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PRODUCTS OF NON- σ -LOWER POROUS SETS

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Abstract. In the present article we provide an example of two closed non- σ -lower porous sets $A, B \subseteq \mathbb{R}$ such that the product $A \times B$ is lower porous. On the other hand, we prove the following: Let X and Y be topologically complete metric spaces, let $A \subseteq X$ be a non- σ -lower porous Suslin set and let $B \subseteq Y$ be a non- σ -porous Suslin set. Then the product $A \times B$ is non- σ -lower porous. We also provide a brief summary of some basic properties of lower porosity, including a simple characterization of Suslin non- σ -lower porous sets in topologically complete metric spaces.

Keywords: topologically complete metric space, abstract porosity, σ -porous set, σ -lower porous set, Cartesian product

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1. INTRODUCTION

In the present article we deal with Cartesian products of σ -lower porous sets. The work is motivated by a paper of L. Zajíček [5] where the following theorem is proved:

Theorem Z [5, Theorem 1]. *Let (X, ϱ) and (Y, σ) be topologically complete metric spaces and let $A \subseteq X$ and $B \subseteq Y$ be non- σ -porous G_δ -sets. Then the Cartesian product $A \times B$ is non- σ -porous in the space $(X \times Y, \varrho_m)$ where ϱ_m is the maximum metric.*

It is a natural question to ask whether an analogous statement holds for lower porosity (i.e. the notion of porosity defined by limes inferior rather than limes superior). In Section 4 we present a counterexample, showing that the answer is negative.

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Theorem 1. *There exist closed non- σ -lower porous sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that the Cartesian product $A \times B$ is lower porous in \mathbb{R}^2 .*

However, if we strengthen the assumptions of the original conjecture, we obtain the following theorem. These two theorems together give us a fairly complete answer to our question.

Theorem 2. *Let (X, ρ) and (Y, σ) be topologically complete metric spaces. Assume that $A \subseteq X$ and $B \subseteq Y$ are Suslin sets in their respective spaces. If A is non- σ -lower porous in X and B is non- σ -porous in Y then the Cartesian product $A \times B$ is non- σ -lower porous in $X \times Y$ (with the maximum metric).*

It is easy to see that both aforementioned notions of σ -porosity are invariant with respect to bi-Lipschitz homeomorphisms. Therefore, in all the previous theorems we can equip the product spaces with any metric which is “bi-Lipschitz equivalent” to the maximum metric and the resulting statement will be true.

It is also fitting to give an explanation as to why in Theorem 2 we only require the sets A and B to be Suslin while in Theorem Z these are assumed to be of the type G_δ . The reason is that we use two inscribing theorems (see 2.5 and 2.6) which, at the time Theorem Z was proved, had not yet been discovered. Of course, this means Theorem Z can be generalized to Suslin sets.

2. SOME FACTS ABOUT σ -LOWER POROSITY AND ABSTRACT POROSITY

The main aim of this section is to provide the reader with a self-contained collection of some basic facts about σ -lower porous sets (with some references to related articles). It might be of some independent interest, but we shall use these facts to prove our main results.

Notation. In the whole paper we shall denote by $B(x, r)$ the open ball with centre x and radius r , by \overline{A} the closure of the set A , and by ∂A the boundary of A . As usual, for a set X the symbol 2^X denotes the power set of X .

Convention. Unless stated otherwise, we shall consider all product spaces equipped with the maximum metric (i.e. for $x_1, x_2 \in (X, \rho)$ and $y_1, y_2 \in (Y, \sigma)$, $\rho_m(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) = \max\{\rho(x_1, x_2), \sigma(y_1, y_2)\}$).

The following standard definitions of σ -porosity originate in a work of A. Denjoy from 1920; however, a systematic investigation of these sets (and the usage of the current nomenclature) has begun in 1967 with an article of E. P. Dolzhenko. For extensive information about σ -porous sets from various viewpoints we refer the reader

to L. Zajíček's survey articles [3] and [6]. The notion of abstract porosity is defined for example in [5].

Definition 2.1. Let (X, ρ) be a metric space, $M \subseteq X$, $x \in X$ and $R > 0$. We define

$$\gamma(x, R, M) = \sup\{r > 0: \text{for some } z \in X, B(z, r) \subseteq B(x, R) \setminus M\},$$

$$\bar{p}(M, x) = \limsup_{R \rightarrow 0^+} \frac{2 \cdot \gamma(x, R, M)}{R}, \quad \underline{p}(M, x) = \liminf_{R \rightarrow 0^+} \frac{2 \cdot \gamma(x, R, M)}{R}.$$

A set $M \subseteq X$ is (*upper*) *porous at x* if $\bar{p}(M, x) > 0$ and *lower porous at x* if $\underline{p}(M, x) > 0$.

Now assume \mathbf{P} is a relation between points and subsets of X (i.e. $\mathbf{P} \subseteq X \times 2^X$). The symbol $\mathbf{P}(x, A)$ where $x \in X$ and $A \subseteq X$ means that $\langle x, A \rangle \in \mathbf{P}$. We say that \mathbf{P} is an *abstract porosity on X* if the following conditions are satisfied (for all $A \subseteq X$, $B \subseteq X$ and $x \in X$):

- (A1) If $A \subseteq B \subseteq X$, $x \in X$ and $\mathbf{P}(x, B)$, then $\mathbf{P}(x, A)$.
- (A2) $\mathbf{P}(x, A)$ if and only if there is an $r > 0$ such that $\mathbf{P}(x, A \cap B(x, r))$.
- (A3) $\mathbf{P}(x, A)$ if and only if $\mathbf{P}(x, \overline{A})$.

Note that the relations which correspond (in the sense of the first point of the following list) to the notions of porosity and lower porosity are clearly abstract porosities. Let \mathbf{P} be an abstract porosity on X . We say that $A \subseteq X$ is

- ▷ \mathbf{P} -porous at $x \in X$ if $\mathbf{P}(x, A)$,
- ▷ \mathbf{P} -porous (*in X*) if A is \mathbf{P} -porous at each of its points,
- ▷ σ - \mathbf{P} -porous (*in X*) if A is a countable union of \mathbf{P} -porous sets,
- ▷ σ - \mathbf{P} -porous at $x \in X$ if there is an $r > 0$ such that $A \cap B(x, r)$ is σ - \mathbf{P} -porous.

In case \mathbf{P} corresponds to lower porosity we say A is *lower porous*, σ -*lower porous* or σ -*lower porous at x* . If \mathbf{P} corresponds to ordinary (upper) porosity, we simply omit the symbol \mathbf{P} and write A is *porous* etc. (however, in some cases we tend to add "upper" to avoid confusion).

Remark 2.2. If (X, ρ) is a metric space and \mathbf{P} is an abstract porosity on X , it is well known that the family \mathcal{S} of all σ - \mathbf{P} -porous sets in X satisfies the following conditions:

- (i) If $A \subseteq B$ and $B \in \mathcal{S}$ then $A \in \mathcal{S}$.
- (ii) If $A_n \in \mathcal{S}$ for all $n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$.

The following two propositions are well known (see the survey article [6]), but we shall provide the proofs for the sake of completeness. Proposition 2.4 gives us a method to recognize non- σ -lower porous sets.

Proposition 2.3. *Let (X, ρ) be a metric space and let $A \subseteq X$ be σ -lower porous. Then A can be covered by a countable family of closed lower porous sets.*

Proof. Without loss of generality we can assume the set A is lower porous. From the definition of lower porosity it is clear that for any $x \in A$ we can choose a positive number $h_0 = h_0(x)$ such that for all $h \in (0, h_0(x))$:

$$\frac{2 \cdot \gamma(x, h, A)}{h} > \frac{\underline{p}(x, A)}{2}.$$

Thus we have chosen a function $h_0: A \rightarrow (0, \infty)$. Set

$$A_n := \left\{ x \in A : h_0(x) > \frac{1}{n} \text{ and } \underline{p}(x, A) > \frac{1}{n} \right\};$$

then, clearly, $A = \bigcup_{n=1}^{\infty} A_n$. We shall now prove that for each $n \in \mathbb{N}$ the set $\overline{A_n}$ is lower porous. And since it is obvious that for any $x \in X$, $R > 0$ and $M \subseteq X$ the equality $\gamma(x, R, M) = \gamma(x, R, \overline{M})$ is true, we only need to show that the set A_n is lower porous at each point $x \in \overline{A_n} \setminus A_n$.

To that end, choose a natural number n and a point $x \in \overline{A_n} \setminus A_n$. Now, for an arbitrary $h \in (0, 1/n)$ there is a point $y \in B(x, \frac{1}{2}h) \cap A_n$ and from the definition of A_n it follows that there is a point $z \in B(y, \frac{1}{2}h)$ such that $B(z, \frac{1}{8}h/n) \subseteq B(y, \frac{1}{2}h) \setminus A_n$. Thus $\gamma(x, h, A_n) \geq \frac{1}{8}h/n$ and

$$\liminf_{h \rightarrow 0^+} \frac{2 \cdot \gamma(x, h, A_n)}{h} \geq \frac{1}{4n} > 0.$$

□

Proposition 2.4. *Let (X, ρ) be a metric space and let $F \subseteq X$ be a topologically complete subspace. Let there exist a set $D \subseteq F$ dense in F such that F is lower porous (in X) at no point of D . Then F is not σ -lower porous in X .*

Proof. Assume to the contrary that F is σ -lower porous. Proposition 2.3 gives us closed lower porous sets F_n ($n \in \mathbb{N}$) such that $F \subseteq \bigcup_{n=1}^{\infty} F_n$. Hence $F = \bigcup_{n=1}^{\infty} (F_n \cap F)$ and the set $F_n \cap F$ is closed in F for each natural n . Using the Baire theorem in the topologically complete space F we obtain an open set $G \subseteq X$ such that $\emptyset \neq G \cap F \subseteq F_{n_0} \cap F$ for some natural number n_0 . Thus $G \cap F$ (being a subset of F_{n_0}) is lower porous in X and it follows that F is lower porous at every point $x \in G \cap F$ (for G is an open set). But the set D is dense in F so there exists a point $x \in D \cap G \cap F$ which is a contradiction to the assumption that F is lower porous at no point of D . □

Now we formulate two rather deep inscribing theorems which will be used on various occasions throughout the paper. Their purpose is to obtain some of our statements about non- σ -porous and non- σ -lower porous sets for all Suslin sets instead of closed (or G_δ) sets only.

Theorem 2.5 [8, Theorem 3.1]. *Let (X, ϱ) be a topologically complete metric space and let $S \subseteq X$ be a non- σ -porous Suslin set. Then there exists a closed non- σ -porous set $F \subseteq S$.*

Theorem 2.6 [7, Corollary 3.4]. *Let (X, ϱ) be a topologically complete metric space and let $S \subseteq X$ be a non- σ -lower porous Suslin set. Then there exists a closed non- σ -lower porous set $F \subseteq S$.*

We continue by recalling several basic definitions (cf. e.g. [4] and [1]) which we need in the following.

Definition 2.7. Let (X, ϱ) be a metric space and let \mathbf{P} be an abstract porosity on X . If $A \subseteq X$ then by $K_{\mathbf{P}}(A)$ we denote the set of all $x \in A$ such that A is not σ - \mathbf{P} -porous at x .

Recall that a family of sets $\mathcal{M} \subseteq 2^X$ is called

- ▷ *locally finite* if for each $x \in X$ there is an $r > 0$ such that the ball $B(x, r)$ intersects at most finitely many elements of \mathcal{M} ,
- ▷ *discrete* if for each $x \in X$ there is an $r > 0$ such that the ball $B(x, r)$ intersects at most one element of \mathcal{M} ,
- ▷ σ -*discrete* if it is a countable union of discrete families.

We say \mathcal{M} is a *cover* of X if $\bigcup \mathcal{M} = X$. Let \mathcal{U} and \mathcal{V} be two covers of X . Then \mathcal{V} is a *refinement* of \mathcal{U} if for each $B \in \mathcal{V}$ there is a set $A \in \mathcal{U}$ such that $B \subseteq A$.

An elementary proof of the following Proposition 2.9 can be found as the proof of Lemma 3 in the article [4]; we give an alternative proof which is more transparent, but is not elementary since it uses the famous theorem of A.H.Stone about the paracompactness of metric spaces ([1, Theorem 4.4.1]). We will use the following easy lemma.

Lemma 2.8. *Let (X, ϱ) be a metric space and let \mathbf{P} be an abstract porosity on X . Then:*

- (i) *If \mathcal{M} is a discrete family of \mathbf{P} -porous sets, then $\bigcup \mathcal{M}$ is \mathbf{P} -porous.*
- (ii) *If \mathcal{M} is a σ -discrete family of σ - \mathbf{P} -porous sets, then $\bigcup \mathcal{M}$ is σ - \mathbf{P} -porous.*

Proof. First, we shall prove the assertion (i). Let \mathcal{M} be a discrete family of \mathbf{P} -porous sets and let $x \in \bigcup \mathcal{M}$ be an arbitrary point; we shall prove that $\bigcup \mathcal{M}$ is

\mathbf{P} -porous at x . Since the family \mathcal{M} is discrete, there is an $r > 0$ and $M \in \mathcal{M}$ such that

$$(1) \quad \left(\bigcup \mathcal{M} \right) \cap B(x, r) = M \cap B(x, r).$$

The set M is \mathbf{P} -porous and from (A1) (see Definition 2.1) we have that so is $M \cap B(x, r)$. It follows from (1) and (A2) that also the sum $\bigcup \mathcal{M}$ is \mathbf{P} -porous at x .

To prove the second assertion, assume (clearly without loss of generality) \mathcal{M} is a discrete family of σ - \mathbf{P} -porous sets. Each $M \in \mathcal{M}$ can be written in the form $M = \bigcup_{n=1}^{\infty} A_n^M$ where the set A_n^M is \mathbf{P} -porous for any $n \in \mathbb{N}$. It is obvious that for each $n \in \mathbb{N}$ the family $\{A_n^M : M \in \mathcal{M}\}$ is discrete. Thus, using the first part of this lemma, we obtain the σ - \mathbf{P} -porosity of

$$\bigcup \mathcal{M} = \bigcup_{n=1}^{\infty} \bigcup_{M \in \mathcal{M}} A_n^M.$$

□

Proposition 2.9. *Let (X, ϱ) be a metric space and let \mathbf{P} be an abstract porosity on X . Assume the set $A \subseteq X$ is σ - \mathbf{P} -porous at each of its points. Then A is σ - \mathbf{P} -porous.*

Proof. Set $A_n := \{x \in A : B(x, 1/n) \cap A \text{ is } \sigma\text{-}\mathbf{P}\text{-porous}\}$; by the assumption, $A = \bigcup_{n=1}^{\infty} A_n$. Let us fix an arbitrary $k \in \mathbb{N}$ and prove that A_k is σ - \mathbf{P} -porous.

To that end, we define the open cover \mathcal{U} of X as $\mathcal{U} := \{B(x, \frac{1}{2}k^{-1}) : x \in X\}$; it is easy to see that for each $B \in \mathcal{U}$ the set $B \cap A_k$ is σ - \mathbf{P} -porous. Using the Stone Paracompactness Theorem we obtain a σ -discrete refinement \mathcal{V} of \mathcal{U} . Since \mathcal{V} is a refinement of \mathcal{U} , we have that for each $G \in \mathcal{V}$ the set $G \cap A_k$ is σ - \mathbf{P} -porous and it follows from Lemma 2.8 that $A_k = \bigcup \{G \cap A_k : G \in \mathcal{V}\}$ is σ - \mathbf{P} -porous. This concludes the proof. □

An immediate consequence of this result is the following.

Corollary 2.10. *Let (X, ϱ) be a metric space and let \mathbf{P} be an abstract porosity on X . Assume the set $A \subseteq X$ is not σ - \mathbf{P} -porous. Then:*

- (i) $K_{\mathbf{P}}(A)$ is nonempty and closed in A .
- (ii) The set $A \setminus K_{\mathbf{P}}(A)$ is σ - \mathbf{P} -porous.
- (iii) $K_{\mathbf{P}}(K_{\mathbf{P}}(A)) = K_{\mathbf{P}}(A)$ (i.e. $K_{\mathbf{P}}(A)$ is σ - \mathbf{P} -porous at none of its points).
- (iv) The set of all points at which $K_{\mathbf{P}}(A)$ is not \mathbf{P} -porous is dense in $K_{\mathbf{P}}(A)$.

The proposition that follows now provides a simple characterization of non- σ -lower porous Suslin sets. It can be regarded as an analogue for lower porosity to a partial converse of the Foran lemma which was proved by L. Zajíček (see [5, Corollary 1]); the mentioned result works for upper porosity and G_δ sets (but can, of course, be generalized to Suslin sets via Theorem 2.5).

Proposition 2.11. *Let (X, ρ) be a topologically complete metric space and let $A \subseteq X$ be a Suslin set. Then the following statements are equivalent:*

- (i) *A is not σ -lower porous.*
- (ii) *There exists a closed set $F \subseteq A$ and a set $D \subseteq F$ dense in F such that F is lower porous at no point of D .*

Proof. To prove the implication (i) \Rightarrow (ii) assume A is a non- σ -lower porous Suslin set; using Theorem 2.6 we can assume without loss of generality that A is closed. Let \mathbf{P} be the abstract porosity which corresponds to lower porosity in X . Now it suffices to take $F := K_{\mathbf{P}}(A)$, as all the desired properties of F follow from Corollary 2.10.

To prove (ii) \Rightarrow (i) suppose that (ii) holds. Then Proposition 2.4 gives that F is non- σ -lower porous, and thus so is A . □

Remark 2.12. (a) It may be interesting to note a connection of Proposition 2.11 to the article [2] (especially Section 5) where the notion of \mathbf{P} -reducible sets is defined and studied. If \mathbf{P} is an abstract porosity on a metric space X , we say that $A \subseteq X$ is \mathbf{P} -reducible if each nonempty closed set $F \subseteq A$ contains a \mathbf{P} -porous subset with nonempty relative interior in F . Now the statement of Proposition 2.11 can be reformulated as follows:

If X is topologically complete and \mathbf{L} is the relation corresponding to the notion of lower porosity on X , then a Suslin set $A \subseteq X$ is σ -lower porous if and only if it is \mathbf{L} -reducible.

(b) Now let us briefly turn our attention to the general case. As Corollary 2.10 (iv) holds for any abstract porosity \mathbf{P} , the following is true:

Let \mathbf{P} be any abstract porosity on a metric space X and let $A \subseteq X$ be closed. Then:

$$A \text{ is non-}\sigma\text{-}\mathbf{P}\text{-porous} \implies A \text{ is not } \mathbf{P}\text{-reducible.}$$

If X is topologically complete and \mathbf{P} corresponds to upper porosity on X , it suffices to assume the set A to be Suslin (due to Theorem 2.5).

However, if \mathbf{P} is such that an analogue of Proposition 2.4 for \mathbf{P} does not hold (e.g., the upper porosity), then the other implication in the previous statement does not necessarily hold (see Example 3.1 or Corollary 5.3 with Proposition 5.1 of [2]).

That is the reason why a more elaborate method of recognizing non- σ -upper porous sets had to be developed in order to prove Theorem Z from the introduction (the method of the Foran Lemma and its partial converse).

3. ONE POSITIVE RESULT

Theorem 3.1. *Let (X, ϱ) and (Y, σ) be topologically complete metric spaces. Assume the Suslin set $A \subseteq X$ is not σ -lower porous and the Suslin set $B \subseteq Y$ is not σ -porous. Then the Cartesian product $A \times B$ is not σ -lower porous in the space $X \times Y$ (with the maximum metric).*

Proof. Let $\mathbf{L} \subseteq X \times 2^X$ be the relation corresponding to the notion of lower porosity on X (i.e. $\mathbf{L}(x, C)$ if and only if C is lower porous at x) and let $\mathbf{U} \subseteq Y \times 2^Y$ be the relation corresponding to upper porosity on Y . Since both these relations are abstract porosities, from Corollary 2.10 we know that $K_{\mathbf{L}}(A) \neq \emptyset$ and $K_{\mathbf{U}}(B) \neq \emptyset$; without loss of generality we shall now assume that $A = K_{\mathbf{L}}(A)$ and $B = K_{\mathbf{U}}(B)$ and using Theorem 2.5 and Theorem 2.6 we may also assume that the sets A and B are closed in their spaces.

Denote by A_1 the set of all points of A at which A is not lower porous and by B_1 the set of all points of B at which B is not porous. From Corollary 2.10 we know that A_1 is dense in A and B_1 is dense in B ; thus $A_1 \times B_1$ is dense in $A \times B$. By Proposition 2.4, it suffices to prove that $A \times B$ is lower porous at no point of $A_1 \times B_1$. However, this is true due to Corollary 4.8, hence the proof is complete. \square

4. A COUNTEREXAMPLE

Definition 4.1. Denote $D_0 := \emptyset$ and for each $n \in \mathbb{N}$ we define the open set $D_n \subseteq (0, 1)$ as

$$D_n := \bigcup_{i=0}^{3^{n-1}-1} \left(\frac{1+3i}{3^n}, \frac{2+3i}{3^n} \right).$$

Furthermore, for each $n \in \mathbb{N} \cup \{0\}$ we define

$$M_n := \partial D_n, \quad A_n := [0, 1] \setminus D_n.$$

Finally, if $I \subseteq \mathbb{N}$ is nonempty, we define

$$D_I := \bigcup_{n \in I} D_n, \quad M_I := \bigcup_{n \in I} M_n, \quad A_I := [0, 1] \setminus D_I.$$

Definition 4.2. Let (X, ϱ) be a metric space and let $\varepsilon > 0$. Recall that $M \subseteq X$ is an ε -net in X , if for each point $x \in X$ there exists some $y \in M$ such that $\varrho(x, y) \leq \varepsilon$.

The following facts are easy to see.

Observation 4.3.

- (i) For each $n \in \mathbb{N}$ the set M_n is a 3^{-n} -net in the interval $[0, 1]$.
- (ii) If $I \subseteq \mathbb{N}$ is infinite, then
 - ▷ $\overline{M_I} = [0, 1]$,
 - ▷ A_I is porous.
- (iii) Whenever $m, n \in \mathbb{N}$, $m \neq n$, then we have $M_m \cap M_n = \emptyset$.
- (iv) $M_n \cap D_m \neq \emptyset$ if and only if $m < n$.
- (v) $A_{\mathbb{N}}$ is the ternary Cantor set.

Lemma 4.4. Let $I \subseteq \mathbb{N}$ be infinite and let $\emptyset \neq J \subseteq \mathbb{N}$. Then $M_I \cap A_J$ is dense in A_J .

Proof. Choose an arbitrary $y \in A_J$ and $\varepsilon > 0$. Now find an $n_0 \in I$ such that $2 \cdot 3^{-n_0} < \varepsilon$ and denote $K := J \cap (0, n_0)$. On account of Observation 4.3 (iv) it is true that $M_{n_0} \cap A_K = M_{n_0} \cap A_J$. Setting $n_1 := \max(K \cup \{0\})$ we have $n_1 < n_0$ and it is obvious that the components of A_K are closed intervals whose length is at least 3^{-n_1} . The set M_{n_0} is a 3^{-n_0} -net in $[0, 1]$ (Observation 4.3 (i)) and $3^{-n_1} > 2 \cdot 3^{-n_0}$; from these two facts it now easily follows that M_{n_0} is a $(2 \cdot 3^{-n_0})$ -net in A_K . This implies the existence of a point $z \in M_{n_0} \cap A_K = M_{n_0} \cap A_J \subseteq M_I \cap A_J$ such that $|z - y| \leq 2 \cdot 3^{-n_0} < \varepsilon$, which concludes the proof. \square

Definition 4.5. Let (X, ρ) be a metric space, let $A \subseteq X$ and let $x \in X$. We define the function $\delta_{A,x}: (0, \infty) \rightarrow [0, \infty)$ as

$$\delta_{A,x}(h) := \frac{2 \cdot \gamma(x, h, A)}{h}.$$

Lemma 4.6. Assume $I \subseteq \mathbb{N}$ is nonempty and let $x \in A_I$ and $n \in I$. Then for each $h \in [4/3^{n+1}, 4/3^n]$ we have $\delta_{A_I,x}(h) \geq \frac{1}{4}$.

Proof. Let $I \subseteq \mathbb{N}$, $x \in A_I$ and $n \in I$ be given. Since $n \in I$, we have $A_I \subseteq [0, 1] \setminus D_n$ and thus $x \in [0, 1] \setminus D_n$. From Observation 4.3 (i) we know that the set $M_n = \partial D_n$ is a 3^{-n} -net in the interval $[0, 1]$ which implies that $\text{dist}(x, D_n) \leq 3^{-n}$. From this and from the fact that D_n consists of pairwise disjoint open intervals of length 3^{-n} , it follows that for all $h \in [1/3^n, 2/3^n]$ there holds the inequality

$$2 \cdot \gamma(x, h, [0, 1] \setminus D_n) \geq h - \frac{1}{3^n}.$$

What is more, for any $h > 2/3^n$

$$2 \cdot \gamma(x, h, [0, 1] \setminus D_n) \geq \frac{1}{3^n}.$$

Consequently

$$\begin{aligned} \delta_{A_I, x}(h) &\geq \frac{2 \cdot \gamma(x, h, [0, 1] \setminus D_n)}{h} \\ &\geq \begin{cases} \frac{1}{h} \left(h - \frac{1}{3^n} \right) \geq 1 - \frac{3^{n+1}}{4} \cdot \frac{1}{3^n} = \frac{1}{4} & \text{for } h \in \left[\frac{4}{3^{n+1}}, \frac{2}{3^n} \right], \\ \frac{1}{h} \cdot \frac{1}{3^n} \geq \frac{3^n}{4} \cdot \frac{1}{3^n} = \frac{1}{4} & \text{for } h \in \left[\frac{2}{3^n}, \frac{4}{3^n} \right]. \end{cases} \end{aligned}$$

□

Proposition 4.7. *Let (X, ρ) and (Y, σ) be metric spaces and let us have sets $A \subseteq X$ and $B \subseteq Y$. Finally, let there be given points $x \in X$ and $y \in Y$. Then:*

- (i) $\gamma(\langle x, y \rangle, h, A \times B) = \max\{\gamma(x, h, A), \gamma(y, h, B)\}$ for any $h > 0$;
- (ii) $\delta_{A \times B, \langle x, y \rangle} = \max\{\delta_{A, x}, \delta_{B, y}\}$.

Proof. We shall prove the assertion (i). Without loss of generality we may assume that $\alpha := \max\{\gamma(x, h, A), \gamma(y, h, B)\} = \gamma(x, h, A) > 0$. Choose arbitrary $h > 0$ and $\varepsilon \in (0, \alpha)$. By the definition of $\gamma(x, h, A)$, there exists a point $x_1 \in X$ such that $B(x_1, \alpha - \varepsilon) \subseteq B(x, h) \setminus A$. Thus,

$$B(\langle x_1, y \rangle, \alpha - \varepsilon) \subseteq B(\langle x, y \rangle, h) \setminus A \times B$$

and this means that

$$\gamma(\langle x, y \rangle, h, A \times B) \geq \alpha - \varepsilon = \max\{\gamma(x, h, A), \gamma(y, h, B)\} - \varepsilon.$$

To prove the opposite inequality we take arbitrary $h > 0$ and $\varepsilon > 0$ again. Setting $\beta := \gamma(\langle x, y \rangle, h, A \times B)$, we can assume that $\varepsilon < \beta$. Now find a point $\langle x_1, y_1 \rangle \in X \times Y$ such that

$$G := B(\langle x_1, y_1 \rangle, \beta - \varepsilon) \subseteq B(\langle x, y \rangle, h) \setminus A \times B.$$

Taking into account that $G = B(x_1, \beta - \varepsilon) \times B(y_1, \beta - \varepsilon)$ (for we consider the space $X \times Y$ with the maximum metric), this yields that

$$B(x_1, \beta - \varepsilon) \subseteq B(x, h) \setminus A \quad \text{or} \quad B(y_1, \beta - \varepsilon) \subseteq B(y, h) \setminus B.$$

This implies the following inequality which concludes the proof of (i):

$$\max\{\gamma(x, h, A), \gamma(y, h, B)\} \geq \beta - \varepsilon = \gamma(\langle x, y \rangle, h, A \times B) - \varepsilon.$$

The second assertion follows immediately from (i). □

Corollary 4.8. *Under the assumptions of Proposition 4.7 we have that if A is not lower porous at x and B is not porous at y , then $A \times B$ is not lower porous at $\langle x, y \rangle$.*

Proof. Let x and y be as above. Then

$$\liminf_{h \rightarrow 0^+} \delta_{A,x}(h) = 0 \quad \text{and} \quad \limsup_{h \rightarrow 0^+} \delta_{B,y}(h) = 0.$$

From Proposition 4.7 we know that $\delta_{A \times B, \langle x, y \rangle} = \max\{\delta_{A,x}, \delta_{B,y}\}$, and so it is easy to see that $\liminf_{h \rightarrow 0^+} \delta_{A \times B, \langle x, y \rangle} = 0$, i.e., $A \times B$ is not lower porous at $\langle x, y \rangle$. \square

Remark 4.9. Let (X, ρ) be a metric space. If the set $A \subseteq X$ is not porous then neither is $A^2 = A \times A$ porous in X^2 . The same statement is true for lower porosity or, in general, for any notion of porosity which is determined solely by the function $\delta_{A,x}(h)$.

Indeed, if we assume that the set A is not porous at a certain point $x \in A$, then, since $\delta_{A,x} = \delta_{A^2, \langle x, x \rangle}$, it is clear that A^2 is not porous at $\langle x, x \rangle$. Clearly, the same argument works for many other notions of porosity—including, for example, lower porosity.

We shall now prove the main result of this section which implies Theorem 1 from the Introduction.

Theorem 4.10. *Let the set $I \subseteq \mathbb{N}$ be defined by the formula*

$$I := \bigcup_{i=1}^{\infty} [i^2, i^2 + i) \cap \mathbb{N}$$

and let $J = \mathbb{N} \setminus I$. Then none of the closed sets A_I and A_J is σ -lower porous while the product $A_I \times A_J$ is lower porous.

Proof. First, we shall prove that the set A_J is not σ -lower porous; of course, the proof for A_I would be analogous. Being a closed subspace of \mathbb{R} , A_J is a topologically complete space. Hence, according to Proposition 2.4 it suffices to find a dense subset of A_J at whose points the set A_J is not lower porous. We claim that $M_I \cap A_J$ is such a set. Indeed, by Lemma 4.4, $M_I \cap A_J$ is dense in A_J ; it only remains to be shown that A_J is lower porous at no point of $M_I \cap A_J$.

To prove that, choose an arbitrary point $x \in M_I \cap A_J$ and let $n_0 \in I$ be the unique natural number such that $x \in M_{n_0}$ (the uniqueness of n_0 is clear from Observation 4.3 (iii)). Now x can be written in the form $k/3^{n_0}$, where $k \in \mathbb{N}$ is not divisible by 3. It follows that for each natural $j > n_0$

$$(2) \quad \text{dist}(x, D_j) = \frac{1}{3^j}.$$

Moreover, since $x \in M_I \cap A_J$, for each natural $j < n_0$ we have

$$(3) \quad \text{dist}(x, D_j) \geq \frac{1}{3^{n_0}}.$$

Now fix a natural number i_0 such that $i_0^2 > n_0$ and choose an arbitrary $i > i_0$. The inequalities (2) and (3) imply that

$$(4) \quad \text{dist}\left(x, \bigcup\{D_n : n \in J, n \leq i^2 - 1\}\right) = \frac{1}{3^{i^2-1}}.$$

From the definition of J we see that $\{i^2, i^2 + 1, \dots, i^2 + i - 1\} \cap J = \emptyset$. This fact, together with (4), implies that the longest interval contained in

$$\left(x - \frac{1}{3^{i^2-1}}, x + \frac{1}{3^{i^2-1}}\right)$$

and disjoint with A_J is a component of D_{i^2+i} (as $i^2 + i \in J$), and therefore its length is $3^{-(i^2+i)}$. That is,

$$\delta_{A_J, x}\left(\frac{1}{3^{i^2-1}}\right) = 3^{i^2-1} \cdot \frac{1}{3^{i^2+i}} = \frac{1}{3^{i+1}};$$

it follows that $\liminf_{h \rightarrow 0_+} \delta_{A_J, x}(h) = 0$ which means that A_J is not lower porous at x .

To prove that the product $A_I \times A_J$ is lower porous, choose an arbitrary point $\langle x, y \rangle \in A_I \times A_J$. By Lemma 4.6 we have

$$\delta_{A_I, x}(h) \geq \frac{1}{4}, \quad \text{whenever } h \in \bigcup_{n \in I} \left[\frac{4}{3^{n+1}}, \frac{4}{3^n}\right] =: F_I,$$

and also $\delta_{A_J, y}(h) \geq \frac{1}{4}, \quad \text{whenever } h \in \bigcup_{n \in J} \left[\frac{4}{3^{n+1}}, \frac{4}{3^n}\right] =: F_J.$

But $I \cup J = \mathbb{N}$, so $F_I \cup F_J = (0, \frac{4}{3}]$, and it immediately follows from Proposition 4.7 that $\liminf_{h \rightarrow 0_+} \delta_{A_I \times A_J, \langle x, y \rangle}(h) \geq \frac{1}{4}$, concluding the proof. \square

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