in F$ . Since  $u \notin L = H \cap \mathbb{C}_G(K)$ , we have  $u \notin \mathbb{C}_G(K)$  and  $q + q_u^{-1} \neq 0_F$ . Thus  $p \neq 0_F$  and  $e^{-1} = p^{-1}(r)$  for some  $e \in K$ ,  $e \neq 1$ . Now,  $p_e(r) = e = (p^{-1}(r))^{-1} = p^{-1}(r^{-1}) = p^{-1}q(r)$ , and so  $p_e = p^{-1}q$  and  $p_e^{-1} = q^{-1}p = qp$ . On the other hand, G = AH, and hence  $(e, v) \in S$  for some  $v \in H$ . The equality [A, A] = 1 implies  $bueu^{-1}uv = buev = evbu = evbv^{-1}uv$  and  $bueu^{-1} = evbv^{-1}$ . From this,  $(q + q_v^{-1}) p_b(r) = b^{-1}vbv^{-1} = e^{-1}ueu^{-1} = (q + q_u^{-1}) p_e(r)$  and  $(q + q_v^{-1}) p_b = (q + q_u^{-1}) p_e$ ,  $p = (q + q_u^{-1}) p_b^{-1} = (q + q_v^{-1}) p_e^{-1} = (q + q_v^{-1}) qp$ ,  $1_F = (q + q_v^{-1}) q = 1_F + q_v^{-1}q$  and  $0_F = q_v^{-1}q$ , a contradiction.

We have proved that  $A \subseteq H \cup KL$ . But, if  $w \in A \cap H$  and  $c \in K$ , then  $cz \in A$  for some  $z \in H$  and wcz = czw = cwz, wc = cw and  $w \in L \subseteq KL$ . Thus  $A \subseteq KL$  and the rest is clear.

#### 3.10 Lemma.

- (i)  $G' \subseteq K$ .
- (ii) If  $H \not = G$ , then G' = K.
- (iii) If  $\mathbb{L}_G(H) = 1 + H$ , then A = G'.

**Proof.** (i)  $G/K \cong H$ .

- (ii) Since  $H \not \supseteq G$ , we must have  $G' \neq 1$ . But  $G' \subseteq G$  and  $G' \subseteq K$ .
- (iii) Combine (ii) and 3.9(ii).

#### 4. Connected transversals to Abelian maximal subgroups

- **4.1** In this part, let H be a (proper) maximal subgroup of a group G such that H is Abelian and not normal in G. Further, let A, B be H-connected pseudotransversals to H in G. By 1.8(iii),  $\mathbb{Z}(G) = \mathbb{L}_G(H) \subseteq H$ .
- **4.2 Theorem.** Suppose that  $(\mathbb{Z}(G) =) \mathbb{L}_G(H) = 1$ . Then A = B = G' is a normal Abelian subgroup of G.

**Proof.** First, let  $a \in A$ . Then  $b^{-1}a \in H$  for some  $b \in B$ , and hence  $b^{-1}a \in H \cap Hb^{-1} = H \cap bHb^{-1} = T$ . If  $\mathbb{N}_G(T) \subseteq H$ , then  $b \in H$  by 1.7, and so  $a \in A \cap H = 1$ ,  $b \in B \cap H = 1$  and a = b = 1 ([1, 3.12(i)]). On the other hand, if  $\mathbb{N}_G(T) \nsubseteq H$ , then  $\mathbb{N}_G(T) = G$  (since H is maximal in G),  $T \subseteq G$ ,  $T \subseteq \mathbb{L}_G(H) = 1$  and, again, a = b. We have proved that  $A \subseteq B$ . Quite similarly,  $B \subseteq A$ , and so A = B.

Now, let  $a, b \in A$ . There is  $c \in A$  such that  $c^{-1}ab \in H$ . We have  $c^{-1}ba = c^{-1}abb^{-1}a^{-1}ba \in H$  and  $a^{-1}c^{-1}aba = a^{-1}c^{-1}ac \cdot c^{-1}ba \in H$ . Consequently,  $c^{-1}ab \in T = H \cap aHa^{-1}$ . Again, if  $\mathbb{N}_G(T) \subseteq H$ , then  $a \in H$  by 1.7, and hence a = 1 and  $ab = b \in A$ . If  $\mathbb{N}_G(T) \not\subseteq H$ , then T = 1 and  $ab = c \in A$ . We have proved that  $AA \subseteq A$ , i.e., A is a subsemigroup of G. Further, if  $a \in A$ , then  $b^{-1}a^{-1} \in H$  for some  $b \in A$ , and then  $ab \in H \cap A$ , and  $a^{-1} = b \in A$ . Thus A is a subgroup of G. Since  $A \cap H = A$  is an Abelian group.

Now, G = AH, and hence G'' = 1 by [3]. Since  $H \not\equiv G$ , we have  $G' \not\equiv H$  and then G = HG'. By 3.10(iii) A = G'.

**4.3 Corollary.** (cf. 2.3 and 2.9.)  $A\mathbb{Z}(G) = B\mathbb{Z}(G) = G'\mathbb{Z}(G) = K$  is a normal subgroup of G,  $K' \subseteq \mathbb{Z}(G)$ ,  $K/\mathbb{Z}(G)$  is an Abelian group, K is nilpotent of class at most 2,  $\langle A, B \rangle \subseteq K \neq G$  and  $[A, B] \subseteq \mathbb{Z}(G)$ . Finally,  $G'' \subseteq \mathbb{Z}(G)$  and G''' = 1.

## 5. Connected transversals to finite Abelian subgroups

- **5.1** In this section, let H be a finite Abelian subgroup of a group G such that there exist H-connected pseudotransversals A, B to H in G.
  - **5.2 Theorem.** The group G is soluble.
- **Proof.** (i) First, assume that G is finite and proceed by induction on card(G). If H = G, then G is Abelian. Hence, let  $H \neq G$  and let  $G_1$  be a subgroup of G such that H is a (proper) maximal subgroup of  $G_1$ . It follows from [1, 3.11(i)] that  $\mathbb{L}_G(G_1) \neq 1$  and then  $G/\mathbb{L}_G(G_1)$  is soluble by induction. Now it remains to show that  $G_1$  is soluble.
  - If  $H \subseteq G_1$ , then  $G'_1 \subseteq H$  and  $G''_1 = 1$ . If  $H \not \supseteq G$ , then  $G'''_1 = 1$  by 4.3.
- (ii) Next, let  $G = \langle A, B \rangle$ . Since  $\mathbb{L}_G(H)$  is a normal Abelian subgroup of G, we may assume that  $\mathbb{L}_G(H) = 1$ . Now,  $\overline{G} = G/\mathbb{Z}(G)$  is a finite group by 1.5 and  $\overline{G}$  is soluble by (i).
- (iii) Finally, consider the general case. Let  $G_1 = \langle A, B \rangle$  and  $H_1 = H \cap G_1$ . Then A, B are  $H_1$ -connected pseudotransversals to  $H_1$  in  $G_1$  and the subgroup  $G_1$  is soluble by (ii). On the other hand, the index  $[G:G_1]$  is finite and there is a subgroup K of G such that  $K \subseteq G$  and [G:K] is finite. Consequently, K is soluble (since  $G_1$  is so) and G/K is soluble (by (i)).
- **5.3 Remark.** Suppose that there exists a subgroup R of G such that  $H \subseteq R$  and  $H \not \supseteq R$  and  $R \subseteq K$ , wherever K is a subgroup of G properly containing H.

- (i) We have  $R \nsubseteq \mathbb{N}_G(H)$ , and hence  $\mathbb{N}_R(H) = \mathbb{N}_G(H) = H$ . Consequently,  $\mathbb{Z}(G) \subseteq H$  and  $\mathbb{Z}(R) \subseteq H$ .
  - (ii) If T is a subgroup of H such that  $\mathbb{N}_G(T) \subseteq H$ , then  $T \subseteq \mathbb{Z}(R)$ .
  - (iii)  $\mathbb{L}_G(H) \subseteq \mathbb{Z}(R) = \mathbb{L}_R(H)$ .
- (iv) Let  $a \in A$ . Then  $b^{-1}a \in T = H \cap bHb^{-1}$  for some  $b \in B$  (see the proof of 4.2) and either  $\mathbb{N}_G(T) \subseteq H$  and  $b \in B \cap H \subseteq \mathbb{L}_G(H) \subseteq \mathbb{Z}(R)$ ,  $a \in A \cap H \subseteq \mathbb{L}_G(H) \subseteq \mathbb{Z}(R)$  (by 1.7 and [1, 3.11(i)]) or  $\mathbb{N}_G(T) \nsubseteq H$  and  $b^{-1}a \in \mathbb{Z}(R)$  by (ii).
- We have proved that  $A \subseteq B\mathbb{Z}(R)$ . Quite similarly,  $B \subseteq A\mathbb{Z}(R)$ , and hence  $A\mathbb{Z}(R) = B\mathbb{Z}(R) = E$ . Clearly,  $H \cap E = \mathbb{Z}(R)$ .
- (v) Let  $a \in A$  and  $b \in B$ . Then  $c^{-1}ab \in H$  for some  $c \in B$  and we have  $c^{-1}ab \in T = H \cap aHa^{-1}$  (see the proof of 4.2). If  $\mathbb{N}_G(T) \subseteq H$ , then  $a \in A \cap H \subseteq \mathbb{L}_G(H) \subseteq Z(R)$ ,  $b^{-1}ab \in \mathbb{L}_G(H)$  and  $ab \in b\mathbb{L}_G(H) \subseteq b\mathbb{Z}(R) \subseteq E$ . If  $\mathbb{N}_G(T) \nsubseteq H$ , then  $c^{-1}ab \in \mathbb{Z}(R)$  and, again  $ab \in E$ .

We have proved that  $AB \subseteq E$ . Quite similarly,  $BA \subseteq E$ .

(vi) Let  $a \in A$ . Then  $b^{-1}a^{-1} \in H$  for some  $b \in H$ , and so  $ab \in AB \cap H \subseteq E \cap H = \mathbb{Z}(R)$ . Now,  $b^{-1}a^{-1} \in \mathbb{Z}(R)$  and  $a^{-1} \in b\mathbb{Z}(R) \subseteq E$ .

We have proved that  $A^{-1} \subseteq E$ . Quite similarly  $B^{-1} \subseteq E$ .

- (vii) Let  $a \in A$  and  $b \in B$ . Then  $c^{-1}a^{-1}b \in H$  for suitable  $c \in B$  and  $c^{-1}a^{-1}bAH = c^{-1}a^{-1}abH = c^{-1}aa^{-1}bH = c^{-1}acH = c^{-1}caH = aH$  (since  $a^{-1}b^{-1}ab \in H$  and  $c^{-1}a^{-1}ca \in H$ ). Now  $a^{-1}c^{-1}a^{-1}ba \in H$  and  $c^{-1}a^{-1}b \in T = H \cap aHa^{-1}$ . Proceeding similarly as in (v), we check that  $ab^{-1} \in E$ . Thus  $A^{-1}B \subseteq E$  and, symetrically,  $B^{-1}A \subseteq E$ .
- (viii)  $AE = AB\mathbb{Z}(R) \subseteq E\mathbb{Z}(R) = A\mathbb{Z}(R)\mathbb{Z}(R) = E$ . Similary  $BE \subseteq E$ ,  $A^{-1}E \subseteq E$  and  $B^{-1}E \subseteq E$ , and hence  $SS \subseteq E$  and  $SE \subseteq E$  where  $S = A \cup A^{-1} \cup B \cup B^{-1}$ . Further, by induction on  $n \ge 1$ ,  $n timesSS ... S \subseteq E$ . On the other hand,  $\langle A, B \rangle = \bigcup_{n \ge 1} SS ... S$ , so that  $\langle A, B \rangle \subseteq E$ .
- (ix) Since  $H \not = R$ , we have  $H \neq \mathbb{Z}(R)$  and we take  $u \in H \setminus \mathbb{Z}(R)$ . If  $u \in E$ , then u = ar for same  $a \in A$ ,  $r \in \mathbb{Z}(R)$ , and then  $a \in A \cap H \subseteq \mathbb{Z}(R)$ ,  $u = ar \in \mathbb{Z}(R)$ , a contradiction. Thus  $u \notin E$  and, in particular,  $\langle A, B \rangle \subseteq E \neq G$ .
  - **5.4 Theorem.** If  $G = \langle A, B \rangle$ , then H is subnormal in G.
- **Proof.** (i) First, assume that G is finite and proceed by induction on card(G). Let K be a subgroup of G such that  $H \subseteq K$  and  $H \neq K$ . Then  $L = \mathbb{L}_G(K) \neq 1$  and K = HL ([1, 3.11(i)]). Further, by induction, K is subnormal in G and, since G is finite, also the subgroup  $R = \bigcap K$  is subnormal in G. If  $H \not \supseteq R$  then  $G = \langle A, B \rangle \neq G$  by 5.3(ix), a contradiction. Thus  $H \trianglelefteq R$  and H is subnormal in G.
- (ii) Next, let G be (possibly) infinite. We can assume that  $\mathbb{L}_G(H) = 1$ . Now, by 1.5, the index  $[G : \mathbb{Z}(G)]$  is finite, and so  $H\mathbb{Z}(G)$  is subnormal in G by (i). However,  $H \leq H\mathbb{Z}(G)$ .
  - **5.5 Corollary.** If  $\mathbb{L}_G(H) = 1 \neq G = \langle A, B \rangle$ , then  $\mathbb{Z}(G) \neq 1$ .

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