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A NOTE ON THE TRANSCENDENCE OF INFINITE PRODUCTS

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Abstract. The paper deals with several criteria for the transcendence of infinite products of the form $\prod_{n=1}^{\infty}[b_n\alpha^{a_n}]/b_n\alpha^{a_n}$ where $\alpha>1$ is a positive algebraic number having a conjugate α^* such that $\alpha\neq|\alpha^*|>1$, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers with some specific conditions.

The proofs are based on the recent theorem of Corvaja and Zannier which relies on the Subspace Theorem (P. Corvaja, U. Zannier: On the rational approximation to the powers of an algebraic number: solution of two problems of Mahler and Mendès France, Acta Math. 193, (2004), 175–191).

Keywords: transcendence, infinite product

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1. Introduction

Following Erdős [4], Corvaja and Zannier [2] we prove

Theorem 1. *The number* $x=\prod_{n=1}^{\infty}[n(\sqrt{5}+1)^{k^n}]/n(\sqrt{5}+1)^{k^n}$ *is transcendental for all integers* k *greater than* 4.

Here $[z]$ means the integer part of the number z. The authors do not know if the number x is also transcendental or irrational for $k=2,3$ and 4.

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In 2000 Zhu [14] proved some criteria for an infinite product to be transcendental. Making use of linear recurrence sequences of the second order Nyblom [11] constructed a set of transcendental valued infinite products. Utilizing theta series Kim [9] and Koo described some interesting infinite products. Recently Corvaja and Hančl [1] established a criterion for an infinite product to be transcendental. Tachiya [12] found some transcendental valued infinite products of algebraic numbers. Zhou [13] worked with similar products and obtained some irrationality results. All this shows that metric properties of infinite products are of current interest.

Erdős [4] proved that if $a = \{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that $\liminf_{n\to\infty} a_n^{1/2^n} = \infty$ then the expressible set $E_a = \left\{\sum_{n=1}^{\infty} 1/a_n c_n, c_n \in \mathbb{N}\right\}$ does not contain rational numbers. Using this idea of Erdős, Hančl, Nair and Šustek [6] found some necessary conditions for the Lebesgue measure of E_a to be equal to zero. For other applications of the method of Erdős see e.g. [5], [7] or [8]. It seems likely that this method still has great potential.

Our main theorem is Theorem 2. Its proof makes use of the main theorem in [2]. See also [3]. Theorem 2 and the method of Erdős yield Theorems 3–7. In all of Theorems 2–7 we suppose that α is a positive algebraic number greater than 1 having a conjugate α^* such that $\alpha \neq |\alpha^*| > 1$ where $|z|$ means the usual absolute value of the number z. Denote by $\mathbb{N}$ and $\mathbb{Q}$ the set of all natural and rational numbers, respectively. If α is an algebraic number then set $d = [\mathbb{Q}(\alpha)\colon \mathbb{Q}]$, the degree of the algebraic number field $\mathbb{Q}(\alpha)$.

2. Main results

Theorem 2. *Let x and γ be real numbers such that $\gamma > 0$. If for infinitely many positive integers n, p and q*

$$0 < \left| x - \frac{p}{q\alpha^n} \right| < \frac{1}{\alpha^{n(1+\gamma)} q^{1+\gamma+d}}, \tag{2.1}$$

then the number x is transcendental.

Theorem 3. *Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers with $\limsup_{n\to\infty} \sqrt[n]{a_n} > 2$. Then the number $x = \prod_{n=1}^{\infty} [\alpha^{a_n}]/\alpha^{a_n}$ is transcendental.*

Theorem 4. *Let $\varepsilon > 0$. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers with $\limsup_{n\to\infty} \sqrt[n]{a_n} > 2 + 1/\varepsilon$. Assume that $\alpha^{a_n} > n^{1+\varepsilon}$ for every sufficiently large n. Then the number $x = \prod_{n=1}^{\infty} [\alpha^{a_n}]/\alpha^{a_n}$ is transcendental.*

Theorem 5. *Let δ and ε be two positive real numbers. Assume that*

$$\frac{1+d+\delta}{1+d}\cdot\frac{\varepsilon}{1+\varepsilon}>1. \tag{2.2}$$

Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that the sequence $\{B_n\}_{n=1}^{\infty}=\{b_n\alpha^{a_n}\}_{n=1}^{\infty}$ is non-decreasing and

$$\limsup_{n\to\infty} B_n^{1/(2+d+\delta)^n}=\infty. \tag{2.3}$$

Assume that $B_n>n^{1+\varepsilon}$ for every sufficiently large n. Then the number $x=\prod_{n=1}^{\infty}[B_n]/B_n=\prod_{n=1}^{\infty}[b_n\alpha^{a_n}]/b_n\alpha^{a_n}$ is transcendental.

Theorem 6. *Assume that s is a non-negative real number. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is strictly increasing, $\{B_n\}_{n=1}^{\infty}=\{b_n\alpha^{a_n}\}_{n=1}^{\infty}$ is non-decreasing,*

$$\limsup_{n\to\infty}\sqrt[n]{a_n}>2+\frac{sd}{s+1} \tag{2.4}$$

and

$$b_n=\alpha^{sa_n}+o(\alpha^{sa_n}). \tag{2.5}$$

Then the number $x=\prod_{n=1}^{\infty}[B_n]/B_n=\prod_{n=1}^{\infty}[b_n\alpha^{a_n}]/b_n\alpha^{a_n}$ is transcendental.

Theorem 7. *Assume that ε and s are real numbers with $s\geqslant 0$ and $\varepsilon>0$. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is non-decreasing,*

$$\limsup_{n\to\infty}\sqrt[n]{a_n}>\Big(1+\frac{sd}{s+1}\Big)\Big(1+\frac{1}{\varepsilon}\Big)+1, \tag{2.6}$$

$$\alpha^{a_n}>n^{1+\varepsilon} \tag{2.7}$$

and

$$b_n=\alpha^{sa_n}+o(\alpha^{sa_n}). \tag{2.8}$$

Then the number $x=\prod_{n=1}^{\infty}[b_n\alpha^{a_n}]/b_n\alpha^{a_n}$ is transcendental.

3. Proofs

P r o o f of Theorem 1. Theorem 1 is an immediate consequence of Theorem 5. It is enough to set $\alpha = \sqrt{5}+1$, $\varepsilon = 7$, $\delta = \frac{1}{2}$, $b_n = n$ and $a_n = 5^n$ for all $n \in \mathbb{N}$. Then $d = 2$ and α has only one conjugate $\alpha^* = -\sqrt{5}+1$. $\square$

P r o o f of Theorem 2. In fact Theorem 2 is a consequence of the main theorem in [2]. Assume Theorem 2 does not hold. Thus x is an algebraic number. Let $H(\alpha)$ be the Weil height for the number α. So $H(\alpha^n) = H^n(\alpha)$ for all $n \in \mathbb{N}$. From this we obtain that there exists a positive real number a such that $a < 1$ and for all $n \in \mathbb{N}$ we have $\alpha^n > H(\alpha^n)^a$. Now, set $\delta := x$, $q := q_n$, $\varepsilon := a\gamma$ and $u := \alpha^n$ where q_n is a suitable integer corresponding to α^n. Hence inequality (1.1) from [2] holds for infinitely many pairs (q, u). Therefore $q_n\alpha^n x$ is a pseudo-Pisot number for infinitely many positive integers n. (A pseudo-Pisot number β is an algebraic number with $|\beta| > 1$, having all absolute values of conjugates strictly less then 1 and with $Tr_{\mathbb{Q}(\beta)/\mathbb{Q}} \in \mathbb{Z}$.) From the definition of α we have that α has a conjugate α^* with $\alpha \neq |\alpha^*| > 1$. Thus there exists an authomorphism σ of the set $\mathbb{K}$ such that $\alpha^* = \sigma(\alpha)$ where $\mathbb{K}$ is the Galois closure over $\mathbb{Q}$ of the field $\mathbb{Q}(\alpha, x)$. (For more information see e.g. [10], chapter 5, page 243, lines 8–12 from the top. See also Lemma 4 from [1].) Hence for all $n \in \mathbb{N}$ the authomorphism σ maps the number $q_n\alpha^n x$ to its conjugate and for infinitely many positive integers n the number $q_n\alpha^n x$ is a pseudo-Pisot number. So for infinitely many n either $q_n\alpha^n x = \sigma(q_n\alpha^n x) = q_n(\alpha^*)^n\sigma(x)$ or $1 > |\sigma(q_n\alpha^n x)| = |q_n||\alpha^*|^n|\sigma(x)|$. But for the number α^* we have $|\alpha^*| > 1$. So the number of n such that $1 > |\sigma(q_n\alpha^n x)| = |q_n||\alpha^*|^n|\sigma(x)|$ is finite. Therefore $x/\sigma(x) = (\alpha^*/\alpha)^n$ for infinitely many $n \in \mathbb{N}$ which is a contradiction with the fact that $|\alpha^*/\alpha|$ is a positive real number which is not equal to 1. $\square$

Lemma 1. *Let y be a positive real number and let $\{a_n\}_{n=1}^{\infty}$ be an non-decreasing sequence of positive real numbers such that*

$$\limsup_{n\to\infty} \sqrt[n]{a_n} > y+1. \tag{3.1}$$

Then $\limsup\limits_{n\to\infty}\Big(a_n/\sum\limits_{j=1}^{n-1} a_j\Big) > y$.

P r o o f of Lemma 1. Let us assume that $\limsup\limits_{n\to\infty}\Big(a_n/\sum\limits_{j=1}^{n-1} a_j\Big) \leqslant y$. Then for every $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $a_n \leqslant \sum\limits_{j=1}^{n-1} a_j(y+\delta)$ for every $n \geqslant n_0$. From this we obtain that for all $n > n_0$

$$a_n \leqslant (y+\delta)\sum_{j=1}^{n-1} a_j = (y+\delta)\Big(a_{n-1} + \sum_{j=1}^{n-2} a_j\Big) \leqslant (y+\delta)\Big((y+\delta)\sum_{j=1}^{n-2} a_j + \sum_{j=1}^{n-2} a_j\Big)$$

$$= (y+\delta)(1+y+\delta)\sum_{j=1}^{n-2} a_j \leqslant \ldots \leqslant (y+\delta)(1+y+\delta)^{n-n_0-1}\sum_{j=1}^{n_0} a_j.$$

Hence $\limsup\limits_{n\to\infty} \sqrt[n]{a_n} \leqslant 1+y$ which contradicts (3.1). □

P r o o f of Theorem 3. Let N_0 be a sufficiently large positive integer. For $m \geqslant N_0$ set $p = p(m) = \prod\limits_{n=1}^{m} [\alpha^{a_n}]$ and $N = N(m) = \sum\limits_{n=1}^{m} a_n$. Then

$$\Big|x - \frac{p}{\alpha^N}\Big| = \Big|\frac{p}{\alpha^N}\Big| \cdot \Big|1 - \prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}}\Big|. \tag{3.2}$$

Using the inequality $|1-t| \leqslant |\log t|$ for $0 < t < 1$ we deduce from the above

$$\Big|1 - \prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}}\Big| \leqslant \Big|\log\Big(\prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}}\Big)\Big|.$$

On the other hand

$$\log\Big(\prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}}\Big) = \sum_{n=m+1}^{\infty} \log\Big(1 - \frac{\{\alpha^{a_n}\}}{\alpha^{a_n}}\Big),$$

where the symbol $\{\cdot\}$ stands for the fractional part. Using the inequality $|\log(1-t)| \leqslant |2t|$ for $0 < t < \frac{1}{2}$, and the fact that the fractional part $\{\cdot\}$ is always < 1, we obtain

$$\sum_{n=m+1}^{\infty} \Big|\log\Big(1 - \frac{\{\alpha^{a_n}\}}{\alpha^{a_n}}\Big)\Big| < \sum_{n=m+1}^{\infty} \frac{2}{\alpha^{a_n}} < \frac{2}{\alpha^{a_{m+1}}} \cdot \frac{\alpha}{\alpha-1}.$$

From the above inequalities, (3.2) and the fact that $p/\alpha^N \leqslant 1$ we obtain that

$$\Big|x - \frac{p}{\alpha^N}\Big| < \frac{2}{\alpha^{a_{m+1}}} \cdot \frac{1}{\alpha-1}. \tag{3.3}$$

We shall now compare the integer $N = \sum\limits_{n=1}^{m} a_n$ with a_{m+1}. From Lemma 1 we obtain that there is a $\gamma > 0$ such that $a_{m+1} \geqslant (1+\gamma)N$ for infinitely many m. This and (3.3) yield that for infinitely many m with $N = \sum\limits_{n=1}^{m} a_n$

$$\Big|x - \frac{p}{\alpha^N}\Big| < \frac{2}{\alpha^{a_{m+1}}} \cdot \frac{1}{\alpha-1} \leqslant \frac{2}{\alpha^{(1+\gamma)N}} \cdot \frac{1}{\alpha-1} \leqslant \frac{1}{\alpha^{(1+\gamma/2)N}}.$$

This and Theorem 2 (setting $q = 1$ in (2.1)) imply that the number x is transcendental. □

Lemma 2. *Let* $\varepsilon > 0$ *and* $\{b_n\}_{n=1}^{\infty}$ *be a non-decreasing sequence of positive real numbers such that* $b_n \geqslant n^{1+\varepsilon}$. *Then* $\sum\limits_{j=n}^{\infty} 1/b_j < (1+2^{\varepsilon}/\varepsilon)/b_n^{\varepsilon/(1+\varepsilon)}$ *for every* $n \geqslant 1$.

P r o o f of Lemma 2. We have

$$\sum_{j=n}^{\infty} \frac{1}{b_j} = \sum_{n+j \leqslant b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} + \sum_{n+j > b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}}. \tag{3.4}$$

We will estimate both sums on the right hand side of the equation (3.4). For the first summand we have

$$\sum_{n+j \leqslant b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} \leqslant \frac{[b_n^{1/(1+\varepsilon)}] - n + 1}{b_n} \leqslant \frac{b_n^{1/(1+\varepsilon)} - n + 1}{b_n}. \tag{3.5}$$

Now we will estimate the second summand.

$$\begin{aligned} \sum_{n+j > b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} &\leqslant \sum_{n+j > b_n^{1/(1+\varepsilon)}} \frac{1}{(n+j)^{1+\varepsilon}} \\ &< \int_{[b_n^{1/(1+\varepsilon)}]}^{\infty} \frac{\mathrm{d}x}{x^{1+\varepsilon}} = \frac{1}{\varepsilon [b_n^{1/(1+\varepsilon)}]^{\varepsilon}} = \frac{1}{\varepsilon} \frac{1}{b_n^{\varepsilon/(1+\varepsilon)}} \frac{b_n^{\varepsilon/(1+\varepsilon)}}{[b_n^{1/(1+\varepsilon)}]^{\varepsilon}} \\ &\leqslant \frac{1}{\varepsilon} \frac{1}{b_n^{\varepsilon/(1+\varepsilon)}} \Bigl(1 + \frac{1}{[b_n^{1/(1+\varepsilon)}]}\Bigr)^{\varepsilon} \leqslant \frac{1}{\varepsilon} \frac{1}{b_n^{\varepsilon/(1+\varepsilon)}} \Bigl(1 + \frac{1}{n}\Bigr)^{\varepsilon}. \end{aligned} \tag{3.6}$$

From (3.4), (3.5) and (3.6) we obtain that

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{1}{b_j} &= \sum_{n+j \leqslant b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} + \sum_{n+j > b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} \\ &< \frac{b_n^{1/(1+\varepsilon)} - n + 1}{b_n} + \frac{1}{\varepsilon} \frac{1}{b_n^{\varepsilon/(1+\varepsilon)}} \Bigl(1 + \frac{1}{n}\Bigr)^{\varepsilon} \\ &= \frac{1 - n/b_n^{1/(1+\varepsilon)} + 1/b_n^{1/(1+\varepsilon)}}{b_n^{1-1/(1+\varepsilon)}} + \frac{\varepsilon^{-1}(1+1/n)^{\varepsilon}}{b_n^{\varepsilon/(1+\varepsilon)}} \\ &\leqslant \frac{1 + \varepsilon^{-1}(1+1/n)^{\varepsilon}}{b_n^{\varepsilon/(1+\varepsilon)}} < \frac{1 + 2^{\varepsilon}/\varepsilon}{b_n^{\varepsilon/(1+\varepsilon)}} \end{aligned}$$

and the proof of Lemma 2 is complete. □

P r o o f of Theorem 4. Let N_0 be a sufficiently large positive integer. For $m \geqslant N_0$ set $p = p(m) = \prod\limits_{n=1}^{m} [\alpha^{a_n}]$ and $N = N(m) = \sum\limits_{n=1}^{m} a_n$. Now we proceed as in the proof of Theorem 3 to obtain that

$$\Bigl| x - \frac{p}{\alpha^N} \Bigr| < \sum_{n=m+1}^{\infty} \frac{2}{\alpha^{a_n}}.$$

This and Lemma 2 yield that

$$\left|x - \frac{p}{\alpha^N}\right| < \sum_{n=m+1}^{\infty} \frac{2}{\alpha^{a_n}} = 2 \cdot \sum_{n=m+1}^{\infty} \frac{1}{\alpha^{a_n}} < 2 \cdot \frac{1 + 2^{\varepsilon}/\varepsilon}{\alpha^{a_{m+1}\varepsilon/(1+\varepsilon)}}.$$

We shall now compare the integer $N = \sum_{n=1}^{m} a_n$ with a_{m+1}. From Lemma 1 we obtain that there is a γ such that for infinitely many n

$$\frac{\varepsilon}{1+\varepsilon}\, a_{m+1} \geqslant \frac{\varepsilon}{1+\varepsilon}\Big(1 + \frac{1}{\varepsilon} + \gamma\Big)N = \Big(1 + \frac{\varepsilon}{1+\varepsilon}\gamma\Big)N.$$

This and Theorem 2 imply that the number x is transcendental. □

P r o o f of Theorem 5. From (2.3) we obtain that there exist infinitely many n such that

$$B_n^{(2+d+\delta)^{-n}} > \max_{j=1,\dots,n-1} B_j^{(2+d+\delta)^{-j}}. \tag{3.7}$$

Otherwise there exist a positive integer n_0 such that for all $n > n_0$

$$B_n^{(2+d+\delta)^{-n}} \leqslant \max_{j=1,\dots,n_0-1} B_j^{(2+d+\delta)^{-j}}$$

which contradicts (2.3). The inequality (3.7) implies that for infinitely many n

$$\begin{aligned} B_n &> \Big(\max_{j=1,\dots,n-1} B_j^{(2+d+\delta)^{-j}}\Big)^{(2+d+\delta)^n} \\ &> \Big(\max_{j=1,\dots,n-1} B_j^{(2+d+\delta)^{-j}}\Big)^{(1+d+\delta)((2+d+\delta)^{n-1}+(2+d+\delta)^{n-2}+\ldots+1)} > \Big(\prod_{j=1}^{n-1} B_j\Big)^{1+d+\delta}. \end{aligned}$$

From this we obtain that for infinitely many n

$$B_n^{\varepsilon/(1+\varepsilon)} > \Big(\prod_{j=1}^{n-1} B_j\Big)^{(1+d+\delta)\varepsilon/(1+\varepsilon)}. \tag{3.8}$$

Now we proceed as in the proof of Theorem 4. Hence we obtain that for all sufficiently large m we have

$$\left|x - \frac{\prod_{k=1}^{m}[B_k]}{\prod_{k=1}^{m} B_k}\right| < \sum_{k=m+1}^{\infty} \frac{2}{B_k} = 2 \cdot \sum_{k=m+1}^{\infty} \frac{1}{B_k} < 2 \cdot \frac{1 + 2^{\varepsilon}/\varepsilon}{B_{m+1}^{\varepsilon/(1+\varepsilon)}} = \frac{s}{B_{m+1}^{\varepsilon/(1+\varepsilon)}} \tag{3.9}$$

where $s = 2(1+2^{\varepsilon}/\varepsilon)$. Set $\varepsilon' = \frac{1}{2}((1+d+\delta)\varepsilon/(1+\varepsilon)-1-d)$. From (2.2) we obtain that $\varepsilon' > 0$. The inequalities (2.2) and (3.8) imply that for infinitely many n

$$\frac{s}{B_n^{\varepsilon/(1+\varepsilon)}} < \frac{s}{\left(\prod_{j=1}^{n-1} B_j\right)^{(1+d+\delta)\varepsilon/(1+\varepsilon)}} = \frac{s}{\left(\prod_{j=1}^{n-1} B_j\right)^{1+d+\varepsilon'}} < \frac{1}{\left(\prod_{j=1}^{n-1} B_j\right)^{1+d+\varepsilon'/2}}.$$

From this, (3.9) and the fact that $B_k = b_k\alpha^{a_k}$ we obtain that for infinitely many n

$$\begin{aligned}\left|x - \frac{\prod_{k=1}^{n-1}[B_k]}{\left(\prod_{k=1}^{n-1} b_k\right)\alpha^{\sum_{k=1}^{n-1} a_k}}\right| &= \left|x - \frac{\prod_{k=1}^{n-1}[B_k]}{\prod_{k=1}^{n-1} b_k\alpha^{a_k}}\right| = \left|x - \frac{\prod_{k=1}^{n-1}[B_k]}{\prod_{k=1}^{n-1} B_k}\right| < \frac{s}{B_n^{\varepsilon/(1+\varepsilon)}} \\ &< \frac{1}{\left(\prod_{j=1}^{n-1} B_j\right)^{1+d+\varepsilon'/2}} = \frac{1}{\left(\prod_{k=1}^{n-1} b_k\alpha^{a_k}\right)^{1+d+\varepsilon'/2}} \\ &\leqslant \frac{1}{\alpha^{(1+\varepsilon'/2)\sum_{k=1}^{n-1} a_k}\left(\prod_{k=1}^{n-1} b_k\right)^{1+d+\varepsilon'/2}}.\end{aligned}$$

This and Theorem 2 imply that the number x is transcendental. □

Proof of Theorem 6. From (2.5) we obtain that there is a sufficiently small positive real number δ such that $\alpha^{(s-\delta/3)a_M} \leqslant b_M \leqslant \alpha^{(s+\delta/3)a_M}$ for all sufficiently large M. Similarly as in the proofs of Theorems 3–5 we have

$$\left|x - \frac{\prod_{n=1}^{m}[B_n]}{\prod_{n=1}^{m} B_n}\right| < \sum_{n=m+1}^{\infty}\frac{K}{B_n}$$

for all sufficiently large positive integers m where K is a suitable positive real constant which does not depend on m. From this and the fact that $\alpha^{(s-\delta/3)a_M} \leqslant b_M$ we obtain that for all sufficiently large positive integers m

$$\begin{aligned}\left|x - \frac{\prod_{n=1}^{m}[B_n]}{\prod_{n=1}^{m} B_n}\right| &< \sum_{n=m+1}^{\infty}\frac{K}{B_n} \leqslant \sum_{n=m+1}^{\infty}\frac{1}{\alpha^{(s+1-\delta/2)a_n}} \\ &\leqslant \frac{1}{\alpha^{(s+1-\delta/2)a_{m+1}}}\cdot\frac{1}{1-1/\alpha^{s+1-\delta/2}} \\ &= \frac{1}{\alpha^{(s+1-\delta/2)a_{m+1}}}\cdot\frac{\alpha^{s+1-\delta/2}}{\alpha^{s+1-\delta/2}-1} \leqslant \frac{1}{\alpha^{(s+1-\delta)a_{m+1}}}.\end{aligned} \tag{3.10}$$

From (2.4) and Lemma 1 we obtain that for infinitely many m

$$a_{m+1} > \left(1+\frac{sd}{s+1}+\delta'\right)\sum_{n=1}^{m} a_n \tag{3.11}$$

where δ' is a real number such that $0 < \delta' < \limsup_{n\to\infty} \sqrt[n]{a_n} - 2 - sd/(s+1)$. From (3.11) and the fact that δ is a sufficiently small positive real number we obtain that for infinitely many m

$$\begin{aligned}(s+1-\delta)a_{m+1} &> (s+1-\delta)\Big(1+\frac{sd}{s+1}+\delta'\Big)\sum_{n=1}^{m} a_n\\ &= \Big(s+1+sd+(s+1)\delta'-\delta\Big(1+\frac{sd}{s+1}+\delta'\Big)\Big)\sum_{n=1}^{m} a_n\\ &= \Big(1+(s+\delta)(d+1)+\delta'+s\delta'-\delta\Big(d+2+\frac{sd}{s+1}+\delta'\Big)\Big)\sum_{n=1}^{m} a_n\\ &\geqslant (1+(s+\delta)(d+1)+\delta')\sum_{n=1}^{m} a_n.\end{aligned}$$

From this, (3.10) and the fact that $b_M \leqslant \alpha^{(s+\delta/3)a_M}$ we obtain that for infinitely many m

$$\begin{aligned}&\left|x-\frac{\prod_{n=1}^{m}[B_n]}{\big(\prod_{n=1}^{m} b_n\big)\alpha^{\sum_{n=1}^{m} a_n}}\right| = \left|x-\frac{\prod_{n=1}^{m}[B_n]}{\prod_{n=1}^{m} B_n}\right| \leqslant \frac{1}{\alpha^{(s+1-\delta)a_{m+1}}}\\ &\leqslant \frac{1}{\alpha^{((s+\delta)(1+d)+1+\delta')\sum_{n=1}^{m} a_n}}\\ &= \frac{1}{\alpha^{\big((s+\delta/2)(1+d+\delta'/(s+\delta+1))+\frac{1}{2}\delta(1+d+\delta'/(s+\delta+1))+1+\delta'/(s+\delta+1)\big)\sum_{n=1}^{m} a_n}}\\ &\leqslant \frac{1}{\big(\prod_{n=1}^{m} b_n\big)^{1+d+\delta'/(s+\delta+1)}\alpha^{(1+\delta'/(s+\delta+1))\sum_{n=1}^{m} a_n}}.\end{aligned}$$

This and Theorem 2 imply that the number x is transcendental. □

Proof of Theorem 7. From (2.8) we obtain that there is a sufficiently small positive real number δ such that $\alpha^{(s-\delta/3)a_M} \leqslant b_M \leqslant \alpha^{(s+\delta/3)a_M}$ for all sufficiently large M. Similarly as in the proofs of Theorems 3–6 we have

$$\left|x-\frac{\prod_{n=1}^{m}[B_n]}{\prod_{n=1}^{m} B_n}\right| < \sum_{n=m+1}^{\infty}\frac{K}{B_n}$$

where K is a suitable positive real constant which does not depend on m. From this, Lemma 2, (2.7) and the fact that $\alpha^{(s-\delta/3)a_M} \leqslant b_M$ we obtain that for all sufficiently large positive integers m

$$\begin{aligned}\left|x-\frac{\prod_{n=1}^{m}[B_n]}{\prod_{n=1}^{m} B_n}\right| &< \sum_{n=m+1}^{\infty}\frac{K}{B_n} \leqslant \sum_{n=m+1}^{\infty}\frac{1}{\alpha^{(s+1-\delta/2)a_n}}\\ &\leqslant \frac{1+2^{\varepsilon}/\varepsilon}{\alpha^{(\varepsilon/(1+\varepsilon))(s+1-\delta/2)a_{m+1}}} \leqslant \frac{1}{\alpha^{(\varepsilon/(1+\varepsilon))(s+1-\delta)a_{m+1}}}.\end{aligned} \tag{3.12}$$

From (2.6) and Lemma 1 we obtain that for infinitely many m

$$a_{m+1} > \left(\left(1+\frac{sd}{s+1}\right)\frac{1+\varepsilon}{\varepsilon}+\delta'\right)\sum_{n=1}^{m} a_n \tag{3.13}$$

where δ' is a real number such that $0<\delta'<\limsup_{n\to\infty}\sqrt[n]{a_n}-1-(1+(sd/(s+1))\times(1+\varepsilon)/\varepsilon)$. From (3.13) and the fact that δ is a sufficiently small positive real number we obtain that for infinitely many m

$$\begin{aligned}\frac{\varepsilon}{1+\varepsilon}(s+1-\delta)\,a_{m+1} &> \frac{\varepsilon}{1+\varepsilon}(s+1-\delta)\left(\left(1+\frac{sd}{s+1}\right)\frac{1+\varepsilon}{\varepsilon}+\delta'\right)\sum_{n=1}^{m} a_n\\ &= \left((s+\delta)(d+1)+1+\frac{\varepsilon(s+1)}{2(1+\varepsilon)}\,\delta'\right)\sum_{n=1}^{m} a_n.\end{aligned}$$

From this, (3.12), and the inequality $b_M \leqslant \alpha^{(s+\delta/3)a_M}$ we obtain that for infinitely many m

$$\begin{aligned}\left|x-\frac{\prod_{n=1}^{m}[B_n]}{\left(\prod_{n=1}^{m} b_n\right)\alpha^{\sum_{n=1}^{m} a_n}}\right| &= \left|x-\frac{\prod_{n=1}^{m}[B_n]}{\prod_{n=1}^{m} B_n}\right| \leqslant \frac{1}{\alpha^{(\varepsilon/(1+\varepsilon))(s+1-\delta)a_{m+1}}}\\ &\leqslant \frac{1}{\alpha^{\left((s+\delta)(1+d)+1+\frac{1}{2}\varepsilon\delta'(s+1)/(1+\varepsilon)\right)\sum_{n=1}^{m} a_n}}\\ &= \frac{1}{\alpha^{\left((s+\delta/2)(1+d+\varepsilon')+1+\varepsilon'+(\frac{1}{2}\varepsilon\delta'(s+1)/(1+\varepsilon)+\frac{1}{2}\delta(1+d)-\varepsilon'(1+s+\delta/2))\right)\sum_{n=1}^{m} a_n}}\\ &\leqslant \frac{1}{\left(\prod_{n=1}^{m} b_n\right)^{(1+d+\varepsilon')}\alpha^{(1+\varepsilon')\sum_{n=1}^{m} a_n}}\end{aligned}$$

where ε' is a real number such that $\frac{1}{2}\delta'\varepsilon(s+1)/(1+\varepsilon)+\frac{1}{2}\delta(1+d) > \varepsilon'(1+s+\frac{1}{2}\delta) > 0$. This and Theorem 2 imply that the number x is transcendental. □

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