Ivan Kolář
On special types of nonholonomic 3-jets

Terms of use:

ON SPECIAL TYPES OF NONHOLONOMIC 3-JETS

Ivan Kolář

Abstract. We deduce a classification of all special types of nonholonomic 3-jets. In the introductory part, we summarize the basic properties of nonholonomic r-jets.

Generally speaking, a very attractive phenomenon of the problem of classifying the special types of nonholonomic 3-jets is that its solution is heavily based on the Weil algebra technique, even though no algebras appear in the formulation of the problem. We start with summarizing some properties of classical r-jets from the viewpoint used in the present paper and then we mention the basic properties of nonholonomic r-jets, [2]. The second part of Section 1 is devoted to the categorial approach to the concept of special type of nonholonomic r-jets from [7]. In Section 2 we describe the nonholonomic 3-jets in detail. Section 3 contains our previous classification results concerning nonholonomic 2-jets, [5], semiholonomic 3-jets, [3], and two lemmas on the invariant homomorphisms of the related Weil algebras. Section 4 is devoted to the fundamental algebraic properties of the Weil algebra $\widetilde{\mathbb{D}}_m^3$ corresponding to the nonholonomic 3-jets. In Section 5 we classify those nonholonomic 3-jets that are not one-semiholonomic. The classification list is completed in the last section.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [8].

1. Nonholonomic r-jets. The classical, or holonomic, r-jets $X = j_x^r \varphi$ of smooth maps $\varphi\colon M \to N$ form a fibered manifold $J^r(M, N) \to M \times N$ with respect to the source and target projections $\alpha X = x \in M$ and $\beta X = \varphi(x) \in N$. All r-jets form a category J^r over pointed manifolds (M, x): if $X \in J_x^r(M, N)_y$ and $Z = j_y^r \psi \in J_y^r(N, Q)_z$, then $Z \circ X = j_x^r(\psi \circ \varphi) \in J_x^r(M, Q)_z$. We write $L_{m,n}^r = J_0^r(\mathbb{R}^m, \mathbb{R}^n)_0$. Then

$$L^r = \bigcup_{m,n \in \mathbb{N}} L_{m,n}^r$$

is a category over integers called the skeleton of J^r. Clearly, J^r can be reconstructed from L^r: if $\dim M = m$ and $\dim N = n$, then $J^r(M, N)$ coincides with the

associated bundle

$$J^r(M, N) = (P^r M \times P^r N)[L^r_{m,n}],$$

where $P^r M(M, G^r_m)$ or $P^r N(N, G^r_n)$ is the r-th order frame bundle of M or N, [8]. Further, J^r is a bundle functor of the product category $\mathcal{M}f_m \times \mathcal{M}f$, where $\mathcal{M}f$ denoted the category of all manifolds and all smooth maps and $\mathcal{M}f_m$ is the category of m-dimensional manifolds and local diffeomorphisms. If $f\colon M \to M'$ is a local diffeomorphism and $g\colon N \to N'$ is a smooth map, then

(1) $$J^r(f, g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1} \in J^r(M', N'),$$

[8]. Clearly, J^r preserves products in the second factor, i.e.

$$J^r(M, N_1 \times N_2) = J^r(M, N_1) \times_M J^r(M, N_2),$$

where the fiber product is constructed with respect to the source projection. Moreover, $\varphi\colon M \to N$ defines a map $j^r\varphi\colon M \to J^r(M, N)$, $(j^r\varphi)(x) = j^r_x\varphi$. Then $j^r_x\varphi$ can be identified with $j^1_x(j^{r-1}\varphi)$.

   The bundle $\widetilde{J}^2(M, N)$ of nonholonomic 2-jets of M into N is the space of 1-jets $X = j^1_x f$ of the α-sections of $J^1(M, N)$, i.e. the maps $f\colon M \to J^1(M, N)$ satisfying $\alpha f(u) = u$, $u \in M$. This is a bundle over $M \times N$ with respect to the source projection $\alpha(j^1_x f) = x$ and the target projection $\beta(j^1_x f) = \beta(f(x)) \in N$, where β on the right hand side is the target projection of $J^1(M, N)$. Local coordinates x^i on M and y^p on N induce the additional coordinates y^p_i on $J^1(M, N)$. It will be useful to write $y^p = y^p_0$. So the coordinate expression of an α-section $f(u)$ is $f^p_0(u)$, $f^p_i(u)$. Then x^i and $y^p_{h_1 h_2}$, $h_1, h_2 = 0, 1 \ldots, m$,

(2) $$y^p_{00} = f^p_0(x), \quad y^p_{i0} = f^p_i(x), \quad y^p_{0i} = \frac{\partial f^p_0(x)}{\partial u^i}, \quad y^p_{ij} = \frac{\partial f^p_i(x)}{\partial u^j}$$

are the induced coordinates on $\widetilde{J}^2(M, N)$. The subset $J^2(M, N) \subset \widetilde{J}^2(M, N)$ is characterized by

(3) $$y^p_{i0} = y^p_{0i} \quad \text{and} \quad y^p_{ij} = y^p_{ji}.$$

   We have two projections $\varrho_1, \varrho_2\colon \widetilde{J}^2(M, N) \to J^1(M, N)$, $\varrho_1(j^1_x f) = f(x)$, $\varrho_2(j^1_x f) = j^1_x(\beta \circ f)$. In coordinates, $\varrho_1(X) = (x^i, y^p_{00}, y^p_{i0})$, $\varrho_2(X) = (x^i, y^p_{00}, y^p_{0i})$. We say that $X \in \widetilde{J}^2(M, N)$ is semiholonomic, if $\varrho_1(X) = \varrho_2(X)$, i.e. $y^p_{i0} = y^p_{0i}$. We write $\bar{J}^2(M, N)$ for the bundle of all semiholonomic 2-jets of M into N, By (3), $J^2(M, N) \subset \bar{J}^2(M, N)$.

   Even the nonholonomic 2-jets form a category $\widetilde{J}^2$ over pointed manifolds. For $X = j^1_x f(u) \in \widetilde{J}^2_x(M, N)_y$ and $Z = j^1_y g(v) \in \widetilde{J}^2_y(N, Q)_z$ one defines

(4) $$Z \circ X = j^1_x\big(g(\beta f(u)) \circ f(u)\big) \in \widetilde{J}^2_x(M, Q)_z$$

with the composition of 1-jets on the right hand side. If X and Z are holonomic, then (4) coincides with the classical composition of 2-jets. Indeed, if $X = j^1_x(j^1_u \varphi)$ and $Z = j^1_y(j^1_v \psi)$, then

(5) $$j^1_x\big(j^1_u(\psi \circ \varphi)\big) = j^1_x\big(j^1_{\varphi(u)}\psi \circ j^1_u\varphi\big)$$

and the right hand sides of (4) and (5) are the same.

Assume by induction that we have constructed the bundle $\widetilde{J}^{r-1}(M, N) \to M \times N$ with the source projection $\alpha\colon \widetilde{J}^{r-1}(M, N) \to M$ and the target projection $\beta\colon \widetilde{J}^{r-1}(M, N) \to N$ and $r-1$ canonical projections $\varrho_1, \ldots, \varrho_{r-1}\colon \widetilde{J}^{r-1}(M, N) \to \widetilde{J}^{r-2}(M, N)$. Then we define the bundle of nonholonomic r-jets of M into N to be the space $\widetilde{J}^r(M, N)$ of 1-jets of α-sections $f\colon M \to \widetilde{J}^{r-1}(M, N)$. This is a bundle over $M \times N$ with respect to the source and target projections $\alpha(j_x^1 f) = x$ and $\beta(j_x^1 f) = \beta(f(x)) \in N$ where β on the right hand side means the target projection of $\widetilde{J}^{r-1}(M, N)$.

Assume by induction that $(x^i, y_{h_1 \ldots h_{r-1}}^p)$, $h_1, \ldots, h_{r-1} = 0, 1, \ldots, m$, are the local coordinates on $\widetilde{J}^{r-1}(M, N)$ induced by x^i on M and y^p on N and the coordinate form of ϱ_s is

$$\varrho_s(x^i, y_{h_1 \ldots h_{r-1}}^p) = \left(x^i, y_{h_1 \ldots h_{r-s-1} 0 h_{r-s+1} \ldots h_{r-1}}^p\right), \quad s = 1, \ldots, r-1.$$

The coordinate expression of f is $f_{h_1 \ldots h_{r-1}}(u)$. This induces the coordinates $(x^i, y_{h_1 \ldots h_r}^p)$, $h_1, \ldots, h_r = 0, 1, \ldots, m$ on $\widetilde{J}^r(M, N)$ by

$$(6) \qquad y_{h_1 \ldots h_{r-1} 0}^p = f_{h_1 \ldots h_{r-1}}^p(x), \quad y_{h_1 \ldots h_{r-1} j}^p = \frac{\partial f_{h_1 \ldots h_{r-1}}(x)}{\partial u^j}.$$

We construct r projections $\beta_s\colon \widetilde{J}^r(M, N) \to \widetilde{J}^{r-1}(M, N)$ by

$$(7) \qquad \beta_1(j_x^1 f) = f(x), \quad \beta_{s+1}(j_x^1 f) = j_x^1(\varrho_s \circ f), \qquad s = 1, \ldots, r-1.$$

Hence the coordinate form of β_s is

$$(8) \qquad \beta_s\left(x^i, y_{h_1 \ldots h_r}^p\right) = \left(x^i, y_{h_1 \ldots h_{r-s} 0 h_{r-s+2} \ldots h_r}^p\right).$$

An element $X \in \widetilde{J}^r(M, N)$ is called semiholonomic, if $\beta_1 X = \cdots = \beta_r X$. The bundle of all semiholonomic r-jets is denoted by $\bar{J}^r(M, N)$. Write $\langle h_1 \ldots h_r \rangle = (i_1 \ldots i_s)$ for the subsequence of all nonzero indices and $|h_1 \ldots h_r|$ for the set $\{i_1, \ldots, i_s\}$. Then the elements of $\bar{J}^r(M, N)$ are characterized by

$$y_{h_1 \ldots h_r}^p = y_{l_1 \ldots l_r}^p \quad \text{whenever} \quad \langle h_1 \ldots h_r \rangle = \langle l_1 \ldots l_r \rangle.$$

The inclusion $J^r(M, N) \hookrightarrow \widetilde{J}^r(M, N)$ is defined by

$$(9) \qquad j_x^r \varphi \mapsto j_x^1(j^{r-1}\varphi)$$

Hence the elements of $J^r(M, N) \subset \bar{J}^r(M, N)$ are characterized by

$$y_{h_1 \ldots h_r}^p = y_{l_1 \ldots l_r}^p \quad \text{whenever} \quad |h_1 \ldots h_r| = |l_1 \ldots l_r|.$$

Remark 1. G. Virsik clarified, [12], that the composition of various canonical projections $\widetilde{J}^r(M, N) \to \cdots \to \widetilde{J}^k(M, N)$ define $\binom{r}{k}$ canonical projections $\widetilde{J}^r(M, N) \to \widetilde{J}^k(M, N)$. They are in bijection with the fixations of $r - k$ zeros in the sequence $h_1, \ldots, h_r$. Given a fixation $\varkappa$ of $r - k$ elements from $1, \ldots, r$, the coordinate form of the corresponding projection is $(x^i, y_{h_1 \ldots h_r}^p) \mapsto (x^i y_{\varkappa(h_1 \ldots h_r)}^p)$, where $\varkappa(h_1 \ldots h_r)$ means that we replace the indices at the prescribe places by zeros. Then we can define an element of $\widetilde{J}^r(M, N)$ to be k-semiholonomic, if all its canonical projections into $\widetilde{J}^k(M, N)$ coincide. We write $\widetilde{J}^{r,k}(M, N)$ for the

bundle of all k-semiholonomic r-jets. Clearly, $(r-1)$-semiholonomic means semi-holonomic in the classical sense. In [4] we deduced some geometric properties of one-semiholonomic r-jets.

The composition of $X = j_x^1 f \in \widetilde{J}^r(M, N)_y$ and $Z = j_y^1 g \in \widetilde{J}^r(N, Q)_z$ is defined by

$$(10) \qquad Z \circ X = j_x^1\big(g(\beta f(u)) \circ f(u)\big) \in \widetilde{J}_x^r(M, Q)_z$$

with the composition of nonholonomic $(r-1)$-jets on the right hand side. The associativity of (10) is proved e.g. in [7]. So $\widetilde{J}^r$ is a category over pointed manifolds. Modifying (5), we deduce that for two holonomic r-jets, (10) coincides with the classical composition.

Lemma 1. *Every projection β_s, $s = 1, \ldots, r$, preserves the composition of nonho-lonomic jets.*

Proof. For β_1 this follows directly from (10). Assume by induction that every ϱ_s, $s = 1, \ldots, r-1$, preserves the composition of nonholonomic jets. Then we have

$$\beta_s\big(j_x^1(g(\beta f(u)) \circ f(u))\big) = j_x^1\big(\varrho_{s-1}(g(\beta f(u)) \circ f(u))\big)$$
$$= j_x^1\big(\varrho_{s-1}(g(\beta f(u))) \circ \varrho_{s-1}(f(u))\big) = \beta_s(j_y^1 g) \circ \beta_s(j_x^1 f).$$

In particular, Lemma 1 implies that the composition of two k-semiholonomic r-jets is a k-semiholonomic r-jet.

Even $\widetilde{J}^r$ can be interpreted as a functor on $\mathcal{M}f_m \times \mathcal{M}f$ by

$$(11) \qquad \widetilde{J}^r(f, g)(X) = (j_y^r g) \circ X \circ (j_x^r f)^{-1}$$

with the composition of nonholonomic r-jets. Using induction, one verifies directly

$$\widetilde{J}^r(M, N_1 \times N_2) = \widetilde{J}^r(M, N_1) \times_M \widetilde{J}^r(M, N_2).$$

The same holds in the k-semiholonomic case. Write

$$\widetilde{L}_{m,n}^r = \widetilde{J}_0^r(\mathbb{R}^m, \mathbb{R}^n)_0, \quad \widetilde{L}^r = \bigcup_{m,n \in \mathbb{N}} \widetilde{L}_{m,n}^r.$$

Then $\widetilde{L}^r$ is a category over integers, called the skeleton of $\widetilde{J}^r$. Analogously to the holonomic case,

$$\widetilde{J}^r(M, N) = (P^r M \times P^r N)[\widetilde{L}_{m,n}^r].$$

We recall that $X \in \widetilde{J}_x^r(M, N)_y$ is said to be regular, if there exists $Z \in \widetilde{J}_y^r(N, M)_x$ such that $Z \circ X = j_x^r \operatorname{id}_M$. In [7], we introduced the following concept. $\qquad \square$

Definition 1. A nonholonomic r-jet category C is a rule transforming every pair (M, N) of manifolds into a fibered submanifold $C(M, N) \subset \widetilde{J}^r(M, N)$ such that

(i) $J^r(M, N) \subset C(M, N)$ is fibered submanifold,

(ii) if $X \in C_x(M, N)_y$ and $Z \in C_y(N, Q)_z$, then $Z \circ X \in C_x(M, Q)_z$,

(iii) if X is regular in $\widetilde{J}^r$, then there exists $Z \in C_y(N, M)_x$ such that $Z \circ X = j_x^r \operatorname{id}_M$,

(iv) $C(M, N_1 \times N_2) = C(M, N_1) \times_M C(M, N_2)$.

Write

$$L^C_{m,n} = C_0(\mathbb{R}^m, \mathbb{R}^n)_0 \subset \widetilde{L}^r_{m,n} \quad \text{and} \quad L^C = \bigcup_{m,n\in\mathbb{N}} L^C_{m,n} \,.$$

This skeleton is a subcategory of $\widetilde{L}^r$ and we have

$$C(M, N) = (P^r M \times P^r N)[L^C_{m,n}] \,.$$

According to [7] and [3], the special types of nonholonomic r-jets are identified with the nonholonomic r-jet categories.

2. Nonholonomic 3-jets. We summarize some facts and formulae concerning nonholonomic 2-jets and 3-jets, that will be applied to our classification problem.

We have three projections $\beta_1, \beta_2, \beta_3 \colon \widetilde{J}^3(M, N) \to \widetilde{J}^2(M, N)$ and two projections $\varrho_1, \varrho_2 \colon \widetilde{J}^2(M, N) \to J^1(M, N)$. According to Remark 1, this gives rise to three projections $\gamma_1, \gamma_2, \gamma_3 \colon \widetilde{J}^3(M, N) \to J^1(M, N)$, $\gamma_1 = \varrho_1 \circ \beta_1 = \varrho_1 \circ \beta_2$, $\gamma_2 = \varrho_2 \circ \beta_1 = \varrho_1 \circ \beta_3$, $\gamma_3 = \varrho_2 \circ \beta_2 = \varrho_2 \circ \beta_3$.

It is sufficient to write the related coordinate expressions on $\widetilde{L}^2_{m,n}$ and $\widetilde{L}^3_{m,n}$. The canonical coordinates of $X \in \widetilde{L}^2_{m,n}$ are $(y^p_{i0}, y^p_{0i}, y^p_{ij})$ and $\varrho_1(X) = (y^p_{i0})$, $\varrho_2(X) = (y^p_{0i})$. If $Z \in \widetilde{L}^2_{n,q}$, $Z = (z^a_{p0}, z^a_{0p}, z^a_{pq})$, then the coordinates $(v^a_{i0}, v^a_{0i}, v^a_{ij})$ of $V = Z \circ X \in \widetilde{L}^2_{m,q}$ are

$$(12) \qquad v^a_{i0} = z^a_{p0} y^p_{i0} \,, \quad v^a_{0i} = z^a_{0p} y^p_{0i} \,, \quad v^a_{i0} = z^a_{pq} y^p_{i0} y^q_{0j} + z^a_{p0} y^p_{ij} \,.$$

In the third order, the coordinate expression of $X \in \widetilde{L}^3_{m,n}$ is $(y^p_{i00}, y^p_{0i0}, y^p_{00i}, y^p_{ij0},$ $y^p_{i0j}, y^p_{0ij}, y^p_{ijk})$. Then

$$\begin{aligned}
\beta_1 X &= (y^p_{i00}, y^p_{0i0}, y^p_{ij0}) \,, \qquad \beta_2 X = (y^p_{i00}, y^p_{00i}, y^p_{i0j}) \,, \\
\beta_3 X &= (y^p_{0i0}, y^p_{00i}, y^p_{0ij}) \,,
\end{aligned}$$

$$(13)$$

$$\gamma_1 X = (y^p_{i00}) \,, \quad \gamma_2 X = (y^p_{0i0}) \,, \quad \gamma_3 X = (y^p_{00i}) \,.$$

If we express $Z \in \widetilde{L}^3_{n,q}$ and $V \in \widetilde{L}^3_{n,q}$ analogously as above, then the coordinate expression of $V = Z \circ X$ is

$$(14) \qquad v^a_{i00} = z^a_{p00} y^p_{i00} \,, \quad v^a_{0i0} = z^a_{0p0} y^p_{0i0} \,, \quad v^a_{00i} = z^a_{00p} y^p_{00i} \,,$$

$$(15) \qquad \begin{aligned}
v^a_{ij0} &= z^a_{pq0} y^p_{i00} y^q_{0j0} + z^a_{p00} y^p_{ij0} \,, \quad v^a_{i0j} = z^a_{p0q} y^p_{i00} y^q_{00j} + z^a_{p00} y^p_{i0j} \\
v^a_{0ij} &= z^a_{0pq} y^p_{0i0} y^q_{00j} + z^a_{0p0} y^p_{0ij}
\end{aligned}$$

$$(16) \qquad \begin{aligned}
v^a_{ijk} &= z^a_{pqr} y^p_{i00} y^q_{0j0} y^r_{00k} + z^a_{pq0} (y^p_{i0k} y^q_{0j0} + y^p_{i00} y^q_{0jk}) \\
&\quad + z^a_{p0q} y^p_{ij0} y^q_{00k} + z^a_{p00} y^p_{ijk} \,.
\end{aligned}$$

Now we describe the related Weil algebras. In general, a Weil algebra is a finite dimensional, commutative, associative and unital algebra of the form $A = \mathbb{R} \times N$, where N is the ideal of all nilpotent elements, [6]. There exists an integer r such

that $N^{r+1} = 0$, the smallest r is called the order of A. The multiplication in A is determined by the multiplication in N, for

$$(x_1 + n_1)(x_2 + n_2) = x_1 x_2 + x_1 n_2 + x_2 n_1 + n_1 n_2 , \quad x_1, x_2 \in \mathbb{R}, \ n_1, n_2 \in N .$$

According to the general theory, $\widetilde{D}_m^r = \widetilde{J}_0^r(\mathbb{R}^m, \mathbb{R}) = \mathbb{R} \times \widetilde{N}_m^r$ is a Weil algebra, [6]. Clearly, $\widetilde{N}_m^r = \widetilde{L}_{m,1}^r$. So $Z \in \widetilde{N}_m^3$ is of the form

$$(17) \qquad\qquad Z = (z_{i00}, z_{0i0}, z_{00i}, z_{ij0}, z_{i0j}, z_{0ij}, z_{ijk}) .$$

The algebra multiplication in $\widetilde{\mathbb{D}}_m^r$ is induced by the multiplication of reals, [6]. This implies that the product Z of $X, Y \in \widetilde{N}_m^3$ is of the form

$$(18) \qquad \begin{aligned} &z_{i00} = 0 , \quad z_{0i0} = 0 , \quad z_{00i} = 0 , \quad z_{ij0} = x_{i00}y_{0j0} + y_{i00}x_{0j0} \\ &z_{i0j} = x_{i00}y_{00j} + y_{i00}x_{00j} , \quad z_{0ij} = x_{0i0}y_{00j} + y_{0i0}x_{00j} , \end{aligned}$$

$$(19) \qquad z_{ijk} = x_{0jk}y_{i00} + x_{0j0}y_{i0k} + y_{0jk}x_{i00} + y_{0j0}x_{i0k} + x_{00k}y_{ij0} + y_{00k}x_{ij0} .$$

The composition of 3-jets defines a right action of the group $G_m^3 = \operatorname{inv} J_0^3(\mathbb{R}^m, \mathbb{R}^m)_0$ on $\widetilde{\mathbb{D}}_m^3$. Its coordinate expression can be obtained by specifying (14)–(16).

Clearly, $\mathbb{D}_m^3 = J_0^3(\mathbb{R}^m, \mathbb{R}) = \mathbb{R} \times N_m^3$ and $\bar{\mathbb{D}}_m^3 = \bar{J}_0^3(\mathbb{R}^m, \mathbb{R}) = \mathbb{R} \times \bar{N}_m^3$ are G_m^3-invariant subalgebras of $\widetilde{\mathbb{D}}_m^3$. Consider the canonical injection $i_{m,n} \colon \mathbb{R}^m \to \mathbb{R}^{m+n}$, $x \mapsto (x,0)$ and write $I_{m,n}^3 = j_0^3 i_{m,n}$. The rule $Z \to Z \circ I_{m,n}^3$ defines an algebra epimorphism $\widetilde{\mathbb{D}}_{m+n}^3 \to \widetilde{\mathbb{D}}_m^3$, whose coordinate expression is

$$(20) \qquad\qquad \bar{z}_{h_1 h_2 h_3} = z_{h_1 h_2 h_3}$$

with no appearence of $z_{q_1 q_2 q_3}$ with at least one q_s greater then m, $q_s = 0, 1, \ldots, m+n$, $s = 1, 2, 3$ on the right hand side.

3. Some previous classification results. In [5] we deduced that the only non-trivial G_m^2-invariant subalgebra of $\widetilde{\mathbb{D}}_m^2$ containing $\mathbb{D}_m^2$ is $\bar{\mathbb{D}}_m^2$. This implies directly that the only nonholonomic 2-jet categories are $\widetilde{J}^2$, $\bar{J}^2$ and J^2, [3].

In [3] we determined all semiholonomic 3-jet categories C, i.e. $C(M, N) \subset \bar{J}^3(M, N)$. We write $\bar{J}^{3,2}(M, N) \subset \bar{J}^3(M, N)$ for the semiholonomic 3-jets holonomic in the second order. For every sixtuple $d = (d_1, d_2, d_3, d_4, d_5, d_6)$ of reals satisfying $d_1 + \cdots + d_6 = 1$, we introduce $\bar{L}_{m,n}^d = \{(y_i^p, y_{ij}^p = y_{ji}^p, d_1 y_{ijk}^p + d_2 y_{ikj}^p + d_3 y_{jik}^p + d_4 y_{jki}^p + d_5 y_{kij}^p + d_6 y_{kji}^p)\} \subset \bar{L}_{m,n}^{3,2} = \bar{J}_0^{3,2}(\mathbb{R}^m, \mathbb{R}^n)_0$. Then

$$\bar{L}^d = \bigcup_{m,n \in \mathbb{N}} \bar{L}_{m,n}^d \quad \text{is a subcategory of} \quad \bar{L}^{3,2} = \bigcup_{m,n \in \mathbb{N}} \bar{L}_{m,n}^{3,2} ,$$

which defines $\bar{J}^d(M, N) \subset \bar{J}^{3,2}(M, N)$. Our result from [3] is: All semiholonomic 3-jet categories are $\bar{J}^3$, $\bar{J}^{3,2}$, J^3 and $\bar{J}^d$ for all d.

In [8], Section 32, we deduced certain properties of second order jet functors, that we now reformulate on the algebra level.

Lemma 2. *The only invariant algebra epimorphism $\mathbb{D}_m^2 \to \mathbb{D}_m^2$ is the identity. The only invariant algebra epimorphism $\widetilde{\mathbb{D}}_m^2 \to \widetilde{\mathbb{D}}_m^2$ transforming $\mathbb{D}_m^2$ into $\mathbb{D}_m^2$ is*

the identity. All invariant algebra epimorphisms $\bar{\mathbb{D}}_m^2 \to \bar{\mathbb{D}}_m^2$ *form the one parameter family* φ_t

$$(21) \qquad \bar{z}_i = z_i\,, \quad \bar{z}_{ij} = tz_{ij} + (1-t)z_{ji}\,, \quad t \in \mathbb{R}\,,\ z_i = z_{i0} = z_{0i}\,.$$

Further, the geometric results from [10] can be reformulated as follows.

Lemma 3. *There is no invariant algebra epimorphism* $\widetilde{\mathbb{D}}_m^2 \to \bar{\mathbb{D}}_m^2$ *or* $\widetilde{\mathbb{D}}_m^2 \to \mathbb{D}_m^2$. *The only invariant algebra epimorphism* $\sigma\colon \bar{\mathbb{D}}_m^2 \to \mathbb{D}_m^2$ *is the symmetrization* $(z_i, z_{ij}) \mapsto (z_i, \frac{1}{2}(z_{ij} + z_{ji}))$.

At the bundle level, σ maps $\bar{J}^2(M, N)$ into $J^2(M, N)$. We remark that σ preserves the jet composition: if $X \in \bar{J}_x^2(M, N)_y$, $Z \in \bar{J}_y^2(N, Q)_z$, then $\sigma(X) \in J_x^2(M, N)_y$, $\sigma(Z) \in J_y^2(N, Q)_z$ and $\sigma(Z \circ X) = \sigma(Z) \circ \sigma(X)$.

Consider a nonholonomic 3-jet category C. Then C defines a bundle functor C_m on $\mathcal{M}f_m \times \mathcal{M}f$. According to the general theory, $\mathbb{D}_m^C = C_0(\mathbb{R}^m, \mathbb{R})$ is a G_m^3-invariant Weil algebra satisfying $\mathbb{D}_m^3 \subset \mathbb{D}_m^C \subset \widetilde{\mathbb{D}}_m^3$, [7], [3]. We can reconstruct C from the sequence $\mathbb{D}_m^C = \mathbb{R} \times N_m^C$ by setting $L_{m,n}^C = (N_m^C)^n$.

Thus the problem of finding all nonholonomic 3-jet categories can be divided into 3 steps.

    I. We determine all invariant subalgebras $\mathbb{D}_m^3 \subset A \subset \widetilde{\mathbb{D}}_m^3$.

    II. We consider sequences of them $\mathbb{D}_m^S = \mathbb{R} \times N_m^S$ satisfying (20).

    III. We define $L_{m,n}^S = (N_m^S)^n$ and we discuss whether $L^S = \bigcup\limits_{m,n \in \mathbb{N}} L_{m,n}^S$ is a subcategory of $\widetilde{L}^3$.

We point out that in all concrete cases mentioned in this sections, condition III is automatically satisfied. However, in Section 5 below we meet a case in which this is not true. This clarifies that III is independent of I and II.

4. The Weil algebra $\widetilde{\mathbb{D}}_m^3$**.** In general, every Weil algebra $A = \mathbb{R} \times N$ of order r defines the underlying Weil algebra $A_k = A/N^{k+1}$ in every order $k \leq r$. Every algebra homomorphism $\mu\colon A \to B$ induces the underlying algebra homomorphism $\mu_k\colon A_k \to B_k$, [6]. We have

$$(\widetilde{\mathbb{D}}_m^3)_2 =: \mathbb{B}_m^{3,2} = \{(X_1, X_2, X_3) \in \widetilde{\mathbb{D}}_m^2 \times \widetilde{\mathbb{D}}_m^2 \times \widetilde{\mathbb{D}}_m^2, \varrho_1 X_1 = \varrho_1 X_2,$$
$$(22) \qquad\qquad \varrho_2 X_1 = \varrho_1 X_3, \varrho_2 X_2 = \varrho_2 X_3\}\,.$$

The injection $\mathbb{D}_m^2 \hookrightarrow \mathbb{B}_m^{3,2}$ is $X \mapsto (X, X, X)$. We write $\mathbb{B}_m^{3,2} = \mathbb{R} \times N_m^{3,2}$. Hence the coordinates on $N_m^{3,2}$ are

$$(23) \qquad\qquad (z_{i00}, z_{0i0}, z_{00i}, z_{ij0}, z_{i0j}, z_{0ij})\,.$$

We have $\widetilde{\mathbb{D}}_m^3 = \mathbb{B}_m^{3,2} \times \overset{3}{\bigotimes} \mathbb{R}^{m*}$. The right action of G_m^3 on $\widetilde{\mathbb{D}}_m^3$ induces a right action of G_m^2 on $\mathbb{B}_m^{3,2}$. We define $\widetilde{\mathbb{D}}_m^{3,1} = \widetilde{J}_0^{3,1}(\mathbb{R}^m, \mathbb{R})$. This is an invariant subalgebra of $\widetilde{\mathbb{D}}_m^3$, that is characterized by $z_{i00} = z_{0i0} = z_{00i}$.

A subalgebra $\mathbb{D}_m^3 \subset A \subset \widetilde{\mathbb{D}}_m^3$ or $\mathbb{D}_m^2 \subset B \subset \mathbb{B}_m^{3,2}$ will be called admissible, if it is G_m^3-invariant or G_m^2-invariant, respectively. Write $\delta\colon \widetilde{\mathbb{D}}_m^3 \to \mathbb{B}_m^{3,2}$ for the factor

projection. We say that $\delta(A)$ is the boundary of A. Consider the canonical injection $GL(m, \mathbb{R}) = G_m^1 \hookrightarrow G_m^3$, $a_j^i \mapsto (a_j^i, 0, 0)$. Using (10), we find immediately

Lemma 4. *If $A \subset \widetilde{\mathbb{D}}_m^3$ is admissible, then the kernel K_A of $\delta \mid A$ is a G_m^1-invariant subspace of $\overset{3}{\bigotimes} \mathbb{R}^{m*}$ containing $S^3 \mathbb{R}^{m*}$.*

Consider an admissible subalgebra $B \subset \mathbb{B}_m^{3,2}$. By (14)–(19) we obtain directly

Lemma 5. *$B \times \overset{3}{\bigotimes} \mathbb{R}^{m*} \subset \mathbb{B}_m^{3,2} \times \overset{3}{\bigotimes} \mathbb{R}^{m*}$ is an admissible subalgebra of $\widetilde{\mathbb{D}}_m^3$.*

We write $B \times \overset{3}{\bigotimes} \mathbb{R}^{m*} = [B]$ and we say that $[B]$ is the inverse of B. So every admissible subalgebra $B \subset \mathbb{B}_m^{3,2}$ is the boundary of an admissible subalgebra of $\widetilde{\mathbb{D}}_m^3$. Thus, we will be interested in the admissible subalgebras of $\mathbb{B}_m^{3,2}$.

Remark 2. The general problem of the underlying lower order bundle functors was studied by M. Doupovec, [1]. In general, every Weil algebra of order r and a group homomorphism $H \colon G_m^r \to \operatorname{Aut} A$ defines a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ by $F(M, N) = P^r M[T^A N, H_N]$, where H_N is the action of G_m^r on the Weil bundle $T^A N$ determined by the natural transformations corresponding to the algebra automorphisms $H(v) \colon A \to A$, $v \in G_m^r$, and $F(f, g) = P^r f[T^A g] \colon P^r M[T^A N] \to P^r M'[T^A N']$, $f \colon M \to M'$, $g \colon N \to N'$. Then the bundle functor $(\delta \widetilde{J}^3)$ corresponding to $\mathbb{B}_m^{3,2}$ is determined by the generalized fiber product

$$(\delta \widetilde{J}^3)(M, N) = \{X_1, X_2, X_3 \in \widetilde{J}^2(M, N), \varrho_1 X_1 = \varrho_1 X_2,$$
$$\varrho_2 X_1 = \varrho_1 X_3, \varrho_2 X_2 = \varrho_2 X_3\}$$

and $\delta \colon \widetilde{\mathbb{D}}_m^3 \to \mathbb{B}_m^{3,2}$ induces a projection

$$\delta_{M,N} \colon \widetilde{J}^3(M, N) \to (\delta \widetilde{J}^3)(M, N), \quad \delta_{M,N}(X) = (\beta_1 X, \beta_2 X, \beta_3 X),$$

[1]. Every admisssible subalgebra $B \subset \mathbb{B}_m^{3,2}$ defines a subbundle $P^2 M[T^B N] \subset (\delta \widetilde{J}^3)(M, N)$. The subbundle of $\widetilde{J}^3(M, N)$ determined by $[B]$ is the inverse image of $P^2 M[T^B N] \subset (\delta \widetilde{J}^3)(M, N)$ with respect to $\delta_{M,N}$.

Let $B = \mathbb{R} \times N_B$ be an admissible subalgebra of $\mathbb{B}_m^{3,2}$. Write π_s, $s = 1, 2, 3$, for the projections of B into the individual components of (22). By invariancy and Section 3, $\pi_s(B)$ is $\widetilde{\mathbb{D}}_m^2$ or $\bar{\mathbb{D}}_m^2$ or $\mathbb{D}_m^2$, $s = 1, 2, 3$. We write D_s for any of them, if suitable. We say that the pair (π_r, π_s), $r \neq s$, is projectable, if there exists an induced map $\psi \colon D_r \to D_s$ satisfying $\psi \circ \pi_r = \pi_s$. By (14)–(16), one deduced easily

Lemma 6. *In the projectable case, ψ is an G_m^2-invariant algebra epimorphism transforming $\mathbb{D}_m^2$ into $\mathbb{D}_m^2$.*

Let $A \subset \widetilde{\mathbb{D}}_m^3$ be an admissible subalgebra with $\delta A = B$. Consider $B_1 = B/N_B^2 = \mathbb{R} \times N_1$. If $\xi \in N_B^2$ and $\eta \in N_1$, then

$$\xi = x + K_A, \quad \eta = y + N_B^2, \quad x \in (\widetilde{N}_m^3)^2, \quad y \in \widetilde{N}_m^3 \quad \text{and}$$

$$\xi\eta = (x + K_A)(y + N_B^2) = xy \in \bigotimes^3 \mathbb{R}^{m*}.$$

Definition 2. The linear span of all these elements will be denoted by Q_B.

Hence Q_B is a linear subspace of $\bigotimes^3 \mathbb{R}^{m*}$ that is determined by B only. Clearly, $Q_B \subset K_A$. According to the definition, the coordinate expression of Q_B is

$$(24) \qquad z_{ijk} = x_{0jk}y_{i00} + x_{0j0}y_{i0k} + y_{0jk}x_{i00} + y_{0j0}x_{i0k} + x_{00k}y_{ij0} + y_{00k}x_{ij0},$$

$x, y \in N_B$. In particular, $S^3 \mathbb{R}^{m*} \subset Q_B$.

Using (14)–(19), we verify easily

Lemma 7. $B \times Q_B$ is an admissible subalgebra of $\widetilde{\mathbb{D}}_m^3$.

Example 1. For $B = \mathbb{B}_m^{3,2}$, we have $Q_B = \bigotimes^3 \mathbb{R}^{m*}$. Indeed, the x's and y's are arbitrary. For $x_{ij0} = 1$, $y_{00k} = 1$ and all zeros otherwise, we obtain that every $z_{ijk} = 1$, all zeros otherwise, belongs to Q_B. Since Q_B is a linear subspace, we have $Q_B = \bigotimes^3 \mathbb{R}^{m*}$.

Example 2. In the case of $B = (\mathbb{D}_m^2, \widetilde{\mathbb{D}}_m^2, \widetilde{\mathbb{D}}_m^2)$ with $D_2 = D_3$, we have $Q_B = S^2\mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$. Indeed, our conditions are $z_{i00} = z_{0i0}$, $z_{ij0} = z_{ji0}$, $z_{i0j} = z_{0ij}$. This implies that (24) is symmetric in i and j.

So, if we start from an admissible subalgebra $B \subset \mathbb{B}_m^{3,2}$, every admissible subalgebra $A \subset \widetilde{\mathbb{D}}_m^3$ with $\delta A = B$ must satisfy

$$S^3\mathbb{R}^{m*} \subset Q_B \subset K_A \subset \bigotimes^3 \mathbb{R}^{m*},$$

where K_A is a G_m^1-invariant linear subspace. In all cases that we shall meet in the sequel, this implies $K_A = Q_B$ or $K_A = \bigotimes^3 \mathbb{R}^{m*}$.

5. Strongly nonholonomic 3-jets. A nonholonomic 3-jet will be called strongly nonholonomic, if it is not one-semiholonomic.

In [5], we deduced: If $A \subset \widetilde{\mathbb{D}}_m^3$ is an admissible subalgebra such that $\gamma_s \mid A$, $s = 1, 2, 3$ do not lie on the same straight line in $\mathrm{Hom}\,(A, \mathbb{D}_m^1)$, then $A = \widetilde{\mathbb{D}}_m^3$. Take $a = (a_1, a_2, a_3) \neq 0$ with $a_1 + a_2 + a_3 = 0$ and define $\widetilde{\mathbb{D}}_m^a = \{Z \in \widetilde{\mathbb{D}}_m^3, a_1 z_{i00} + a_2 z_{0i0} + a_3 z_{00i} = 0\}$. Using (14)–(16) and (18), one verifies directly that this is an admissible subalgebra. (Hence $\widehat{\mathbb{D}}_m^a = \mathbb{R} \times \widetilde{N}_m^a$ defines a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$, [5].) Write

$$\widetilde{L}_{m,n}^a = (\widetilde{N}_m^a)^n \quad \text{and} \quad \widetilde{L}^a = \bigcup_{m,n \in \mathbb{N}} \widetilde{L}_{m,n}^a.$$

Lemma 8. $\widetilde{L}^a$ is a subcategory of $\widetilde{L}^3$ if and only if $z_{i00} = z_{0i0}$ or $z_{i00} = z_{00i}$ or $z_{0i0} = z_{00i}$.

Proof. We start with the case $a_1 \neq 0$, so that

$$z_{i00} = bz_{0i0} + cz_{00i} \qquad b + c = 1 \,.$$

Then the elements of $\widetilde{L}^a_{m,n}$ satisfy $y^p_{i00} = by^p_{0i0} + cy^p_{00i}$. In the notation of Section 2, we also have $z^a_{p00} = bz^a_{0p0} + cz^a_{00p}$ and $v^a_{i00} = bv^a_{0i0} + cv^a_{00i}$. Then (14) implies $v^a_{i00} = (bz^a_{0p0} + cz^a_{00p})(by^p_{0i0} + cy^p_{00i}) = b^2 v^a_{0i0} + bcz^a_{0p0}y^p_{00i} + bcz^a_{00p}y^p_{0i0} + c^2 v^a_{00i}$. This yields $bc = 0$, $b^2 = b$, $c^2 = c$, $b + c = 1$. Hence $b = 1$, $c = 0$ or $b = 0$, $c = 1$. If we start with $a_2 \neq 0$ or $a_3 \neq 0$, we obtain the remaining possibility $z_{0i0} = z_{00i}$. $\qquad\square$

In general, we write $D_{12} = \{(X_1, X_2) \in D_1 \times D_2, \varrho_1 X_1 = \varrho_1 X_2\}$. We define $\pi_{12} \colon B \to D_{12}$, $X \mapsto (\pi_1 X, \pi_2 X)$. Since the algebras $\pi_s(B)$ can be $\widetilde{\mathbb{D}}^2_m$, $\bar{\mathbb{D}}^2_m$ and $\mathbb{D}^2_m$ only, we shall indicate them by writing simply n, s and h in the expression for B of the form (22).

Lemma 9. *Let $B \subset (n, n, s)$ be an admissible subalgebra that is not projectable over π_1, $\pi_2 \colon B \to \widetilde{\mathbb{D}}^2_m$. Then $\pi_{12} \colon B \to D_{12}$ is surjective. The same holds in the cases of (n, n, h), (s, n, n) and (h, n, n).*

Proof. Consider $(x_{i00}, x_{0i0} = x_{00i}, x_{ij0}, x_{i0j}, x_{0ij}) \in N_B$. In the nonprojectable case, there exist x, $y \in B$ such that $\pi_1(x) = \pi_1(y)$ and $\pi_2(x) \neq \pi_2(y)$. Hence y is of the form

$$(x_{i00}, x_{0i0} = x_{00i}, x_{ij0}, x_{i0j} + w_{i0j}, x_{0ij} + w_{0ij}) \quad \text{with} \quad (w_{i0j}) \neq o \,.$$

By continuity, this holds on a neighbourhood of y. Hence π_{12} is surjective on an open subset. But π_{12} is a linear map, so that π_{12} is surjective everywhere. The cases (n, n, h), (s, n, n) and (h, n, n) are discussed in the same way. $\qquad\square$

In what follows, the inverses satisfying some additional conditions are indicated by adding them into the square bracket. Further, in the case of $Q_B \neq \overset{3}{\bigotimes}\mathbb{R}^{m*}$, we write all data determining A into a round bracket.

Proposition 1. *All strongly nonholonomic 3-jet categories are determined by the inverses*

(i)	$[n, n, n]$,	*which corresponds to*	$\widetilde{J}^3$,
(ii)	$[s, n, n]$,	—„—	$\beta_1^{-1}(\bar{J}^2)$,
(iii)	$[n, s, n]$,	—„—	$\beta_2^{-1}(\bar{J}^2)$,
(iv)	$[n, n, s]$,	—„—	$\beta_3^{-1}(\bar{J}^2)$,
(v)	$[s, n, n, \pi_2 = \pi_3]$,	—„—	$J^1 \bar{J}^2$,
(vi)	$[n, n, s, \pi_1 = \pi_2]$,	—„—	$\bar{J}^2 J^1$,
(vii)	$[h, n, n]$,	—„—	$\beta_1^{-1}(J^2)$,
(viii)	$[n, h, n]$,	—„—	$\beta_2^{-1}(J^2)$,
(ix)	$[n, n, h]$,	—„—	$\beta_3^{-1}(J^2)$,
(x)	$[h, n, n, \pi_2 = \pi_3]$,	—„—	$\beta_1^{-1}(J^2) \cap J^1 \bar{J}^2$,
(xi)	$[n, n, h, \pi_1 = \pi_2]$,	—„—	$\beta_3^{-1}(J^2) \cap \bar{J}^2 J^1$

and there are two more cases with nontrivial Q_B

(xii)	$(h, n, n, \pi_2 = \pi_3, z_{ijk} = z_{jik})$	*corresponding to* $J^1 J^2$,
(xiii)	$(n, n, h, \pi_1 = \pi_2, z_{ijk} = z_{ikj})$	*corresponding to* $J^2 J^1$.

Proof. By the above mentioned result from [5] and Lemma 8, the only nonholonomic 3-jet category corresponding to an algebra A with $\beta_1(A) = \widetilde{\mathbb{D}}_m^2$, $\beta_2(A) = \widetilde{\mathbb{D}}_m^2$ and $\beta_3(A) = \widetilde{\mathbb{D}}_m^2$ is $\widetilde{J}^3$. By Lemmas 2 and 9, the cases (ii)–(xi) correspond to all possible inverses in this situation. We point out that the case $[n, s, n, \pi_1 = \pi_3]$ is characterized by $(z_{i00}, z_{0i0}, z_{ij0}) = (z_{0i0}, z_{00i}, z_{0ij})$, so that it is one-semiholonomic.

In the cases (ii)–(ix), we deduce, analogously to Example 1, that $Q_B = \overset{3}{\otimes}\mathbb{R}^{m*}$ is the only possibility. The case (x) was discussed in Example 2. The conditions characterizing (xi) are $(z_{i00}, z_{0i0}, z_{ij0}) = (z_{i00}, z_{00i}, z_{i0j})$ and $z_{0ij} = z_{0ji}$. If we put these data into Example 2, we obtain $z_{ijk} = z_{ikj}$. One verifies directly that all algebras in question determine subcategories of $\widetilde{L}^3$. $\qquad\square$

6. Properly one-semiholonomic 3-jets. It remains to discuss the 3-jets that are properly one-semiholonomic, i.e. not semiholonomic. We write $\mathbb{B}_m^{3,1} \subset \mathbb{B}_m^{3,2}$ for the subset $z_{i00} = z_{0i0} = z_{00i}$.

Lemma 10. *Let $B \subset \mathbb{B}_m^{3,1}$ be an admissible subalgebra that is not projectable over π_1, $\pi_2\colon B \to \bar{\mathbb{D}}_m^2$. Then $\pi_{12}\colon B \to D_{12}$ is surjective. The same holds in the case π_1, $\pi_2\colon B \to \mathbb{D}_m^2$.*

The proof is a replica of the proof of Lemma 9.

The description of the inverses in Propositions 2–5 are quite analogous to Proposition 1, so that we do not express them explicitly.

Proposition 2. *In the case of three s's, we have the categories determined by the inverses*
(i) $[s, s, s]$,
(ii) $[s, s, s, \pi_2 = \varphi_t \circ \pi_3]$,
(iii) $[s, s, s, \pi_1 = \varphi_t \circ \pi_3]$,
(iv) $[s, s, s, \pi_1 = \varphi_t \circ \pi_2]$,
(v) $[s, s, s, \pi_2 = \varphi_t \circ \pi_1, \pi_3 = \varphi_\tau \circ \pi_1]$, $t \neq 1$ *or* $\tau \neq 1$.

Proof. This is a direct consequence of Lemma 2 and the evaluations similar to Example 1. In the case $\pi_1 = \pi_2$ and $\pi_2 = \pi_3$ we obtain $\bar{J}^3$. To verify that all algebras in question determine subcategories of $\widetilde{L}^3$, one uses the following geometric fact. At the bundle level, φ_t induces a map $\varphi_t\colon \bar{J}^2(M, N) \to \bar{J}^2(M, N)$ for every t. These maps preserve the jet composition: if $X \in \bar{J}_x^2(M, N)_y$ and $Z \in \bar{J}_y^2(N, Q)_z$, then $\varphi_t(Z \circ X) = \varphi_t(Z) \circ \varphi_t(X)$, $t \in \mathbb{R}$. $\qquad\square$

The proofs of Propositions 3–5 are quite analogous to Propositions 1 and 2.

Proposition 3. *In the case of two s's, we have the categories determined by the inverses*

(i)	$[h, s, s]$,	(vii)	$[s, s, h, \pi_3 = \sigma \circ \pi_2]$,
(ii)	$[h, s, s, \pi_1 = \sigma \circ \pi_2]$,	(viii)	$[s, s, h, \pi_1 = \varphi_t \circ \pi_2]$,
(iii)	$[h, s, s, \pi_1 = \sigma \circ \pi_3]$,	(ix)	$[s, h, s]$,
(iv)	$[h, s, s, \pi_2 = \varphi_t \circ \pi_3]$,	(x)	$[s, h, s, \pi_2 = \sigma \circ \pi_1]$,
(v)	$[s, s, h]$,	(xi)	$[s, h, s, \pi_2 = \sigma \circ \pi_3]$,
(vi)	$[s, s, h, \pi_3 = \sigma \circ \pi_1]$,	(xii)	$[s, h, s, \pi_1 = \varphi_t \circ \pi_3]$

and there are two more cases with nontrivial Q_B

(xiii) $(h, s, s, \pi_2 = \pi_3, z_{ijk} = z_{jik})$ *corresponding to* $J^1 J^2 \cap \widetilde{J}^{3,1}$,

(xiv) $(s, s, h, \pi_1 = \pi_2, z_{ijk} = z_{ikj})$ *corresponding to* $J^2 J^1 \cap \widetilde{J}^{3,1}$.

Proposition 4. *In the case of two h's, we have the categories determined by the inverses*

(i)	$[s, h, h]$,	(vii)	$[h, h, s, \pi_1 = \sigma \circ \pi_3]$,
(ii)	$[s, h, h, \pi_2 = \pi_3]$,	(viii)	$[h, h, s, \pi_2 = \sigma \circ \pi_3]$,
(iii)	$[s, h, h, \pi_2 = \sigma \circ \pi_1]$,	(ix)	$[h, s, h]$,
(iv)	$[s, h, h, \pi_3 = \sigma \circ \pi_1]$,	(x)	$[h, s, h, \pi_1 = \pi_3]$,
(v)	$[h, h, s]$,	(xi)	$[h, s, h, \pi_1 = \sigma \circ \pi_2]$,
(vi)	$[h, h, s, \pi_1 = \pi_2]$,	(xii)	$[h, s, h, \pi_3 = \sigma \circ \pi_2]$.

In the following assertion, $(\beta_2 = \beta_3)^{-1}(J^2)$ or $(\beta_1 = \beta_2)^{-1}(J^2)$ means all $X \in \widetilde{J}^{3,1}(M, N)$ satisfying $\beta_2 X = \beta_3 X \in J^2(M, N)$ or $\beta_1 X = \beta_2 X \in J^2(M, N)$, respectively.

Proposition 5. *In the case of three h's, we have the categories determined by the inverses*

(i) $[h, h, h]$,

(ii) $[h, h, h, \pi_2 = \pi_3]$,

(iii) $[h, h, h, \pi_1 = \pi_2]$,

(iv) $[h, h, h, \pi_1 = \pi_3]$

and there are two more cases with nontrivial Q_B

(v) $(h, h, h, \pi_2 = \pi_3, z_{ijk} = z_{jik})$ *corresponding to* $J^1 J^2 \cap (\beta_2 = \beta_3)^{-1}(J^2)$,

(vi) $(h, h, h, \pi_1 = \pi_2, z_{ijk} = z_{ikj})$ *corresponding to* $J^2 J^1 \cap (\beta_1 = \beta_2)^{-1}(J^2)$.

Clearly, the case $[h, h, h, \pi_1 = \pi_2 = \pi_3]$ corresponds to $\bar{J}^{3,2}$.

Since the special types of semiholonomic 3-jets are determined in [3], we have classified all special types of nonholonomic 3-jets.

References

[1] Doupovec, M., *On the underlying lower order bundle functors*, Czechoslovak Math. J. **55** (2005), 901–916.

[2] Ehresmann, C., *Oeuvres complètes et commentés, Parties I–1 et I–2*, Cahiers Topol. Géom. Diff. (Suppl. 1 et 2) **24** (1983).

[3] Kolář, I., *On special types of semiholonomic 3–jets*, to appear in Czechoslovak Math. J.

[4] Kolář, I., *Bundle functors of the jet type*, Differential Geom. and Applications (Brno, 1998), 231–237, Masaryk Univ. Brno, 1999.

[5] Kolář, I., *A general point of view to nonholonomic jet bundles*, Cahiers Topol. Géom. Diff. Catégoriques **44** (2) (2003), 149–160.

[6] Kolář, I., *Weil Bundles as Generalized Jet Spaces*, Handbook of Global Analysis, Elsevier, Amsterdam, 2008, pp. 625–665.

[7] Kolář, I., *On special types of nonholonomic contact elements*, Differential Geom. Appl. **29** (2011), 135–140.

[8] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer Verlag, 1993.

[9] Kolář, I., Mikulski, W. M., *On the fiber product preserving bundle functors*, Differential Geom. Appl. **11** (1999), 105–115.

[10] Kolář, I., Vosmanská, G., *Natural operations with second order jets*, Rend. Circ. Mat. Palermo (2) Suppl. **21** (1989), 179–186.

[11] Libermann, P., *Introduction to the theory of semi-holonomic jets*, Arch. Math. (Brno) **33** (1997), 173–189.

[12] Virsik, G., *Total connections in Lie groupoids*, Arch. Math. (Brno) **31** (1995), 183–200.

INSTITUTE OF MATHEMATICS AND STATISTICS, MASARYK UNIVERSITY, KOTLÁŘSKÁ 2, CZ 611 37 BRNO, CZECH REPUBLIC
E-mail: `kolar@math.muni.cz`