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Mathematica Bohemica, Vol. 137 (2012), No. 2, 113–122

Persistent URL: http://dml.cz/dmlcz/142856

ON SOME BOUNDARY VALUE PROBLEMS FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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(Received October 15, 2009)

Abstract. We investigate two boundary value problems for the second order differential equation with p-Laplacian

$$(a(t)\Phi_p(x'))' = b(t)F(x), \quad t \in I = [0, \infty),$$

where a, b are continuous positive functions on I. We give necessary and sufficient conditions which guarantee the existence of a unique (or at least one) positive solution, satisfying one of the following two boundary conditions:

$$\text{i) } x(0) = c > 0, \ \lim_{t\to\infty} x(t) = 0; \quad \text{ii) } x'(0) = d < 0, \ \lim_{t\to\infty} x(t) = 0.$$

Keywords: boundary value problem, p-Laplacian, half-linear equation, positive solution, uniqueness, decaying solution, principal solution

MSC 2010: 34C10

1. Introduction

Consider the second order nonlinear differential equation

$$(a(t)\Phi_p(x'))' = b(t)F(x), \quad t \in I = [0, \infty), \tag{1.1}$$

[1] Author supported by the Research Project 0021622409 of the Ministry of Education of the Czech Republic and Grant 201/08/0469 of the Czech Grant Agency.

[2] Authors supported by the Research Project PRIN07-Area 01, n. 37 of the Italian Ministry of Education.

where a, b are continuous and positive functions on I, $\Phi_p(u) = |u|^{p-2}u$, $p > 1$, and F is a continuous function on $\mathbb{R}$ such that $uF(u) > 0$ for $u \neq 0$ and

$$\lim_{u\to 0} \frac{F(u)}{\Phi_p(u)} = L, \quad 0 \leqslant L < \infty. \tag{1.2}$$

We study the existence and uniqueness of positive decreasing solutions of (1.1) on the whole half-line, satisfying

$$x(0) = c, \quad \lim_{t\to\infty} x(t) = 0, \tag{1.3}$$

or

$$x'(0) = d, \quad \lim_{t\to\infty} x(t) = 0, \tag{1.4}$$

where $c > 0$ and $d < 0$ are constants.

Boundary value problems (BVPs) associated with (1.1) appear in studying radial solutions for nonlinear elliptic systems with the p-Laplacian operator $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$, and have been extensively considered in literature; see, e.g., the papers [2], [6], [12], [13], the monograph [1], and references therein.

As usual, by a solution of (1.1) we mean a function x which is continuously differentiable together with its quasiderivative $x^{[1]}$, $x^{[1]}(t) = a(t)\Phi_p(x'(t))$, and satisfies (1.1) on I. In view of (1.2), using some results by Chanturia [8], [9] (see also [4, Theorem 6]), (1.1) has solutions x such that

$$x(t)x'(t) < 0 \quad \text{for } t \in [0,\infty) \tag{1.5}$$

(the so called Kneser solutions). The problem whether these solutions converge to zero as $t \to \infty$ and are unique in some sense depends on the convergence of the integral limits

$$J_1 = \lim_{T\to\infty} \int_0^T \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\Phi_{p^*}\Big(\int_0^t b(s)\,\mathrm{d}s\Big)\,\mathrm{d}t,$$
$$J_2 = \lim_{T\to\infty} \int_0^T \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\Phi_{p^*}\Big(\int_t^T b(s)\,\mathrm{d}s\Big)\,\mathrm{d}t,$$

where p^* is the conjugate number to p, i.e., $p^* = p/(p-1)$. When $J_1 = \infty$ and $J_2 < \infty$, Kneser solutions of (1.1) tend to a nonzero constant, while, when $J_2 = \infty$, any Kneser solution tends to zero ([8, Theorem 1], [4, Theorem 6]). Moreover, in both cases, for any $c \neq 0$ there exists a unique Kneser solution x satisfying $x(0) = c$

([3, Theorem 4]). Finally, if $J_1 + J_2 < \infty$, Kneser solutions converging to zero or to a nonzero constant coexist ([4, Theorem 2]).

Our aim here is to complete this result by considering the solvability and uniqueness of the BVPs (1.1), (1.3) and (1.1), (1.4) when

$$A_\infty = \int_0^\infty \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\,\mathrm{d}t < \infty.$$

The following relation between A_∞, J_1, J_2 will be useful.

Lemma 1.1. *If $A_\infty < \infty$, then either $J_1 < \infty$ or $J_1 = J_2 = \infty$.*

2. Statement of the main result

Our main result is

Theorem 2.1. *Assume $A_\infty < \infty$. Then the BVPs* (1.1), (1.3) *and* (1.1), (1.4) *have at least one positive solution x for any $c > 0$ and $d < 0$, respectively. Moreover, x satisfies on I the inequality*

$$x(t) \leqslant \frac{c}{A_\infty}\int_t^\infty \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\,\mathrm{d}s, \tag{2.1}$$

or

$$x(t) \leqslant -\Phi_p^*(a(0))d\int_t^\infty \Phi_p^*\Big(\frac{1}{a(s)}\Big)\,\mathrm{d}s, \tag{2.2}$$

respectively. In addition, if F is nondecreasing, then this solution is unique.

R e m a r k 1. As already claimed, when $J_1 < \infty, J_2 = \infty$ or $J_1 = J_2 = \infty$, the solvability of the BVP (1.1), (1.3) follows also from previous results in [4], [8].

Theorem 2.1 completes also the characterization of the so-called *minimal set* of (1.1), introduced in [5] as the set of solutions x satisfying (1.5) and $\lim\limits_{t\to\infty} x(t) = 0$. Moreover, Theorem 2.1 plays a crucial role in solving the BVP

$$\begin{cases} (a(t)\Phi_p(x'))' = \tilde{b}(t)F(x), & t \in I,\\ x(0) = x(\infty) = 0,\ x(t) > 0, & t \in (0,\infty),\end{cases}$$

where $\tilde{b}$ is a continuous function which changes sign on I, see [10].

The approach that we use is based on a comparison result concerning the principal solutions of the corresponding half-linear differential equation

$$(a(t)\Phi_p(x'))' = b(t)\Phi_p(x), \tag{2.3}$$

and on a general fixed point theorem for operators defined in a Fréchet space by Schauder's linearization device ([7, Theorem 1.3]). In particular, this result reduces the existence of solutions of a BVP for differential equations on noncompact intervals to the existence of suitable a priori bounds and is useful mainly when the fixed point operator, associated with the BVP, is not known in an explicit form. We recall it in the form that will be used in the sequel.

Theorem 2.2. *Consider the BVP*

$$\begin{cases} (a(t)\Phi_p(x'))' = b(t)F(x), \quad t \in I, \\ x \in S, \end{cases} \tag{2.4}$$

where S is a nonempty subset of the Fréchet space $C[0,\infty)$ of the continuous real functions defined in $[0,\infty)$.

Let F be a restriction to the diagonal of a real continuous function G defined on $\mathbb{R}^2$, that is $F(c) = G(c,c)$ for any $c \in \mathbb{R}$. Let there exist a nonempty, closed, convex and bounded subset $\Omega \subset C[0,\infty)$ such that for any $u \in \Omega$, the BVP

$$\begin{cases} (a(t)\Phi_p(x'))' = b(t)G(u(t), x(t)), \quad t \in I, \\ x \in S, \end{cases}$$

admits a unique solution x_u. Let $\mathcal{T}$ be the operator $\mathcal{T}(u) = x_u$. Assume

i$_1$) $\mathcal{T}(\Omega) \subset \Omega$;

i$_2$) *if $\{u_n\} \subset \Omega$ is a sequence converging in Ω and $\mathcal{T}(u_n) \to x$, then $x \in S$.*

Then the BVP (2.4) *has at least one solution.*

3. A comparison result for half-linear equations

This section is devoted to the properties of principal solutions of the half-linear equation (2.3). It is known, see, e.g., [6], that any nontrivial solution x of (2.3) satisfies either $x(t)x'(t) > 0$ for large t or $x(t)x'(t) < 0$ for $t \geqslant 0$. Moreover, following Mirzov, or Elbert and Kusano, see, e.g., [11, Chapter 4.2], a solution u of (2.3) is called a *principal solution* of (2.3) if for every solution x of (2.3) such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$,

$$\frac{u'(t)}{u(t)} < \frac{x'(t)}{x(t)}$$

for large t. The set of principal solutions is nonempty and principal solutions are determined uniquely up to a constant factor. In [5], a complete characterization of principal solutions of (2.3) is given. In particular, the following properties will be used in the sequel.

Theorem 3.1. i$_1$) *Among all solutions x of* (2.3) *such that $x(0) = x_0 \neq 0$, or $x'(0) = x'_0 \neq 0$, there exists a unique principal solution.*

i$_2$) *Any principal solution u of* (2.3) *satisfies $u(t)u'(t) < 0$ for $t \geqslant 0$ and either $\lim_{t\to\infty} u(t) \neq 0$ if $J_1 = \infty$ and $J_2 < \infty$, or $\lim_{t\to\infty} u(t) = 0$ otherwise.*

i$_3$) *Let u be a solution of* (2.3). *If $\lim_{t\to\infty} u(t) = 0$, then u is a principal solution.*

Now we give a comparison result for principal solutions of the half-linear differential equations

$$(a(t)\Phi_p(x'))' = b_1(t)\Phi_p(x), \tag{3.1}$$

$$(a(t)\Phi_p(y'))' = b_2(t)\Phi_p(y), \tag{3.2}$$

where $b_i, i = 1, 2$ are positive continuous functions for $t \in I$. The following result extends [4, Theorem 5].

Theorem 3.2. *Consider the equations* (3.1), (3.2), *and assume*

$$b_1(t) \leqslant b_2(t) \quad \text{for } t \in I. \tag{3.3}$$

Let $\bar{x}$ and $\overline{y}$ be positive principal solutions of (3.1) *and* (3.2), *respectively, such that either $\bar{x}(0) = \overline{y}(0) = c > 0$ or $\bar{x}'(0) = \overline{y}'(0) = d < 0$. Then*

$$\overline{y}(t) \leqslant \bar{x}(t) \qquad \text{for } t \geqslant 0.$$

P r o o f. First assume $\bar{x}(0) = \overline{y}(0) = c > 0$. The argument is similar to the one given in [4, Theorem 5]. Set

$$w(t) = \bar{x}(t) - \overline{y}(t).$$

We claim that w does not have a negative minimum. Let $T > 0$ be a point of a negative minimum for w and set

$$H(t) = a(t)\left(\Phi_p(\bar{x}'(t)) - \Phi_p(\overline{y}'(t))\right). \tag{3.4}$$

Hence, $H(T) = 0$. Since

$$H'(t) \leqslant b_2(t)\left(\Phi_p(\bar{x}(t)) - \Phi_p(\overline{y}(t))\right), \tag{3.5}$$

we have $H'(t) < 0$ in a right neighborhood I_T of T and so $H(t) < 0$ for $t \in I_T$, that is, in view of (3.4), w is decreasing on I_T, and this is a contradiction.

We claim that $w(t) \geqslant 0$ for $t \geqslant 0$. Assume there exists $t_1 > 0$ such that $w(t_1) < 0$. Since $w(0) = 0$ and w does not have negative minima, we have $\lim_{t\to\infty} w(t) < 0$. Thus, $\lim_{t\to\infty} \overline{y}(t) > 0$. Since $\overline{y}$ is a principal solution, by Theorem 3.1.i$_2$) and using (3.3) we obtain $J_1 = \infty$ and $J_2 < \infty$. Thus, Lemma 1.1 gives $A_\infty = \infty$, and from [4, Lemma 3] we have $\lim_{t\to\infty} x^{[1]}(t) = 0$, so $\lim_{t\to\infty} H(t) = 0$. From (3.5) we get $H'(t) < 0$ for $t > t_1$, and so $H(t) > 0$ for large t, i.e. w is eventually increasing. This is a contradiction because w would have a negative minimum, and so the assertion is proved when $\bar{x}(0) = \overline{y}(0) = c > 0$.

Now assume $\bar{x}'(0) = \overline{y}'(0) = d < 0$. By Theorem 3.1.i$_1$) the principal solutions $\bar{x}$, $\overline{y}$ are positive for $t \geqslant 0$. Let $\overline{z}$ be the principal solution of (3.1) such that $\overline{z}(0) = \overline{y}(0)$. In virtue of the first part of the proof, we have $\overline{y}(t) \leqslant \overline{z}(t)$ for $t \geqslant 0$. Hence, $d = \overline{y}'(0) \leqslant \overline{z}'(0)$. Since both $\bar{x}$ and $\overline{z}$ are principal solutions of (3.1), there exists $\lambda \neq 0$ such that $\overline{z} = \lambda\bar{x}$. Thus, we have

$$0 > d = \overline{y}'(0) \leqslant \overline{z}'(0) = \lambda\bar{x}'(0) = \lambda d,$$

which gives $0 < \lambda \leqslant 1$ and the assertion again follows. □

4. Proof of the main result

Proof of Theorem 2.1. Step 1. First we prove that the BVP (1.1), (1.3) is solvable for any $c > 0$. Let $\overline{z}$ be the principal solution of the half-linear differential equation

$$(a(t)\Phi_p(z'))' = M_c b(t)\Phi_p(z)$$

such that $\overline{z}(0) = c$, where

$$M_c = \sup_{0\leqslant u\leqslant c} \frac{F(u)}{\Phi_p(u)}. \tag{4.1}$$

Let Ω be the subset of $C[0,\infty)$ given by

$$\Omega = \{u \in C[0,\infty)\colon\ \overline{z}(t) \leqslant u(t) \leqslant c\},$$

and for any $u \in \Omega$ consider the BVP

$$(a(t)\Phi_p(y'))' = b(t)\frac{F(u(t))}{\Phi_p(u(t))}\Phi_p(y), \tag{4.2}$$

$$y(0) = c > 0,\ y(t) > 0,\ y'(t) < 0,\ \lim_{t\to\infty} y(t) = 0. \tag{4.3}$$

Fix $u \in \Omega$. Since $A_\infty < \infty$, by Lemma 1.1 and Theorem 3.1 i_1), i_2), the principal solution y_u of (4.2) such that $y_u(0) = c$ satisfies (4.3). Moreover, y_u is the unique solution of the BVP (4.2), (4.3). Indeed, assume that (4.2), (4.3) admits another solution different from y_u, say $\tilde{y}_u$. Since $\lim_{t\to\infty} \tilde{y}_u(t) = 0$, by Theorem 3.1.$\mathrm{i}_3$) the solution $\tilde{y}_u$ should be a principal solution, which is a contradiction. Hence the BVP (4.2), (4.3) is uniquely solvable for any $u \in \Omega$.

Let $\mathcal{T}$ be the operator which associates with any $u \in \Omega$ the unique solution y_u of (4.2), (4.3). Since y_u is a positive principal solution, from Theorem 3.1.i_1) we have $y_u'(t) < 0$, so $y_u(t) \leqslant c$. Moreover, from (4.1) we get

$$\frac{F(u(t))}{\Phi_p(u(t))} \leqslant M_c,$$

and Theorem 3.2 gives

$$\overline{z}(t) \leqslant y_u(t), \tag{4.4}$$

that is $\mathcal{T}(\Omega) \subset \Omega$.

Let $\{u_n\}$ be a convergent sequence in Ω and let $\lim \mathcal{T}(u_n) = x$. We prove that $x \in S$, where S is the set of functions $y \in C[0,\infty)$ satisfying (4.3). Set

$$w_u(t) = \frac{c}{A_\infty} \int_t^\infty \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\,\mathrm{d}s - y_u(t).$$

Hence, $w_u(0) = \lim_{t\to\infty} w_u(t) = 0$. Using an argument similar to the one given in the first part of the proof of Theorem 3.2, it is easy to verify that $w_u(t) \geqslant 0$ on I, i.e.,

$$y_u(t) \leqslant \frac{c}{A_\infty} \int_t^\infty \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\,\mathrm{d}s. \tag{4.5}$$

Indeed, if $w_u(t)$ becomes negative for some t, then w_u has a point $T > 0$ of negative minimum. Thus,

$$a(T)\Phi_p(y_u'(T)) = -\Phi_p\Big(\frac{c}{A_\infty}\Big). \tag{4.6}$$

Consider the function

$$H_u(t) = -\Phi_p\Big(\frac{c}{A_\infty}\Big) - a(t)\Phi_p(y_u'(t)).$$

Since $H_u'(t) = -b(t)\Phi_p(y_u(t))$ and, in view of (4.6), $H_u(T) = 0$, we have $H_u(t) < 0$ for $t > T$. Thus, $w_u'(t) < 0$ for $t > T$, which is a contradiction and so (4.5) holds.

In virtue of (4.4) and (4.5), $x \in S$ and the condition i_2) of Theorem 2.2 is satisfied. By applying Theorem 2.2 with

$$G(u,v) = \begin{cases} F(u)\Phi_p(v)/\Phi_p(u), & \text{if } u \neq 0, \\ L\Phi_p(v), & \text{if } u = 0, \end{cases} \tag{4.7}$$

where L is given in (1.2), the BVP (1.1), (2.1) is solvable for any $c > 0$, so the same holds for the BVP (1.1), (1.3).

S t e p 2. Now we prove that the BVP (1.1), (1.4) is solvable for any $d < 0$. Set

$$D = \Phi_p^*(a(0))|d|A_\infty, \ M_D = \sup_{0 \leqslant u \leqslant D} \frac{F(u)}{\Phi_p(u)}. \tag{4.8}$$

Let $\overline{v}$ be the principal solution of the half-linear differential equation

$$(a(t)\Phi_p(v'))' = M_D b(t)\Phi_p(v)$$

such that $\overline{v}'(0) = d$. Let Ω_2 be the subset of $C[0,\infty)$ given by

$$\Omega_2 = \{u \in C[0,\infty)\colon \ \overline{v}(t) \leqslant u(t) \leqslant D\}.$$

For any $u \in \Omega_2$, consider the BVP given by (4.2) and

$$y'(0) = d < 0, \ y(t) > 0, \ \lim_{t\to\infty} y(t) = 0. \tag{4.9}$$

Fix $u \in \Omega_2$. Since $A_\infty < \infty$, by Lemma 1.1 and Theorem 3.1.i_1), i_2), the principal solution y_u of (4.2) such that $y_u'(0) = d$ satisfies (4.9). Moreover, following the same argument as the one in Step 1, y_u is the only solution of this BVP. Hence, (4.2), (4.9) is uniquely solvable for any $u \in \Omega$.

Let $\mathcal{T}$ be the operator which associates with any $u \in \Omega$ the unique solution y_u of (4.2), (4.9). Since y_u is a positive principal solution, and

$$0 \leqslant \frac{F(u(t))}{\Phi_p(u(t))} \leqslant M_D$$

for every $u \in \Omega_2$, Theorem 3.2 gives $\overline{v}(t) \leqslant y_u(t)$. Since y_u is a positive principal solution, Theorem 3.1.i_1) implies that its quasiderivative is negative increasing, i.e.,

$$a(t)\Phi_p(y_u'(t)) \geqslant a(0)\Phi_p(d).$$

Taking into account that $\lim_{t\to\infty} y_u(t) = 0$, by integration we obtain

$$y_u(t) \leqslant -\Phi_p^*(a(0))d \int_t^\infty \Phi_p^*\Big(\frac{1}{a(s)}\Big)\,\mathrm{d}s, \tag{4.10}$$

which gives, in view of (4.8), $y_u(t) \leqslant D$, that is $\mathcal{T}(\Omega) \subset \Omega$. In view of $\overline{v}(t) \leqslant y_u(t)$ and (4.10), the condition i_2) of Theorem 2.2 is satisfied, where S is the set of functions $y \in C[0, \infty)$ satisfying (4.9). Now, by applying Theorem 2.2 with G given in (4.7), the problem (1.1), (1.4) is solvable for any $d < 0$ and, in view of (4.10), this solution satisfies (2.2).

S t e p 3. In order to complete the proof, it remains to show that the BVPs (1.1), (1.3) and (1.1), (1.4) are uniquely solvable. Let $\bar{x}$, $\overline{y}$ be two solutions of (1.1), (1.3) and set $w(t) = \bar{x}(t) - \overline{y}(t)$. Using the same argument as the one given in the proof of Theorem 3.2 and the monotonicity of F, we obtain that w has neither a positive maximum nor a negative minimum. Thus, $w(t) \equiv 0$ on I which gives the assertion. The uniqueness of the BVP (1.1), (1.4) follows by using a similar argument, with minor changes. The details are left to the reader. □

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