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BOUNDS FOR THE (LAPLACIAN) SPECTRAL RADIUS OF GRAPHS WITH PARAMETER α

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Abstract. Let G be a simple connected graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Denote $({}^{\alpha}t)_i = \sum_{j\colon i\sim j} d_j^{\alpha}$, $({}^{\alpha}m)_i = ({}^{\alpha}t)_i/d_i^{\alpha}$ and $({}^{\alpha}N)_i = \sum_{j\colon i\sim j} ({}^{\alpha}t)_j$, where α is a real number. Denote by $\lambda_1(G)$ and $\mu_1(G)$ the spectral radius of the adjacency matrix and the Laplacian matrix of G, respectively. In this paper, we present some upper and lower bounds of $\lambda_1(G)$ and $\mu_1(G)$ in terms of $({}^{\alpha}t)_i$, $({}^{\alpha}m)_i$ and $({}^{\alpha}N)_i$. Furthermore, we also characterize some extreme graphs which attain these upper bounds. These results theoretically improve and generalize some known results.

Keywords: graph, adjacency matrix, Laplacian matrix, spectral radius, bound

MSC 2010: 05C50, 15A18

1. Introduction

We only consider simple undirected graphs which have no loops and multiple edges. Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. For any two vertices $v_i, v_j \in V$, we write $i \sim j$ if v_i and v_j are adjacent. For any vertex $v_i \in V$, denote the *degree* of v_i by d_i; and denote by t_i the 2-*degree* of v_i, which is the sum of the degrees of the vertices adjacent to v_i; and denote by m_i the *average degree* of v_i, which is t_i/d_i. Furthermore, denote by N_i the sum of the 2-degrees of vertices adjacent to v_i. In [8], the following notations are introduced:

$$({}^{\alpha}t)_i = \sum_{j\colon i\sim j} d_j^{\alpha}, \quad ({}^{\alpha}m)_i = \frac{({}^{\alpha}t)_i}{d_i^{\alpha}}$$

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where α is a real number. For convenience, $({}^{\alpha}t)_i$ and $({}^{\alpha}m)_i$ are called the *generalized 2-degree* and the *generalized average degree* of v_i, respectively. Here we define

$$({}^{\alpha}N)_i = \sum_{j\,:\,i\sim j} ({}^{\alpha}t)_j.$$

Note that $d_i = ({}^{0}t)_i = ({}^{0}m)_i$, $t_i = ({}^{1}t)_i = ({}^{0}N)_i$, $m_i = ({}^{1}m)_i$ and $N_i = ({}^{1}N)_i$. In addition, for a particular value α, a graph is called *generalized pseudo-regular* if all vertices have the same generalized average degrees. A bipartite graph is called *generalized pseudo-semiregular* if all vertices in the same part of a bipartition have the same generalized average degrees. Clearly, if $\alpha = 1$, then generalized pseudo-regular graph and generalized pseudo-semiregular graph are the usual pseudo-regular graph and pseudo-semiregular graph (see, for example, [7], [13]).

The following definitions come from [3]. Let $A(G) = (a_{ij})$ be the adjacency matrix of G and $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be the degree diagonal matrix. Then $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. Clearly, $A(G)$ and $L(G)$ are real symmetric matrices. Hence their eigenvalues are real numbers. The eigenvalues of $A(G)$ are called the *eigenvalues* of G and denoted by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_n(G)$; and the eigenvalues of $L(G)$ are called the *Laplacian eigenvalues* of G and denoted by $\mu_1(G) \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G) = 0$. In particular, $\lambda_1(G)$ and $\mu_1(G)$ are called the *spectral radius* of G and the *Laplacian spectral radius* of G, respectively.

Up to now, the spectral radius $\lambda_1(G)$ and the Laplacian spectral radius $\mu_1(G)$ of G have been extensively investigated for a long time (see, for example, [1], [2], [4], [5], [6], [7], [8], [11], [12], [13] and the references therein). Recently, Liu and Lu [8] introduced two new notations $({}^{\alpha}t)_i$ and $({}^{\alpha}m)_i$ and obtained some new bounds for the Laplacian spectral radius $\mu_1(G)$ of G. Motivated by this technique, we present some new bounds of $\lambda_1(G)$ and $\mu_1(G)$ with parameter α and characterize some extreme graphs which attain these upper bounds. These results theoretically improve and generalize some known results. Hence they are worthy of being retained in terms of precedence (that is, for a given set of graphs, how often does the bound yield the best value among a given set of bounds, more information see [2]).

2. The spectral radius of graphs

For a simple connected graph G, this section shall present some upper and lower bounds on the spectral radius of G, which improve and generalize some known results.

Theorem 2.1. *Let $G = (V, E)$ be a simple connected graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Then for any real number α,*

$$\lambda_1(G) \leqslant \max_{i \sim j} \sqrt{({}^{\alpha}m)_i ({}^{\alpha}m)_j}. \tag{2.1}$$

Moreover, the equality in (2.1) holds for some particular value α if and only if G is either a generalized pseudo-regular graph or a generalized pseudo-semiregular graph.

P r o o f. Let $\tilde{D} = \operatorname{diag}\{d_1^{\alpha}, d_2^{\alpha}, \ldots, d_n^{\alpha}\}$. Then $\lambda_1(G) = \lambda_1(\tilde{D}^{-1}A(G)\tilde{D})$. Now the (i,j)th element of $\tilde{D}^{-1}A(G)\tilde{D}$ is

$$\begin{cases} \dfrac{d_j^{\alpha}}{d_i^{\alpha}} & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, \ldots, x_n)^{\top}$ be a positive eigenvector of $\tilde{D}^{-1}A(G)\tilde{D}$ corresponding to the eigenvalue $\lambda_1(\tilde{D}^{-1}A(G)\tilde{D})$, where x_i corresponds to the vertex v_i. Let $x_s = \max\limits_{1 \leqslant i \leqslant n}\{x_i\}$ and $x_t = \max\limits_{i\colon i \sim s}\{x_i\}$. From $\tilde{D}^{-1}A(G)\tilde{D}x = \lambda_1(\tilde{D}^{-1}A(G)\tilde{D})x = \lambda_1(G)x$, we get

$$\lambda_1(G)x_s = \sum_{k\colon k \sim s} \frac{d_k^{\alpha}}{d_s^{\alpha}} x_k \leqslant ({}^{\alpha}m)_s x_t \tag{2.2}$$

and

$$\lambda_1(G)x_t = \sum_{k\colon k \sim t} \frac{d_k^{\alpha}}{d_t^{\alpha}} x_k \leqslant ({}^{\alpha}m)_t x_s. \tag{2.3}$$

Hence, from (2.2) and (2.3), one has

$$(\lambda_1(G))^2 \leqslant ({}^{\alpha}m)_s ({}^{\alpha}m)_t,$$

which implies that the inequality (2.1) holds.

Now assume that the equality in (2.1) holds for some particular value α, then the above equalities in both (2.2) and (2.3) hold. Hence for any vertex $v_i \in V$ satisfying $i \sim s$, $x_i = x_t$ and for any vertex v_i satisfying $i \sim t$, $x_i = x_s$. Since G is connected, by repeated using the equalities in both (2.2) and (2.3), it is easy to see that for any $v_i \in V$, $x_i = x_s$ or x_t when the distance between the vertices v_i and v_s is even or odd, respectively. If $x_s = x_t$, then x is a constant vector. It follows from $\tilde{D}^{-1}A(G)\tilde{D}x = \lambda_1(G)x$ that each vertex v_i of G has equal generalized average degree $({}^{\alpha}m)_i$, that is, G is a generalized pseudo-regular graph.

If $x_s > x_t$, let $V_1 = \{v_i\colon x_i = x_s\}$ and $V_2 = \{v_i\colon x_i = x_t\}$. Thus $V = V_1 \cup V_2$ and the subgraphs induced by V_1 and V_2 respectively are empty graphs. Hence G is bipartite. It follows from $\tilde{D}^{-1}A(G)\tilde{D}x = \lambda_1(G)x$ that, for any $v_k, v_l \in V_1$, $\lambda_1(G)x_k = ({}^{\alpha}m)_k x_t$ and $\lambda_1(G)x_l = ({}^{\alpha}m)_l x_t$, which implies that $({}^{\alpha}m)_k = ({}^{\alpha}m)_l$. Similarly, we have $({}^{\alpha}m)_i = \tau$ for any $v_i \in V_2$, where τ is a positive constant. Hence G is a generalized pseudo-semiregular graph.

Conversely, if G is a generalized pseudo-regular graph with equal generalized average degree $({}^{\alpha}m)_i = \tau$ for each vertex $v_i \in V$, where τ is a constant, then $\tilde{D}^{-1}A(G)\tilde{D}e_n = \tau e_n$, where e_n is the column vector of all ones. By the Perron-Frobenius Theorem, one has $\lambda_1(G) = \tau$, which implies that the equality in (2.1) holds for some particular value α. Now suppose that G is a generalized pseudo-semiregular graph, that is, G is bipartite and there exists a partition V_1, V_2 of V such that each vertex $v_i \in V_1$ has equal generalized average degree $({}^{\alpha}m)_i = \tau_1$ and each vertex $v_i \in V_2$ has equal generalized average degree $({}^{\alpha}m)_i = \tau_2$, where τ_1, τ_2 are two positive constants. Without loss of generality, we may assume that

$$\tilde{D}^{-1}A(G)\tilde{D} = \begin{pmatrix} 0_{n_1\times n_1} & B_{n_1\times n_2} \\ C_{n_2\times n_1} & 0_{n_2\times n_2} \end{pmatrix},$$

where $0_{n_1\times n_1}$ is an $n_1 \times n_1$ matrix with all entries zeros and $|V_1| = n_1$, $|V_2| = n_2$. Note that the row sums of $B_{n_1\times n_2}$ and $C_{n_2\times n_1}$ are τ_1 and τ_2, respectively. Let $x = (\sqrt{\tau_1}e_{n_1}^{\top}, \sqrt{\tau_2}e_{n_2}^{\top})^{\top}$. Then $\tilde{D}^{-1}A(G)\tilde{D}x = \sqrt{\tau_1\tau_2}x$, which implies that $\lambda_1(G) = \sqrt{\tau_1\tau_2}$. Hence the equality in (2.1) holds for some particular value α. This completes the proof. $\square$

Remark 2.1. If $\alpha = 0$, then the inequality (2.1) becomes

$$\lambda_1(G) \leqslant \max_{i\sim j} \sqrt{d_i d_j},$$

which is Berman and Zhang's bound [1]. If $\alpha = 1$, then the inequality (2.1) becomes

$$\lambda_1(G) \leqslant \max_{i\sim j} \sqrt{m_i m_j},$$

which is Das and Kumar's bound [4]. Hence the inequality (2.1) improves and generalizes some results in [1], [4].

Corollary 2.1. *Let $G = (V, E)$ be a simple connected graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Then for any real number α,*

$$\lambda_1(G) \leqslant \max_{1\leqslant i\leqslant n} ({}^{\alpha}m)_i. \tag{2.4}$$

Moreover, the equality in (2.4) holds for some particular value α if and only if G is a generalized pseudo-regular graph.

Proof. From the proof of Theorem 2.1, we easily get the required result. $\square$

Remark 2.2. If $\alpha = 1$, then the inequality (2.4) becomes

$$\lambda_1(G) \leqslant \max_{1 \leqslant i \leqslant n} m_i,$$

which is the bound by Favaron et al. [5].

In the following, we shall present a new lower bound for the spectral radius of graphs with parameter α.

Theorem 2.2. *Let $G = (V, E)$ be a simple connected graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

$$\lambda_1(G) \geqslant \max_{\alpha} \sqrt{\frac{\sum_{i=1}^{n} ({}^{\alpha}N)_i^2}{\sum_{i=1}^{n} ({}^{\alpha}t)_i^2}}, \tag{2.5}$$

where α ranges over all real numbers. Moreover, the equality in (2.5) *holds for some particular value α if and only if $({}^{\alpha}N)_i/({}^{\alpha}t)_i$ is the same for each vertex $v_i \in V$ or G is a bipartite graph with a bipartition $V = V_1 \cup V_2$ such that $({}^{\alpha}N)_i/({}^{\alpha}t)_i$ is the same for each vertex $v_i \in V_1$ and similarly for V_2.*

Proof. Let $x = (x_1, x_2, \ldots, x_n)^{\top}$ be a positive eigenvector of $A(G)$ corresponding to the eigenvalue $\lambda_1(G)$, where x_i corresponds to the vertex v_i. By the Raleigh principle, we get

$$(\lambda_1(G))^2 = \lambda_1(A(G)^2) = \frac{x^{\top}(A(G))^2 x}{x^{\top} x}. \tag{2.6}$$

Take $C = (({}^{\alpha}t)_1, ({}^{\alpha}t)_2, \ldots, ({}^{\alpha}t)_n)^{\top}$. Then

$$\begin{aligned} A(G)C &= \Bigl(\sum_{j=1}^{n} a_{1j}({}^{\alpha}t)_j, \sum_{j=1}^{n} a_{2j}({}^{\alpha}t)_j, \ldots, \sum_{j=1}^{n} a_{nj}({}^{\alpha}t)_j\Bigr)^{\top} \\ &= \bigl(({}^{\alpha}N)_1, ({}^{\alpha}N)_2, \ldots, ({}^{\alpha}N)_n\bigr)^{\top} \end{aligned}$$

and $C^{\top}C = \sum\limits_{i=1}^{n} ({}^{\alpha}t)_i^2$. It follows from (2.6) that

$$\lambda_1(G) = \sqrt{\frac{x^{\top}(A(G))^2 x}{x^{\top} x}} \geqslant \sqrt{\frac{C^{\top}(A(G))^2 C}{C^{\top} C}} = \sqrt{\frac{\sum_{i=1}^{n} ({}^{\alpha}N)_i^2}{\sum_{i=1}^{n} ({}^{\alpha}t)_i^2}},$$

which implies that the inequality (2.5) holds.

Now suppose that the equality in (2.5) holds for some particular value α. Note that the following proof is similar to that of Theorem 3.1 in [7]. Then C is an eigenvector of $A(G)^2$ corresponding to $\lambda_1(A(G)^2)$, which implies that the multiplicity of $\lambda_1(A(G)^2)$ is either one or two. If the multiplicity of $\lambda_1(A(G)^2)$ is one, then C is an eigenvector of $A(G)$ corresponding to $\lambda_1(G)$, that is, $A(G)C = \lambda_1(G)C$. This implies $({}^{\alpha}N)_i/({}^{\alpha}t)_i$ is the same for each vertex $v_i \in V$.

If the multiplicity of $\lambda_1(A(G)^2)$ is two, then $-\lambda_1(G)$ is also an eigenvalue of G. This implies that G is bipartite (see [3]). Without loss of generality, we may assume that

$$A(G) = \begin{pmatrix} 0_{n_1\times n_1} & B_{n_1\times n_2} \\ B^{\top}_{n_2\times n_1} & 0_{n_2\times n_2} \end{pmatrix} \tag{2.7}$$

where $V = V_1 \cup V_2$, $n_1 = |V_1|$, $n_2 = |V_2|$ with $n_1 + n_2 = n$. From $A(G)^2C = \lambda_1(A(G)^2)C$, we have

$$BB^{\top}C_1 = \lambda_1(A(G)^2)C_1, \quad B^{\top}BC_2 = \lambda_1(A(G)^2)C_2, \tag{2.8}$$

where $C_1 = (({}^{\alpha}t)_1, \dots, ({}^{\alpha}t)_{n_1})^{\top}$ and $C_2 = (({}^{\alpha}t)_{n_1+1}, \dots, ({}^{\alpha}t)_n)^{\top}$. Since $BB^{\top}$ and $B^{\top}B$ have the same nonzero eigenvalues, $\lambda_1(A(G)^2)$ is the spectral radius of both $BB^{\top}$ and $B^{\top}B$ with multiplicity one. From (2.8), one has

$$BB^{\top}BC_2 = \lambda_1(A(G)^2)BC_2, \quad B^{\top}BB^{\top}C_1 = \lambda_1(A(G)^2)B^{\top}C_1.$$

Thus BC_2 and $B^{\top}C_1$ are eigenvectors of $BB^{\top}$ and $B^{\top}B$ respectively corresponding to $\lambda_1(A(G)^2)$. Hence $BC_2 = \tau_1C_1$ and $B^{\top}C_1 = \tau_2C_2$, where τ_1, τ_2 are positive constants. These imply that $({}^{\alpha}N)_i/({}^{\alpha}t)_i = \tau_1$ for any vertex $v_i \in V_1$ and $({}^{\alpha}N)_i/({}^{\alpha}t)_i = \tau_2$ for any vertex $v_i \in V_2$.

Conversely, if $({}^{\alpha}N)_i/({}^{\alpha}t)_i = \tau$ is the same for each vertex $v_i \in V$, then $A(G)C = \tau C$. Since C is a positive vector, then

$$\lambda_1(G) = \tau = \sqrt{\frac{\sum_{i=1}^{n} ({}^{\alpha}N)_i^2}{\sum_{i=1}^{n} ({}^{\alpha}t)_i^2}}.$$

Now assume that G is a bipartite graph with a bipartition $V = V_1 \cup V_2$ such that $({}^{\alpha}N)_i/({}^{\alpha}t)_i = \tau_1$ is the same for any vertex $v_i \in V_1$ and $({}^{\alpha}N)_i/({}^{\alpha}t)_i = \tau_2$ is the same for any vertex $v_i \in V_2$. Without loss of generality, we may assume that $A(G)$ has the form (2.7). Let $C = (C_1^{\top}, C_2^{\top})^{\top}$, where $C_1 = (({}^{\alpha}t)_1, \dots, ({}^{\alpha}t)_{n_1})^{\top}$ and $C_2 = (({}^{\alpha}t)_{n_1+1}, \dots, ({}^{\alpha}t)_n)^{\top}$. By a simple calculation, the ith element of $BB^{\top}C_1$ is $\tau_1\tau_2({}^{\alpha}t)_i$ for any vertex $v_i \in V_1$ and the jth element of $B^{\top}BC_2$ is $\tau_1\tau_2({}^{\alpha}t)_j$ for any vertex $v_j \in V_2$. Hence $A(G)^2C = \tau_1\tau_2 C$. Since C is a positive vector, then $\lambda_1(A(G)^2) = \tau_1\tau_2 = C^\top A(G)^2 C$. which implies that

$$\lambda_1(G) = \sqrt{\tau_1\tau_2} = \sqrt{\frac{\sum_{i=1}^n ({}^\alpha N)_i^2}{\sum_{i=1}^n ({}^\alpha t)_i^2}}.$$

This completes the proof. □

Remark 2.3. If $\alpha = 0$, then the inequality (2.5) beecomes

$$\lambda_1(G) \geqslant \sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}}, \tag{2.9}$$

which is the bound of Yu et al. [13]. If $\alpha = 1$, then the inequality (2.5) becomes

$$\lambda_1(G) \geqslant \sqrt{\frac{\sum_{i=1}^n N_i^2}{\sum_{i=1}^n t_i^2}}, \tag{2.10}$$

which is Hong and Zhang's bound [7]. Hence the inequality (2.5) improves and generalizes some results in [7], [13].

Corollary 2.2. *Let $G = (V, E)$ be a simple connected graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

$$\lambda_1(G) \geqslant \max_\alpha \sqrt{\frac{\sum_{i=1}^n ({}^\alpha t)_i^2}{\sum_{i=1}^n d_i^{2\alpha}}}, \tag{2.11}$$

where α ranges over all real numbers. Moreover, the equality in (2.11) holds for some particular value α if and only if G is a generalized pseudo-regular graph or a generalized pseudo-semiregular graph.

Proof. By a simple calculation, we have

$$\begin{aligned}\Big(\sum_{i=1}^n ({}^\alpha t)_i^2\Big)^2 &= \Big(\sum_{i=1}^n \Big(\sum_{j\colon j\sim i} d_j^\alpha\Big)({}^\alpha t)_i\Big)^2 = \Big(\sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij} d_j^\alpha\Big)({}^\alpha t)_i\Big)^2 \\ &= \Big(\sum_{i=1}^n \sum_{j=1}^n a_{ij} d_j^\alpha ({}^\alpha t)_i\Big)^2 = \Big(\sum_{j=1}^n \Big(\sum_{i=1}^n a_{ij}({}^\alpha t)_i\Big) d_j^\alpha\Big)^2 \\ &= \Big(\sum_{j=1}^n ({}^\alpha N)_j \cdot d_j^\alpha\Big)^2.\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n} ({}^{\alpha}t)_i^2\right)^2 \leqslant \sum_{i=1}^{n} ({}^{\alpha}N)_i^2 \cdot \sum_{i=1}^{n} d_i^{2\alpha} \tag{2.12}$$

with equality if and only if there exists a positive constant l such that $({}^{\alpha}N)_i/d_i^{\alpha} = l$ for each $v_i \in V$. Following from (2.12) and Theorem 2.2, the inequality (2.11) holds.

By Theorem 2.2, the equality in (2.5) holds for some particular value α if and only if $({}^{\alpha}N)_i/({}^{\alpha}t)_i = t$ is the same for each vertex $v_i \in V$ or G is a bipartite graph with a bipartition $V = V_1 \cup V_2$ such that $({}^{\alpha}N)_i/({}^{\alpha}t)_i = t_1$ is the same for each vertex $v_i \in V_1$ and $({}^{\alpha}N)_i/({}^{\alpha}t)_i = t_2$ is the same for each vertex $v_i \in V_2$. Hence the equality in (2.11) holds for some particular value α if and only if $({}^{\alpha}m)_i = ({}^{\alpha}t)_i/d_i^{\alpha} = l/t \triangleq \tau$ is the same for each vertex $v_i \in V$ or G is a bipartite graph with a bipartition $V = V_1 \cup V_2$ such that $({}^{\alpha}m)_i = ({}^{\alpha}t)_i/d_i^{\alpha} = l/t_1 \triangleq \tau_1$ is the same for each vertex $v_i \in V_1$ and $({}^{\alpha}m)_i = ({}^{\alpha}t)_i/d_i^{\alpha} = l/t_2 \triangleq \tau_2$ is the same for each vertex $v_i \in V_2$, that is, G is a generalized pseudo-regular graph or a generalized pseudo-semiregular graph. □

Remark 2.4. If $\alpha = 0$, then the inequality (2.11) becomes

$$\lambda_1(G) \geqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}, \tag{2.13}$$

which is Hofmeister's bound [6] (see also [7], [13]). If $\alpha = 1/2$, then the inequality (2.11) becomes

$$\lambda_1(G) \geqslant \sqrt{\frac{\sum_{i=1}^{n}\left(\sum_{j\,:\,j\sim i}\sqrt{d_j}\right)^2}{2m}}, \tag{2.14}$$

which is Shi's bound [11]. If $\alpha = 1$, then the inequality (2.11) is the bound (2.9) in Remark 2.3. Hence the inequality (2.11) improves and generalizes some known results in [6], [7], [11], [13].

Example 2.1. Let G be the graph shown in Figure 1. Values of $\lambda_1(G)$ and of the various lower bounds for $\lambda_1(G)$ are given (to four decimal places) in Table 1. Taking $\alpha \in [-10, 10]$ in both (2.5) and (2.11), the lower bounds of $\lambda_1(G)$ given in Figure 2 also show that (2.5) is never worse than (2.11).

$\lambda_1(G)$	$\alpha = 0.7$ in (2.5)	(2.9)	(2.10)	(2.13)	(2.14)
2.9032	2.9023	2.8868	2.8983	2.8284	2.8859

Table 1. Values of $\lambda_1(G)$ and of the various lower bounds for $\lambda_1(G)$.

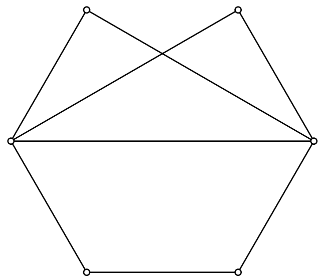

Figure 1. The graph G in Example 2.1.

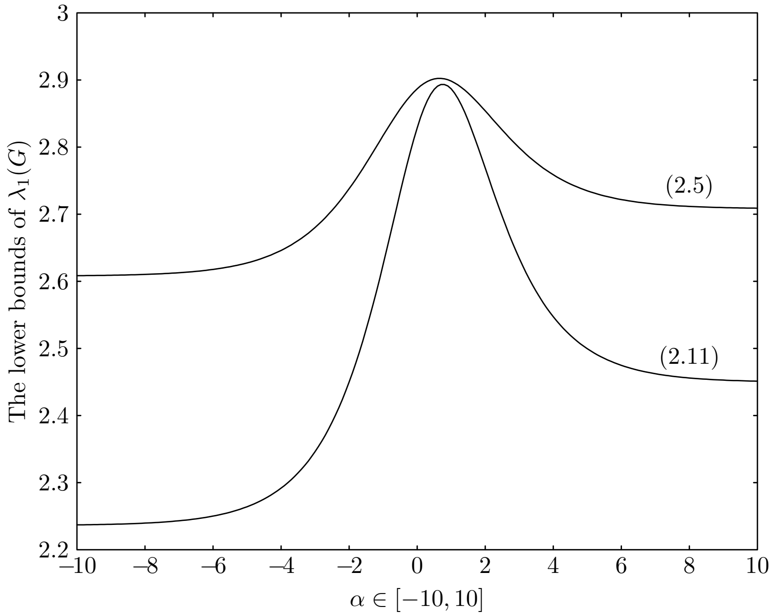

Figure 2. The comparison of the inequalities (2.5) and (2.11).

Remark 2.5. As pointed out in Remark 1 at the end of [10], every lower bound on $\lambda_1(G)$ gives a corresponding upper bound on the energy of G (the *energy* of G is defined as the sum of the absolute values of the eigenvalues of G). Thus, using a similar technique as in the proof of Theorem 2.5 in [9], together with Theorem 2.2 and Corollary 2.2, we may get some new upper bounds on the energy of G, which theoretically improve some results obtained in [9], [14], [15]. These results are omitted here.

3. The Laplacian spectral radius of bipartite graphs

In this section, we shall give a new lower bound for the Laplacian spectral radius of bipartite graphs. Some known bounds are shown to be the consequences of our bounds.

Theorem 3.1. *Let G be a simple connected bipartite graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

$$\mu_1(G) \geqslant \max_{\alpha} \sqrt{\frac{\sum_{i=1}^{n} \left(d_i(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{j\colon j\sim i} (d_j^{\alpha+1} + ({}^{\alpha}t)_j)\right)^2}{\sum_{i=1}^{n} \left(d_i^{\alpha+1} + ({}^{\alpha}t)_i\right)^2}}, \tag{3.1}$$

where α ranges over all real numbers. Moreover, the equality in (3.1) *holds for some particular value α if and only if there exists a positive constant τ such that, for any vertex $v_i \in V$,*

$$\frac{d_i(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{j\colon j\sim i} (d_j^{\alpha+1} + ({}^{\alpha}t)_j)}{d_i^{\alpha+1} + ({}^{\alpha}t)_i} = \tau.$$

Proof. Since G is a bipartite graph, $L(G) = D(G) - A(G)$ and $D(G) + A(G)$ have the same eigenvalues. Note that $D(G) + A(G)$ is a nonnegative irreducible symmetric matrix.

Suppose that $x = (x_1, x_2, \ldots, x_n)^\top$ is the positive eigenvector of $D(G) + A(G)$ corresponding to $\mu_1(G)$. From the Raleigh principle, one has

$$(\mu_1(G))^2 = (\mu_1(D(G) + A(G)))^2 = \frac{x^\top (D(G) + A(G))^2 x}{x^\top x}. \tag{3.2}$$

Take $C = (d_1^{\alpha+1} + ({}^{\alpha}t)_1, d_2^{\alpha+1} + ({}^{\alpha}t)_2, \ldots, d_n^{\alpha+1} + ({}^{\alpha}t)_n)^\top$. Then

$$\begin{aligned}
(D(G) + A(G))C &= D(G)C + A(G)C \\
&= \begin{pmatrix} d_1(d_1^{\alpha+1} + ({}^{\alpha}t)_1) + \sum\limits_{j=1}^{n} a_{1j}(d_j^{\alpha+1} + ({}^{\alpha}t)_j) \\ \vdots \\ d_n(d_n^{\alpha+1} + ({}^{\alpha}t)_n) + \sum\limits_{j=1}^{n} a_{nj}(d_j^{\alpha+1} + ({}^{\alpha}t)_j) \end{pmatrix} \\
&= \begin{pmatrix} d_1(d_1^{\alpha+1} + ({}^{\alpha}t)_1) + \sum\limits_{j\colon j\sim 1} (d_j^{\alpha+1} + ({}^{\alpha}t)_j) \\ \vdots \\ d_n(d_n^{\alpha+1} + ({}^{\alpha}t)_n) + \sum\limits_{j\colon j\sim n} (d_j^{\alpha+1} + ({}^{\alpha}t)_j) \end{pmatrix}.
\end{aligned}$$

Note that $C^\top C = \sum\limits_{i=1}^{n}(d_i^{\alpha+1} + ({}^\alpha t)_i)^2$. It follows from (3.2) that

$$\begin{aligned}
\mu_1(G) &= \sqrt{\frac{x^\top(D(G)+A(G))^2 x}{x^\top x}} \\
&\geqslant \sqrt{\frac{C^\top(D(G)+A(G))^2 C}{C^\top C}} \\
&= \sqrt{\frac{\sum\limits_{i=1}^{n}(d_i(d_i^{\alpha+1} + ({}^\alpha t)_i) + \sum_{j\colon j\sim i}(d_j^{\alpha+1} + ({}^\alpha t)_j))^2}{\sum_{i=1}^{n}(d_i^{\alpha+1} + ({}^\alpha t)_i)^2}}.
\end{aligned}$$

Hence the inequality (3.1) holds.

Now assume that the equality in (3.1) holds for some particular value α. Then C is the eigenvector of $(D(G)+A(G))^2$ corresponding to $\mu_1((D(G)+A(G))^2)$. Thus the multiplicity of $(\mu_1(G))^2 = \mu_1((D(G)+A(G))^2)$ is either one or two. Since $(D(G)+A(G))^2$ is a nonnegative irreducible positive semidefinite matrix, then the multiplicity of $(\mu_1(G))^2 = \mu_1((D(G)+A(G))^2)$ must be one, and C is the eigenvector of $D(G)+A(G)$ corresponding to $\mu_1(G)$. Hence $(D(G)+A(G))C = \mu_1(G)C$, which implies, for any vertex $v_i \in V$,

$$\frac{d_i(d_i^{\alpha+1} + ({}^\alpha t)_i) + \sum_{j\colon j\sim i}(d_j^{\alpha+1} + ({}^\alpha t)_j)}{d_i^{\alpha+1} + ({}^\alpha t)_i} = \mu_1(G).$$

Conversely, if there exists a positive constant τ such that, for any vertex $v_i \in V$,

$$\frac{d_i(d_i^{\alpha+1} + ({}^\alpha t)_i) + \sum_{j\colon j\sim i}(d_j^{\alpha+1} + ({}^\alpha t)_j)}{d_i^{\alpha+1} + ({}^\alpha t)_i} = \tau,$$

then $(D(G)+A(G))C = \tau C$, where $C = (d_1^{\alpha+1} + ({}^\alpha t)_1, d_2^{\alpha+1} + ({}^\alpha t)_2, \ldots, d_n^{\alpha+1} + ({}^\alpha t)_n)^\top$. It follows from the Perron-Frobenius Theorem that the equality in (2.10) holds for some particular value α. This completes the proof. □

Remark 3.1. If $\alpha = 0$, then the inequality (3.1) becomes

$$\mu_1(G) \geqslant \sqrt{\frac{\sum_{i=1}^{n}(d_i^2 + t_i)^2}{\sum_{i=1}^{n} d_i^2}},$$

which is the bound by Yu et al. [13]. If $\alpha = 1$, then the inequality (3.1) becomes

$$\mu_1(G) \geqslant \sqrt{\frac{\sum_{i=1}^{n}(d_i(d_i^2 + t_i) + \sum_{j\colon j\sim i}(d_j^2 + t_j))^2}{\sum_{i=1}^{n}(d_i^2 + t_i)^2}},$$

which is the bound by Tian et al. [12]. Hence the inequality (3.1) improves and generalizes some results in [12], [13].

Corollary 3.1 ([8]). *Let G be a simple connected bipartite graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

$$\mu_1(G) \geqslant \max_{\alpha} \sqrt{\frac{\sum_{i=1}^{n} (d_i^{\alpha+1} + ({}^{\alpha}t)_i)^2}{\sum_{i=1}^{n} d_i^{2\alpha}}}, \tag{3.3}$$

where α ranges over all real numbers. Moreover, the equality in (3.3) holds for some particular value α if and only if there exists a positive constant τ such that $d_i + ({}^{\alpha}m)_i = \tau$ for any vertex $v_i \in V$. In particular, if $\alpha < 1$, then the equality in (3.3) holds if and only if G is a regular bipartite graph; if $\alpha = 1$, then the equality in (3.3) holds if and only if G is a semiregular bipartite graph; if $\alpha > 1$ and G is a regular bipartite graph, then the equality in (3.3) holds.

Proof. By a simple calculation, one has

$$\begin{aligned}
&\left(\sum_{i=1}^{n}(d_i^{\alpha+1} + ({}^{\alpha}t)_i)^2\right)^2 \\
&= \left(\sum_{i=1}^{n} d_i^{\alpha+1}(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{j=1}^{n} (d_j^{\alpha+1} + ({}^{\alpha}t)_j)({}^{\alpha}t)_j\right)^2 \\
&= \left(\sum_{i=1}^{n} d_i^{\alpha+1}(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{j=1}^{n} (d_j^{\alpha+1} + ({}^{\alpha}t)_j)\sum_{i=1}^{n} a_{ij} d_i^{\alpha}\right)^2 \\
&= \left(\sum_{i=1}^{n} d_i^{\alpha+1}(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{i=1}^{n} d_i^{\alpha}\sum_{j=1}^{n} a_{ij}(d_j^{\alpha+1} + ({}^{\alpha}t)_j)\right)^2 \\
&= \left(\sum_{i=1}^{n} \left(d_i^{\alpha+1}(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + d_i^{\alpha}\sum_{j\colon j\sim i} (d_j^{\alpha+1} + ({}^{\alpha}t)_j)\right)\right)^2 \\
&= \left(\sum_{i=1}^{n} \left(d_i(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{j\colon j\sim i} (d_j^{\alpha+1} + ({}^{\alpha}t)_j)\right)\cdot d_i^{\alpha}\right)^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality, one has

$$\begin{aligned}
&\left(\sum_{i=1}^{n}(d_i^{\alpha+1} + ({}^{\alpha}t)_i)^2\right)^2 \\
&\quad \leqslant \sum_{i=1}^{n}\left(d_i(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{j\colon j\sim i} (d_j^{\alpha+1} + ({}^{\alpha}t)_j)\right)^2 \cdot \sum_{i=1}^{n} d_i^{2\alpha}
\end{aligned} \tag{3.4}$$

with equality if and only if there exists a positive constant l such that, for any vertex $v_i \in V$,

$$\frac{d_i(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{j\colon j\sim i} (d_j^{\alpha+1} + ({}^{\alpha}t)_j)}{d_i^{\alpha}} = l.$$

Following from (3.4) and Theorem 3.1, the inequality (3.3) holds.

By Theorem 3.1, the equality in (3.1) holds for some particular value α if and only if there exists a positive constant t such that, for any vertex $v_i \in V$,

$$\frac{d_i(d_i^{\alpha+1} + ({}^{\alpha}t)_i) + \sum_{j\,:\,j\sim i}(d_j^{\alpha+1} + ({}^{\alpha}t)_j)}{d_i^{\alpha+1} + ({}^{\alpha}t)_i} = t.$$

Hence the equality in (3.3) holds for some particular value α if and only if there exists a positive constant τ such that, for any vertex $v_i \in V$,

$$\frac{d_i^{\alpha+1} + ({}^{\alpha}t)_i}{d_i^{\alpha}} = d_i + ({}^{\alpha}m)_i = \frac{l}{t} \triangleq \tau.$$

The rest of the proof is similar to those of Theorem 4.2 in [8] and Theorem 9 in [13]. □

Remark 3.2. Corollary 3.1 shows that Theorem 3.1 is an improvement on Theorem 4.2 in [8]. In addition, as pointed out in Note 4.3 at the end of [8], Corollary 3.1 improves and generalizes some results in [7], [11], [13].

Remark 3.3. Using a similar technique as in the proofs of Theorems 7 and 8 in [12], together with Theorem 3.1, we may obtain some new upper and lower bounds on the sum of powers of the Laplacian eigenvalues of bipartite graphs, which theoretically improve some results obtained in [12]. We omit these results here.

4. Conclusion

Theorems 2.1 and 2.2 give two upper and lower bounds on the spectral radius $\lambda_1(G)$ of a graph G in terms of $({}^{\alpha}t)_i$, $({}^{\alpha}m)_i$ and $({}^{\alpha}N)_i$. Theorem 3.1 gives a lower bound on the Laplacian spectral radius $\mu_1(G)$ of G in terms of d_i and $({}^{\alpha}t)_i$. Furthermore, we characterize some extreme graphs which attain these upper and lower bounds. These results not only improve and generalize some known results theoretically, but also imply some upper bounds on the energy of graphs and the sum of powers of the Laplacian eigenvalues of bipartite graphs. In addition, given a graph G, by choosing the parameter α, we may get the optimal bounds on $\lambda_1(G)$ and $\mu_1(G)$. Hence these bounds are worthy of being retained.

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