Acta Universitatis Carolinae. Mathematica et Physica

Petr Lachout

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 45 (2004), No. 1, 3--8

Persistent URL: `http://dml.cz/dmlcz/142729`

A Solution of an Equation for Indexed Functions

PETR LACHOUT

Praha

Received 6. March 2003

We present a characterization of indexed real functions (4) fulfilling an equation (5). Consequently, we are receiving a description of those linear transformations of a Wiener process which result in a time-changed Wiener process, Brownian bridge or Ornstein–Uhlenbeck process.

1. Problem setting and examples

Assymptotic investigation of statistical estimators and test statistics often leads to a linear transformation of a Wiener process that turns out to be a timed-changed Wiener process, a Brownian bridge or Ornstein–Uhlenbeck process. Let us recall some typical examples of such transformations.

Given a Wiener process $\left(W(t), t \geq 0\right)$, the process $\left(tW(\frac{1}{t}), t > 0\right)$ is a Wiener processes, $\left(W(t) - tW(1), t \in [0, 1]\right)$ and $\left(tW(\frac{1-t}{t}), t \in (0, 1)\right)$ are Brownian bridges and $\left(\mathrm{e}^{-t}W(\mathrm{e}^{2t}), t \in \mathbb{R}\right)$ is an Ornstein–Uhlenbeck process.

In [3] we treated a collection of stochastic integrals of non-random real functions w.r.t. a Wiener process $\left(W(t), t \geq 0\right)$, i.e.

$$\left(\int a_\lambda \,\mathrm{d}W, \lambda \in \Lambda\right), \text{ where } a_\lambda \in \mathsf{L}_2(\mathfrak{m}) \quad \forall \lambda \in \Lambda \tag{1}$$

and $\mathfrak{m}$ denotes the Lebesgue measure on $[0, +\infty)$.

Department of Probability and Statistics, Charles University in Prague, Sokolovská 83, 180 75 Praha 8, and Institute of Information Theory and Automation, Czech Academy of Sciences, Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic

The research has been partially supported by the project CEZ: MSM 113200008 and the Grant Agency of the Czech Republic under Grant 201/03/1027.

We seek conditions under which (1) satisfies

$$\int a_\lambda \, \mathrm{d}W = V(\xi(\lambda)) \quad \text{a.s.} \quad \forall \lambda \in \Lambda, \tag{2}$$

where V is a prescribed Gaussian process, e.g. a Wiener process, a Brownian bridge, an Ornstein–Uhlenbeck process, Λ is a non-empty set and $\xi : T \to \mathbb{R}_+$ is an appropriate function.

Because (1) is always a Gaussian process, one can verify (2) computing the covariance function of (1), only. Applying that, we have proved in [3] that (2) with V being a Wiener process is equivalent to

$$\int a_\lambda a_\psi \, \mathrm{dm} = \min \left\{ \int a_\lambda^2 \, \mathrm{dm}, \int a_\psi^2 \, \mathrm{dm} \right\} \forall \lambda, \ \psi \in \Lambda. \tag{3}$$

Of course in (2) we set $\xi(\lambda) = \int a_\lambda^2 \, \mathrm{dm}$.

In [3] we also present some examples of function families satisfying (3). Especially, we consider families of functions which are constant till a point and zero after that. For these families we succeeded to determine a complete description to satisfy (3). Two particular families keeping (3) are shown in [1], also.

Inspired by (3) we consider a measure space $(E, \mathscr{E}, \mu)$ in this paper and indexed real functions

$$(f_\lambda, \lambda \in \Lambda), \quad \text{where} \quad f_\lambda : (E, \mathscr{E}) \to (\mathbb{R}, \mathbb{B}) \in \mathsf{L}_2(\mu), \tag{4}$$

fulfilling

$$\forall \lambda, \psi \in \Lambda : \int f_\lambda f_\psi \, \mathrm{d}\mu = \min \left\{ \int f_\lambda^2 \, \mathrm{d}\mu, \int f_\psi^2 \, \mathrm{d}\mu \right\}. \tag{5}$$

Let us start with two examples of indexed real functions fulfilling (5).

Example 1. Let $f \in \mathsf{L}_2(\mu)$, $\Lambda \subset \mathbb{R}$ and $A_\lambda \in \mathscr{E}$ for each $\lambda \in \Lambda$. If $A_\lambda \subset A_\psi$ whenever $\lambda, \psi \in \Lambda$, $\lambda \leq \psi$ then the collection of restrictions $(f \mathbb{1}_{A_\lambda}, \lambda \in \Lambda)$ fulfills (5).

Evidently, $f \mathbb{1}_{A_\lambda} \in \mathsf{L}_2(\mu)$ for each $\lambda \in \Lambda$. The property (5) can be also easily checked since for $\lambda, \psi \in \Lambda$, $\lambda \leq \psi$ we are receiving

$$\int f \mathbb{1}_{A_\psi} f \mathbb{1}_{A_\lambda} \, \mathrm{d}\mu = \int f^2 \mathbb{1}_{A_\lambda} \, \mathrm{d}\mu \leq \int f^2 \mathbb{1}_{A_\psi} \, \mathrm{d}\mu. \qquad \triangle$$

Example 2. Let μ be a probability measure, $f \in \mathsf{L}_2(\mu)$, $\Lambda \subset \mathbb{R}$ and $\mathscr{A}_\lambda \subset \mathscr{E}$ be a σ-algebra for each $\lambda \in \Lambda$. Further let $\mathscr{A}_\lambda \subset \mathscr{A}_\psi$ whenever $\lambda, \psi \in \Lambda$, $\lambda \leq \psi$.

Hence, the collection of conditional mean $(\mathsf{E}[f | \mathscr{A}_\lambda], \lambda \in \Lambda)$ fulfills (5).

It is known that $\mathsf{E}[f | \mathscr{A}_\lambda] \in \mathsf{L}_2(\mu)$ whenever $f \in \mathsf{L}_2(\mu)$.

The condition (5) follows properties of the conditional mean, especially Jensen inequality. Taking $\lambda, \psi \in \Lambda$, $\lambda \leq \psi$ we are receiving

$$\int \mathsf{E}[f|\mathscr{A}_\psi]\,\mathsf{E}[f|\mathscr{A}_\lambda]\,\mathrm{d}\mu = \int (\mathsf{E}[f|\mathscr{A}_\lambda])^2\,\mathrm{d}\mu \le \int (\mathsf{E}[f|\mathscr{A}_\psi])^2\,\mathrm{d}\mu. \qquad \triangle$$

2. A solution

We start the section with observations allowing a simplification of the problem.

Lemma 1. *Let a collection of indexed real functions (4) fulfills (5). Then*

1. *For* $\lambda \in \Lambda$, $\int f_\lambda^2\,\mathrm{d}\mu = 0$ *implies* $\mu(f_\lambda \neq 0) = 0$.
2. *For* $\lambda, \psi \in \Lambda$, $\int f_\lambda^2\,\mathrm{d}\mu = \int f_\psi^2\,\mathrm{d}\mu$ *implies* $\mu(f_\lambda \neq f_\psi) = 0$.
3. *For a net* $\lambda_\iota \in \Lambda$, $\iota \in I$
   f_{λ_ι}, $\iota \in I$ *is convergent in* $\mathsf{L}_2(\mu)$ *iff* $\int f_{\lambda_\iota}^2\,\mathrm{d}\mu$, $\iota \in I$ *is convergent.*
4. *If* $\lambda, \psi, \varphi \in \Lambda$, $\int f_\lambda^2\,\mathrm{d}\mu < \int f_\psi^2\,\mathrm{d}\mu < \int f_\varphi^2\,\mathrm{d}\mu$ *then* $f_\varphi - f_\psi$, f_λ *are orthogonal in* $\mathsf{L}_2(\mu)$.

Proof. The first statement is evident. The other statements need short proofs.
(a) For $\lambda, \psi \in \Lambda$, we have

$$\int (f_\lambda - f_\psi)^2\,\mathrm{d}\mu = \int f_\lambda^2\,\mathrm{d}\mu - 2\int f_\lambda f_\psi\,\mathrm{d}\mu + \int f_\psi^2\,\mathrm{d}\mu$$

$$= \left|\int f_\lambda^2\,\mathrm{d}\mu - \int f_\psi^2\,\mathrm{d}\mu\right|, \quad \text{accordingly to (5).}$$

Hence, the property 2 is evident and, clearly, a convergence of indexed real functions in $\mathsf{L}_2(\mu)$ is equivalent with convergence of their second powers integrals w.r.t. to μ.
(b) Let $\lambda, \psi, \varphi \in \Lambda$, $\int f_\lambda^2\,\mathrm{d}\mu < \int f_\psi^2\,\mathrm{d}\mu < \int f_\varphi^2$ then

$$\int (f_\varphi - f_\psi)\,f_\lambda\,\mathrm{d}\mu = \int f_\lambda^2\,\mathrm{d}\mu - \int f_\lambda^2\,\mathrm{d}\mu = 0 \quad \text{according to (5).}$$

Q.E.D.

The solution we want to present is based on an integration w.r.t. a process with orthogonal increments.

Definition 2. *A mapping* $U : [0, +\infty) \to \mathsf{L}_2(\mu) : t \mapsto U_t$ *being right-continuous in* $\mathsf{L}_2(\mu)$ *and having* $U_v - U_s$, U_t *orthogonal in* $\mathsf{L}_2(\mu)$ *whenever* $0 \le t < s < v$ *will be called on o.i.-process in* $\mathsf{L}_2(\mu)$.
The process possesses the reference function defined by

$$F_U : [0, +\infty) \to [0, +\infty) : t \to \int U_t^2\,\mathrm{d}\mu. \tag{6}$$

(The abbreviation "o.i.-process" stands for "process with orthogonal increments".)

Lemma 3. *The reference function of an o.i.-process is always a non-decreasing non-negative right-continuous function.*

Proof. Non-negativity is evident. The reference function is non-decreasing since for each $0 \leq t < s$.

$$F_U(s) = \int U_s^2 \,\mathrm{d}\mu = \int U_t^2 \,\mathrm{d}\mu + 2\int (U_s - U_t)\, U_t \,\mathrm{d}\mu + \int (U_s - U_t)^2 \,\mathrm{d}\mu$$

$$= F_U(t) + \int (U_s - U_t)^2 \,\mathrm{d}\mu .$$

Right-continuity follows the same equality and the fact that the o.i.-process is right-continuous in $\mathsf{L}_2(\mu)$ by definition.

Q.E.D.

Hence, we can employ Lebesgue–Stieltjes integral w.r.t. F_U and integrate w.r.t. an o.i.-process U.

Proposition 4. *Let U be an o.i.-process in $\mathsf{L}_2(\mu)$. Then an integral w.r.t. U can be defined such that*

1. *The integral is defined for all functions from $\mathsf{L}_2(F_U)$ and its values are in $\mathsf{L}_2(\mu)$.*
2. *$\forall f, g \in \mathsf{L}_2(F_U), a, b \in \mathbb{R}$: $\int af + bg \,\mathrm{d}U = a \int f \,\mathrm{d}U + b \int g \,\mathrm{d}U \quad \mu-a.e.$*
3. *$\forall t, s \in [0, +\infty)$: $\int \mathbb{1}_{(t,s]} \,\mathrm{d}U = U_s - U_t \quad \mu-a.e.$*
4. *$\forall f, g \in \mathsf{L}_2(F_U)$: $\int (\int f \,\mathrm{d}U)(\int g \,\mathrm{d}U) \,\mathrm{d}\mu = \int fg \,\mathrm{d}F_U$.*

The integral is correctly defined and its values are modulo μ uniquely determined.

A proof for a finite (probability) measure μ is given in [2], Chap. 2, § 3. The same arguments are also valid for an arbitrary measure. The crucial point of the proof, i.e. 12. lemma in [2], concludes the proof for an arbitrary measure μ, too, since $\mathsf{L}_2(\mu)$ is always a Banach space.

Now, we formulate a solution of the considered problem.

Theorem 1. *Indexed functions (4) fulfill (5) iff there are an o.i.-process U in $\mathsf{L}_2(\mu)$, a function $h : [0, +\infty) \to \mathbb{R}$ and a collection of sets $A_\lambda \subset (0, +\infty)$, $\lambda \in \Lambda$ such that*

$$h \mathbb{1}_{A_\lambda} \in \mathsf{L}_2(F_U) \ \forall \lambda \in \Lambda , \tag{7}$$

$$A_\lambda \subset A_\psi \quad \textit{whenever} \quad \int f_\lambda^2 \,\mathrm{d}\mu \leq \int f_\psi^2 \,\mathrm{d}\mu \ \forall \lambda, \psi \in \Lambda , \tag{8}$$

$$f_\lambda = \int h \mathbb{1}_{A_\lambda} \,\mathrm{d}U \quad \mu-a.e. \quad \forall \lambda \in \Lambda . \tag{9}$$

Proof.

1. Let (9) be fulfilled, $\lambda, \psi \in \Lambda$ and $\int f_\lambda^2 \,\mathrm{d}\mu \leq \int f_\psi^2 \,\mathrm{d}\mu$. Hence,

$$\int f_\lambda f_\psi \, d\mu = \int\left(\int h\mathbb{1}_{A_\lambda} \, dU\right)\left(\int h\mathbb{1}_{A_\psi} \, dU\right) d\mu = \int h\mathbb{1}_{A_\lambda} h\mathbb{1}_{A_\psi} \, dF_U = \int h^2\mathbb{1}_{A_\lambda} \, dF_U$$

$$= \int\left(\int h\mathbb{1}_{A_\lambda} \, dU\right)^2 d\mu = \int f_\lambda^2 \, d\mu .$$

Thus, (5) is satisfied.

2. Let (5) be fulfilled.

Accordingly to Lemma 1, without any loss of generality we may suppose a closed set $\Lambda \subset [0, +\infty)$, $0 \in \Lambda$ and $\int f_\lambda^2 \, d\mu = \lambda$ for all $\lambda \in \Lambda$.

Then, we define $U : [0, +\infty) \to \mathsf{L}_2(\mu)$ by the formula

$$\forall t \in [0, +\infty) \;\; U_t = f_{\lambda_t} \quad \text{where} \quad \lambda_t = \max\{\lambda \in \Lambda : \lambda \leq t\}. \tag{10}$$

Accordingly to Lemma 1, U is an o.i.-process and, evidently,

$$f_\lambda = U_\lambda = U_\lambda - U_0 = \int \mathbb{1}_{(0,\lambda]} \, dU \quad \text{whenever} \quad \lambda \in \Lambda .$$

Hence, we set $h \equiv 1$ and $A_\lambda = (0, \lambda]$ for all $\lambda \in \Lambda$ to show (9).

Q.E.D.

3. A solution for the original problem

The previous section gives a solution of the original task when (1) is fulfilling (2).

Theorem 2. *Indexed functions (1) fulfill (2) with V being a Wiener process iff there are an o.i.-process U in $\mathsf{L}_2(\mathfrak{m})$, a function $h : [0, +\infty) \to \mathbb{R}$ and a collection of sets $A_\lambda \subset (0, +\infty)$, $\lambda \in \Lambda$ such that*

$$h\mathbb{1}_{A_\lambda} \in \mathsf{L}_2(F_U) \; \forall \lambda \in \Lambda , \tag{11}$$

$$\xi(\lambda) = \int h^2\mathbb{1}_{A_\lambda} \, dF_U \; \forall \lambda \in \Lambda , \tag{12}$$

$$A_\lambda \subset A_\psi \quad \textit{whenever} \quad \xi(\lambda) \leq \xi(\psi) \; \forall \lambda, \psi \in \Lambda , \tag{13}$$

$$a_\lambda = \int h\mathbb{1}_{A_\lambda} \, dU \quad \mathfrak{m}-a.e. \quad \forall \lambda \in \Lambda . \tag{14}$$

Proof. The theorem is a particular case of Theorem 1 with $\mu = \mathfrak{m}$.

Q.E.D

Theorem 3. *Indexed functions (1) fulfill (2) with V being a Brownian bridge iff there are an o.i.-process U in $\mathsf{L}_2(\mathfrak{m})$, a function $h : [0, +\infty) \to \mathbb{R}$ and a collection of sets $A_\lambda \subset (0, +\infty)$, $\lambda \in \Lambda$ such that*

$$h\mathbb{1}_{A_\lambda} \in \mathsf{L}_2(F_U) \;\; \forall \lambda \in \Lambda , \tag{15}$$

$$\xi(\lambda) = \int h^2 \mathbb{1}_{A_\lambda} \, \mathrm{d}F_U \leq 1 \ \forall \lambda \in \Lambda, \tag{16}$$

$$A_\lambda \subset A_\psi \quad \textit{whenever} \quad \xi(\lambda) \leq \xi(\psi) \ \forall \lambda, \psi \in \Lambda, \tag{17}$$

$$a_\lambda = \int h(\mathbb{1}_{A_\lambda} - \xi(\lambda)\, \mathbb{1}_{A_1}) \, \mathrm{d}U \quad \mathfrak{m}-a.e. \quad \forall \lambda \in \Lambda. \tag{18}$$

Proof. Theorem follows immediately Theorem 2 because the transformation $(W(t) - tW(1),\ t \in [0, 1])$ transforms a Wiener process to a Brownian bridge and the transformation $(B(t) + tN,\ t \in [0, 1])$, where N is a standard Gaussian r.v. independent with B, reverses a Brownian bridge to a Wiener process.

Q.E.D.

Theorem 4. *Indexed functions (1) fulfill (2) with V being an Ornstein–Uhlenbeck process iff there are an o.i.-process U in $\mathsf{L}_2(\mathfrak{m})$, a function $h : [0, +\infty) \to \mathbb{R}$ and a collection of sets $A_\lambda \subset (0, +\infty)$, $\lambda \in \Lambda$ such that*

$$h \mathbb{1}_{A_\lambda} \in \mathsf{L}_2(F_U) \ \forall \lambda \in \Lambda, \tag{19}$$

$$\xi(\lambda) = \frac{1}{2} \log \left(\int h^2 \mathbb{1}_{A_\lambda} \, \mathrm{d}F_U \right) \forall \lambda \in \Lambda, \tag{20}$$

$$A_\lambda \subset A_\psi \quad \textit{whenever} \quad \xi(\lambda) \leq \xi(\psi) \ \forall \lambda, \psi \in \Lambda, \tag{21}$$

$$a_\lambda = \mathrm{e}^{-\xi(\lambda)} \int h \mathbb{1}_{A_\lambda} \, \mathrm{d}U \quad \mathfrak{m}-a.e. \quad \forall \lambda \in \Lambda. \tag{22}$$

Proof. Theorem follows immediately Theorem 2 because the transformation $(\mathrm{e}^{-t} W(\mathrm{e}^{2t}),\ t \in \mathbb{R})$ transforms a Wiener process to an Ornstein–Uhlenbeck process and the transformation $(\sqrt{t} U(\frac{1}{2} \log(t)),\ t \in \mathbb{R}_+)$ alters an Ornstein–Uhlenbeck process to a Wiener process.

Q.E.D.

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