Kybernetika

Dong Qiu; Weiquan Zhang

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Kybernetika, Vol. 49 (2013), No. 1, 141--148

Persistent URL: http://dml.cz/dmlcz/143245

KYBERNETIKA — VOLUME 49 (2013), NUMBER 1, PAGES 141–148

THE STRONGEST T-NORM FOR FUZZY METRIC SPACES

Dong Qiu, Weiquan Zhang

In this paper, we prove that for a given positive continuous t-norm there is a fuzzy metric space in the sense of George and Veeramani, for which the given t-norm is the strongest one. For the opposite problem, we obtain that there is a fuzzy metric space for which there is no strongest t-norm. As an application of the main results, it is shown that there are infinite non-isometric fuzzy metrics on an infinite set.

Keywords: fuzzy metric space, t-norm, isometry, analysis

Classification: 93E12, 62A10

1. INTRODUCTION

Triangular norms (t-norms for short), introduced by Schweizer and Sklar [9], play a key role in the theory of fuzzy metric spaces. For a given fuzzy metric space in the sense of George and Veeramani [2] (GV fuzzy metric space for short), there is generally more than one t-norm to choose from. Specifically, if (X, M, T) is a GV fuzzy metric space and a continuous t-norm T' is weaker than T, then (X, M, T') is also a GV fuzzy metric space [8]. Furthermore, the properties of the triangle inequality are greatly dependent on the corresponding t-norm in the GV fuzzy metric space. It follows, in general, that the stronger t-norm, the more information one has about the structure of the GV fuzzy metric space. Hence it is natural to ask whether, for a given t-norm, there is a corresponding GV fuzzy metric space for which the given t-norm is strongest. Similarly it is interesting to know whether, for a given GV fuzzy metric space, there is a strongest t-norm.

It should be noted that the same issues have been discussed for probabilistic metric spaces in [11]. Since any fuzzy metric space in the sense of Kramosil and Michálek [6] (KM fuzzy metric space for short) is equivalent to a Menger space [6], the results of [11] hold for the KM fuzzy metric spaces. However, because a GV fuzzy metric space has the stricter "separation" condition and continuity condition than a KM fuzzy metric space, the conclusions and the constructive proofs in [11] no longer apply to the GV fuzzy metric spaces. In this paper, we solve the former problem for all of the positive continuous t-norms by a new constructive way which is different from the one in [11]. We also investigate the latter problem. Unfortunately, the answer is negative for continuous t-norms. Finally, as an application of the main results, it is shown that there are infinite non-isometric GV fuzzy metrics on an infinite set.

2. PRELIMINARIES

Throughout this paper the letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ will denote the set of all positive integers, rational numbers and real numbers, respectively.

Definition 2.1. (Klement et al. [5]) A t-norm is a binary operation on the unit interval $[0,1]$, i. e., a function $T : [0,1]^2 \to [0,1]$, such that for all $a,b,c,d \in [0,1]$ the following four axioms are satisfied:

(T-1) $T(a,1) = a$. (boundary condition)
(T-2) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$. (monotonicity)
(T-3) $T(a,b) = T(b,a)$. (commutativity)
(T-4) $T(a,T(b,c)) = T(T(a,b),c)$. (associativity)

A t-norm T is said to be continuous if it is a continuous function on $[0,1]^2$.

Example 2.1. (Klement et al. [5]) The following are the four basic t-norms T_M, T_P, T_L and T_D given by, respectively, $T_M(x,y) = \min\{x,y\}, T_P(x,y) = x \cdot y, T_L(x,y) = \max\{x+y-1,0\}$ and

$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

The t-norms T_M, T_P and T_L are continuous but T_D is not.

Definition 2.2. (Klement et al. [5]) If, for two t-norms T_1 and T_2, $T_1(a,b) \leq T_2(a,b)$ holds for all $(a,b) \in [0,1]^2$, then we say that T_2 is stronger than T_1, and we write in this case $T_1 \leq T_2$. We write $T_1 < T_2$ whenever $T_1 \leq T_2$ and $T_1 \neq T_2$, and we say that T_2 is strictly stronger than T_1.

Although it is true that $T_D < T_L < T_P < T_M$ and, for an arbitrary t-norm T, $T_D \leq T \leq T_M$, the relation $\leq$ is not a total order, i. e., there are incomparable t-norms.

Example 2.2. The family $T^D_{\lambda\in[0,\infty]}$ of Dombi t-norms [1] is given by

$$T^D_\lambda(x,y) = \begin{cases} T_D(x,y) & \text{if } \lambda = 0, \\ T_M(x,y) & \text{if } \lambda = \infty, \\ \frac{1}{1+((\frac{1-x}{x})^\lambda+(\frac{1-y}{y})^\lambda)^{\frac{1}{\lambda}}} & \text{otherwise.} \end{cases}$$

Now we show that T^D_λ and T_P are incomparable whenever $\lambda \in]0,1[$. In fact, the incomparability of the diagonal sections is sufficient for the incomparability of the corresponding t-norms. For this reason, by solving the equation

$$\frac{1}{1+2^{\frac{1}{\lambda}}(\frac{1-x}{x})} - x^2 = 0,$$

we get a solution $x_0 = 1/(2^{\frac{1}{\lambda}} - 1) \in]0,1[$, which means that $T^D_\lambda(x,x) > T_P(x,x)$ if $0 < x < x_0$, but $T^D_\lambda(x,x) < T_P(x,x)$ if $x_0 < x < 1$.

Definition 2.3. (Schweizer and Sklar [10]) A t-norm T is said to be positive if $T(a,b) > 0$ whenever $a, b \in]0,1]$.

Remark 2.1. Since the family $T^D_{\lambda \in [0,\infty]}$ of Dombi t-norms is strictly increasing with respect to λ [5], Example 2.2 also shows that there are infinite comparable positive continuous t-norms.

Definition 2.4. (Schweizer and Sklar [9]) A probabilistic metric space(briefly, a PM space) is a pair $(X, \mathcal{F})$ such that X is a nonempty set and $\mathcal{F}$ is a mapping from $X \times X$ into Δ the set of distribution functions, whose value $\mathcal{F}(x,y)$ denoted by F_{xy}, satisfies for all $x, y, z \in X$:

(PM-1) $F_{xy}(0) = 0$,
(PM-2) $F_{xy}(t) = 1$ for all $t > 0$ if and only if $x = y$,
(PM-3) $F_{xy}(t) = F_{yx}(t)$,
(PM-4) If $F_{xy}(t) = 1$ and $F_{yz}(s) = 1$, then $F_{xz}(t+s) = 1$.

Definition 2.5. (Schweizer and Sklar [9]) A Menger space is a triple $(X, \mathcal{F}, T)$ such that $(X, \mathcal{F})$ is a PM space and T is a t-norm such that for all $x, y, z \in X$ and $t, s \geq 0$:

(PM-4)' $F_{xz}(t+s) \geq T(F_{xy}(t), F_{yz}(s))$.

Remark 2.2. In the original formulation [7] the operation T doesn't need to meet the request of associativity. To permit the extension of (PM-4)' to a polygonal inequality, Schweizer and Sklar added the associativity condition to T [9]. Since we will need these operations which meet all the other conditions of t-norm but might not meet the associativity, we call such operations symmetric semicopulas.

Definition 2.6. (Kramosil and Michálek [6]) The 3-tuple (X, M, T) is said to be a KM fuzzy metric space if X is an arbitrary nonempty set, T is a t-norm and M is a fuzzy set on $X^2 \times [0, \infty[$ satisfying the following conditions, for all $x, y, z \in X, t, s > 0$:

(KM-1) $M(x, y, 0) = 0$,
(KM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(KM-3) $M(x, y, t) = M(y, x, t)$,
(KM-4) $M(x, z, t+s) \geq T(M(x, y, t), M(y, z, s))$,
(KM-5) $M(x, y, \cdot) : [0, \infty[\to [0,1]$ is left continuous.

Remark 2.3. (Kramosil and Michálek [6], Corollary of Theorem 1) Any KM fuzzy metric space is equivalent to a Menger space.

Definition 2.7. (George and Veeramani [2]) The 3-tuple (X, M, T) is said to be a GV fuzzy metric space if X is an arbitrary nonempty set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X, t, s > 0$:

(GV-1) $M(x, y, t) > 0$,
(GV-2) $M(x, y, t) = 1$ if and only if $x = y$,
(GV-3) $M(x, y, t) = M(y, x, t)$,
(GV-4) $M(x, z, t+s) \geq T(M(x, y, t), M(y, z, s))$,
(GV-5) $M(x, y, \cdot) :]0, \infty[\to]0,1]$ is continuous.

Definition 2.8. (Gregori and Romaguera [3]) Let (X, M, T_1) and (Y, N, T_2) be two GV fuzzy metric spaces. A mapping f from X to Y is called an isometry if for each $x, y \in X$ and each $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$.

Definition 2.9. (Gregori and Romaguera [3]) Two GV fuzzy metric spaces (X, M, T_1) and (Y, N, T_2) are called isometric if there is an isometry from X onto Y.

3. MAIN RESULTS

Problem 3.1. For any continuous t-norm T, is there a nonempty set X and a fuzzy set M on $X^2 \times]0, \infty[$ for which T is the strongest t-norm which makes (X, M, T) to be a GV (or KM) fuzzy metric space?

Since any KM fuzzy metric space is equivalent to a Menger space, the definite answer in [11] is also true for KM fuzzy metric spaces. However, the conclusions and the constructive proofs in [11] no longer apply to GV fuzzy metric spaces because the GV fuzzy metric spaces have the stricter conditions (GV-2)' and (GV-5) than the conditions (KM-2) and (KM-5). For the GV fuzzy metric spaces we solve this problem for some subclass of the continuous t-norms.

Theorem 3.1. Let T be an arbitrary positive continuous t-norm. Then there is a nonempty set X and a fuzzy set M on $X^2 \times]0, \infty[$ for which T is the strongest t-norm which makes (X, M, T) to be a GV fuzzy metric space.

Proof. Let X be a set with an infinite number of elements and $f : X \to]0, 1]$ be a function such that the range $f(X)$ of f is dense in $[0, 1]$ and only one point x_0 of X takes the value 1. Define M by ([4] Example 14):

$$M(x, y, t) = \begin{cases} 1 & \text{if } x = y, \\ T(f(x), f(y)) & \text{if } x \neq y, \text{ and } t > 0. \end{cases}$$

Then (X, M, T) is a GV fuzzy metric space [4].

Next, we show that for this GV fuzzy metric space the given T is the strongest possible. Suppose there is a continuous t-norm T' such that T' is stronger than T and (X, M, T') is also a GV fuzzy metric space. For any two-tuple $(a, b) \in \{(c, d) \in f(X) \times f(X) : c \neq d, \text{ and } c, d \in]0, 1[\}$, there are two different points $x, y \in X$ such that $f(x) = a$ and $f(y) = b$. Then for the three different points x_0, x and y, in the GV fuzzy metric space (X, M, T') we have that

$$\begin{aligned} T(a, b) = T(f(x), f(y)) = M(x, y, t) &\geq T'(M(x, x_0, t), M(x_0, y, t)) \\ &= T'(T(f(x), f(x_0)), T(f(x_0), f(y))) \\ &= T'(T(f(x), 1), T(1, f(y))) \\ &= T'(f(x), f(y)) \\ &= T'(a, b) \end{aligned}$$

which implies $T(a, b) \geq T'(a, b)$ for all $(a, b) \in \{(c, d) \in f(X) \times f(X) : c \neq d, \text{ and } c, d \in]0, 1[\}$. In addition, we have assumed that T' is stronger than T. So we obtain that

$T(a,b) = T'(a,b)$ for all $(a,b) \in \{(c,d) \in f(X) \times f(X) : c \neq d$, and $c,d \in]0,1[\}$. Since the set $\{(c,d) \in f(X) \times f(X) : c \neq d$, and $c,d \in]0,1[\}$ is dense in $[0,1]^2$, by the continuity of t-norms T and T' we have that $T = T'$. □

Remark 3.1. It should be noted that the associativity of t-norms is not used in the proof of Theorem 3.1. Thus Theorem 3.1 is also true for all of the positive continuous symmetric semicopulas. The number of points in the resulting GV fuzzy metric space is at least countable infinite in Theorem 3.1. Indeed we can construct a GV fuzzy metric space with a countable infinite number of points.

Example 3.1. Let T be a positive continuous t-norm, $X =]0,1] \cap \mathbb{Q}$ and f be the identity function in $]0,1]$. Define M as the same as the one in Theorem 3.1. Similarly, we can prove that T is the strongest t-norm which makes (X,M,T) to be a GV fuzzy metric space.

From Theorem 3.1 and Example 3.1, we have actually proved the following theorem.

Theorem 3.2. Let T be a positive continuous t-norm and X be a set which has at least a countable infinite number of elements. Then there exists a fuzzy set M on $X^2 \times]0,\infty[$ for which T is the strongest t-norm which makes (X,M,T) to be a GV fuzzy metric space.

Example 3.2. Let $X =]1,+\infty[$ and $f(x) = 1/x$. Define M by

$$M(x,y,t) = \begin{cases} 1 & \text{if } x = y, \\ \frac{1}{xy} & \text{if } x \neq y, \text{ and } t > 0. \end{cases}$$

In [4], Gregori, Morillas and Sapena have shown that (M,T_P) is a GV fuzzy metric but (M,T_M) is not. In fact, by Theorem 3.1 we have that T_P is the strongest t-norm for M.

Next we will investigate the opposite problem of Problem 3.1:

Problem 3.2. For a given GV fuzzy metric space (X,M,T'), is there a strongest t-norm T which makes (X,M,T) is a GV fuzzy metric space?

Unfortunately, the answer is negative for continuous t-norms.

Lemma 3.1. [8] Let X be a nonempty set. If (M,T) is a GV fuzzy metric and T' is a continuous t-norm (or symmetric semicopula) such that $T' \leq T$, then (M,T') is a GV fuzzy metric on X.

Theorem 3.3. There exists a GV fuzzy metric space for which there is no strongest t-norm.

Proof. We have shown that T_λ^D and T_P are incomparable whenever $\lambda \in]0,1[$ in Example 2.2. Now for $\lambda = 1/2$, let $T = \max\{T_{1/2}^D, T_P\}$. It is easy to see that T satisfies the conditions (T-1), (T-2) and (T-3) of Definition 2.1, which implies that T is a symmetric semicopula. Nevertheless, it is not associative since $T(T(1/4,3/7),7/9) = T(3/28,7/9) = 1/12 \neq T(1/4,T(3/7,7/9)) = T(1/4,1/3) = 1/(6+2\sqrt{6})$.

Furthermore, since T is obviously positive and continuous, by Remark 3.1 we can apply the construction of Theorem 3.1 to yield a GV fuzzy metric space which makes T is the strongest symmetric semicopula satisfying the conditions (T-1), (T-2) and (T-3) of Definition 2.1 and (GV-4) of Definition 2.7. By Lemma 3.1 this GV fuzzy metric space is also a GV fuzzy metric space under $T^D_{1/2}$ and T_P. Thus if there were a strongest t-norm it would be stronger than T. Since this is impossible, we have completed the proof. □

Now we give an application of the main results to show that there are infinite non-isometric GV fuzzy metrics on an infinite set.

Theorem 3.4. If two GV fuzzy metric spaces (X, M, T_1) and (Y, N, T_2) are isometric, then (X, M, T_2) and (Y, N, T_1) are GV fuzzy metric spaces.

Proof. From the hypothesis, it follows that there is an one-to-one mapping f from X to Y such that $M(x, y, t) = N(f(x), f(y), t)$ for each $x, y \in X$ and each $t > 0$. Thus for any $x, y, z \in X, t, s > 0$, we have that

$$\begin{aligned} M(x,z,t+s) = N(f(x),f(z),t+s) &\geq T_2(N(f(x),f(y),t),N(f(y),f(z),s)) \\ &= T_2(M(x,y,t),M(y,z,s)) \end{aligned}$$

which implies (X, M, T_2) is a GV fuzzy metric space. Similarly, we can prove that (Y, N, T_1) is a GV fuzzy metric space. □

From Theorem 3.4, we can easily obtain the following criterion.

Corollary 3.1. For any two GV fuzzy metric spaces (X, M, T_1) and (Y, N, T_2), if either (X, M, T_2) or (Y, N, T_1) is not a GV fuzzy metric space, then (X, M, T_1) and (Y, N, T_2) are non-isometric.

Example 3.3. The 3-tuple $(\mathbb{N}, M_d, T_M)$ where $M_d(x, y, t) = t/(t + d(x, y))$, is a GV fuzzy metric space [2]. In addition, we define a fuzzy set N by

$$N(x,y,t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y, \\ \frac{y}{x} & \text{if } y \leq x, \end{cases}$$

for all $x, y \in \mathbb{N}$ and $t > 0$. Then $(\mathbb{N}, N, T_P)$ is also a GV fuzzy metric space [2]. Since $(\mathbb{N}, N, T_M)$ is not a GV fuzzy metric space [2], by Corollary 3.1 we have that $(\mathbb{N}, M_d, T_M)$ and $(\mathbb{N}, N, T_P)$ are non-isometric.

Theorem 3.5. Let X be a set that has at least a countable infinite number of elements. Then there exist infinite non-isometric GV fuzzy metrics on it.

Proof. For any two positive continuous t-norms T_1 and T_2 with $T_1 < T_2$, by Theorem 3.2, we have that there exist two corresponding GV fuzzy metric spaces (X, M_1) and (X, M_2) such that T_1 and T_2 are, respectively, the strongest t-norms. Then (X, M_1, T_2) is not a GV fuzzy metric space. By Corollary 3.1, we have that M_1 and M_2 are two non-isometric GV fuzzy metrics on X. From Remark 2.1, it follows that there are infinite comparable positive continuous t-norms. Thus there are infinite non-isometric GV fuzzy metrics on X. □

4. CONCLUSION

In this work we have discussed the relationships between GV fuzzy metric spaces and continuous t-norms. As an application of the main results, it is shown that there are infinite non-isometric GV fuzzy metrics on an infinite set. However, in general, the following problem remains open:

Open problem 4.1. For any continuous t-norm T, is there a nonempty set X and a fuzzy set M on $X^2\times]0,\infty[$ for which T is the strongest t-norm which makes (X, M, T) to be a GV fuzzy metric space?

Even the following problem but except for the trivial case in which the set is of no more than two elements still remains open:

Open problem 4.2. For a given positive continuous t-norm T and a given nonempty finite set X, is there a fuzzy set M on $X^2\times]0,\infty[$ for which T is the strongest t-norm which makes (X, M, T) to be a GV fuzzy metric space?

ACKNOWLEDGEMENT

The authors thank the anonymous reviewers for their valuable comments. This work was supported by The Mathematical Tianyuan Foundation of China (Grant no. 11126087), The National Natural Science Foundation of China (Grant no.11201512) and The Natural Science Foundation Project of CQ CSTC (cstc2012jjA00001).

(Received November 16, 2011)

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Dong Qiu, College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Nanan, Chongqing, 400065. P. R. China.
e-mail: dongqiu@yahoo.cn

Weiquan Zhang, School of Information Engineering, Guangdong Medical College, Dongguan, Guangdong, 523808. P. R. China.
e-mail: w9zhang@126.com