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ESTIMATES FOR k-HESSIAN OPERATOR AND SOME APPLICATIONS

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Abstract. The k-convex functions are the viscosity subsolutions to the fully nonlinear elliptic equations $F_k[u]=0$, where $F_k[u]$ is the elementary symmetric function of order k, $1\leqslant k\leqslant n$, of the eigenvalues of the Hessian matrix D^2u. For example, $F_1[u]$ is the Laplacian Δu and $F_n[u]$ is the real Monge-Ampère operator $\det D^2u$, while 1-convex functions and n-convex functions are subharmonic and convex in the classical sense, respectively. In this paper, we establish an approximation theorem for negative k-convex functions, and give several estimates for the mixed k-Hessian operator. Applications of these estimates to the k-Green functions are also established.

Keywords: k-convex function, k-Hessian operator, k-Hessian measure, k-Green function

MSC 2010: 47J20, 58C35, 31A15, 31A05

1. Introduction

The k-Hessian operator F_k and the k-Hessian measure μ_k were introduced by N. S. Trudinger and X. J. Wang for k-convex functions. In [9], they proved the weak continuity of the associated k-Hessian measure μ_k with respect to the convergence in L^1, which is generalized in [10] to the weak continuity of the mixed k-Hessian measure $\tilde{\mu}_k$ associated with k-tuples of k-convex functions. And they gave some applications to the corresponding capacity, quasicontinuity and the Dirichlet problem. On the other hand, the pluripotential theory in several complex variables, especially the complex Monge-Ampère operator, has attracted considerable studies. There are many applications to complex analysis, complex differential geometry and number theory (cf. [3], [6] and references therein). It is interesting to use tools and ideas

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from the theory of the complex Monge-Ampère operator to study the real k-Hessian operator. In [8], [9], [10] N. S. Trudinger, X. J. Wang and D. Labutin have given some corresponding results in pluripotential theory.

The weak continuity of the k-Hessian measure with respect to the convergence in L^1 is not valid for the complex Monge-Ampère measure in the plurisubharmonic case. With help of this stronger result for the k-Hessian operator, it is natural to ask whether we can prove some new results for k-convex functions. The purpose of this paper is to establish a global approximation theorem for negative k-convex functions, and give several estimates for the mixed k-Hessian operator. Moreover, we show how these estimates are applied to the convergence of the k-Green function.

We shall adopt definitions and notation from [9] and [10]. Let Ω be an open set in the n-dimensional Euclidean space $\mathbb{R}^n$. For $k = 1,\ldots,n$ and $u \in C^2(\Omega)$, let $\lambda = (\lambda_1,\ldots,\lambda_n)$ denote the eigenvalues of the Hessian matrix of the second derivatives D^2u and let S_k be the kth elementary symmetric function on $\mathbb{R}^n$, given by

$$S_k(\lambda) = \sum_{i_1<\ldots<i_k} \lambda_{i_1}\ldots\lambda_{i_k}. \tag{1.1}$$

The *k-Hessian operator* F_k is defined by $F_k[u] = S_k(\lambda(D^2u))$. We may write $F_k[u] = [D^2u]_k$, where for any $n\times n$ matrix $\mathcal{A}$, $[\mathcal{A}]_k$ denotes the sum of its $k\times k$ principal minors. $F_1[u] = \Delta u$ is the Laplace operator and $F_n[u] = \det D^2u$ is the real Monge-Ampère operator.

In the viscosity sense, an upper semi-continuous function, $u\colon \Omega \to [-\infty,\infty)$ is called *k-convex* in Ω if $F_k[q] \geqslant 0$ for all quadratic polynomials q for which the difference $u-q$ has a finite local maximum in Ω. In particular, a function $u \in C^2(\Omega)$ is k-convex in Ω if and only if $F_j[u] \geqslant 0$ in Ω, $j = 1,\ldots,k$. A k-convex function is called *proper* if it does not assume the value $-\infty$ identically on any component of Ω. We denote by $\Phi^k(\Omega)$ the class of proper k-convex functions in Ω. The definition of a 1-convex function is equivalent to the usual definition of a subharmonic function. For $j \leqslant k$, $\Phi^k(\Omega) \subseteq \Phi^j(\Omega)$. And a function $u \in \Phi^n(\Omega)$ if and only if it is convex on each component of Ω. A k-convex function is therefore subharmonic, in particular, it is locally integrable. Corresponding to the k-Hessian operator F_k, there exists a mixed k-Hessian operator $\tilde{F}_k$ in [10], determined by the polarized form $\tilde{S}_k$ of the k-homogeneous polynomial S_k. For $u^1,\ldots,u^k \in C^2(\Omega)$, the *mixed k-Hessian operator* is defined by

$$\tilde{F}_k[u^1,\ldots,u^k] = \tilde{S}_k(\lambda(D^2u^1),\ldots,\lambda(D^2u^k)). \tag{1.2}$$

The k-*Hessian measure* associated with a function $u \in \Phi^k(\Omega)$ is a non-negative Borel measure defined by

$$\int_\Omega \eta \,\mathrm{d}\mu_k[u] = \lim_{h\to 0} \int_\Omega \eta F_k[u_h] \tag{1.3}$$

for any $\eta \in C_0^\infty(\Omega)$, where u_h is the smooth regularization of u. The *mixed* k-*Hessian measure* associated with $u^1, \ldots, u^k \in \Phi^k(\Omega)$ is defined by

$$\int_\Omega \eta \,\mathrm{d}\tilde{\mu}_k[u^1, \ldots, u^k] = \lim_{h\to 0} \int_\Omega \eta \tilde{F}_k[u_h^1, \ldots, u_h^k] \tag{1.4}$$

for any $\eta \in C_0^\infty(\Omega)$, where u_h^i is the smooth regularization of u^i, $i = 1, \ldots, k$. The measure is independent of the choice of the smooth regularization u_h^i of u^i by virtue of the approximation theorem (cf. Theorem 2.4 in [10]).

We say a bounded domain $\Omega \subseteq \mathbb{R}^n$ is k-*hyperconvex* if it is connected and there is a continuous k-convex function $\varrho\colon\ \Omega \to [-\infty, 0)$ such that the set $\{x \in \Omega\colon\ \varrho(x) < c\}$ is a relatively compact subset of Ω, for each $c \in (-\infty, 0)$. A function ϱ is called an *exhaustion function* of Ω. It follows that $\varrho(x) \to 0$ as $x \to \partial\Omega$. Note that ϱ may assume $-\infty$ as a value. In this paper, Ω will always denote a k-hyperconvex domain.

We establish a global approximation of negative k-convex functions by decreasing sequences of negative k-convex functions which are continuous on $\bar{\Omega}$, equal to zero on $\partial\Omega$ and have bounded k-Hessian mass. These functions serve as test functions.

Theorem 1.1. *For any negative k-convex function u, there is a decreasing sequence of functions $u_j \in \Phi^k(\Omega) \cap C(\bar{\Omega})$ with $u_j|_{\partial\Omega} = 0$, $j \in \mathbb{N}$, $\mu_k[u_j](\Omega) < +\infty$, and $\lim\limits_{j\to+\infty} u_j(x) = u(x)$ for $x \in \Omega$.*

We remark here that the same approximation of negative plurisubharmonic functions was given by U. Cegrell in [3]. And F. Wikström showed in [13] that upper bounded plurisubharmonic functions can be approximated from above by plurisubharmonic functions continuous up to the boundary on B-regular domains.

The following estimates were established for the complex Monge-Ampère operator by Z. Blocki [1], U. Cegrell [3], and U. Cegrell, L. Persson [4], respectively. Here we give these estimates for the mixed k-Hessian operator. They are a different type of relationship between the k-Hessian operator and the mixed k-Hessian operator.

Theorem 1.2. *Suppose that u, h are k-convex functions satisfying $u \leqslant h$ in Ω, and $\lim\limits_{x\to\partial\Omega}(h(x) - u(x)) = 0$, $v_1, \ldots, v_k$ are nonpositive bounded k-convex functions in Ω. Then we have*

$$\int_\Omega (h-u)^k \,\mathrm{d}\tilde{\mu}_k[v_1, \ldots, v_k] \leqslant k!\|v_1\|_\Omega \ldots \|v_{k-1}\|_\Omega \int_\Omega |v_k| \,\mathrm{d}\mu_k[u]. \tag{1.5}$$

Theorem 1.3. *Suppose that $h, u_1, u_2, v_1, \ldots, v_{k-p-q}$ are nonpositive k-convex functions of x in Ω, $1 \leqslant p$, $q \leqslant k$, $p+q \leqslant k$, each function converges to zero as x tends to $\partial\Omega$ and $\mu_k[u_i](\Omega) < +\infty$, $i = 1, 2$, $\mu_k[v_j](\Omega) < +\infty$, $j = 1, \ldots, k-p-q$. Then*

$$
\begin{aligned}
(1.6)\qquad & \int (-h)\,\mathrm{d}\tilde{\mu}_k[\underbrace{u_1, \ldots, u_1}_{p}, \underbrace{u_2, \ldots, u_2}_{q}, v_1, \ldots, v_{k-p-q}] \\
& \leqslant \Big(\int (-h)\,\mathrm{d}\tilde{\mu}_k[\underbrace{u_1, \ldots, u_1}_{p+q}, v_1, \ldots, v_{k-p-q}] \Big)^{p/(p+q)} \\
& \quad \times \Big(\int (-h)\,\mathrm{d}\tilde{\mu}_k[\underbrace{u_2, \ldots, u_2}_{p+q}, v_1, \ldots, v_{k-p-q}] \Big)^{q/(p+q)}.
\end{aligned}
$$

Corollary 1.1. *Let $h, u_1, \ldots, u_k$ be nonpositive k-convex functions of x in Ω satisfying $\mu_k[u_i](\Omega) < +\infty$, $i = 1, \ldots, k$. Suppose that each function converges to zero as x tends to $\partial\Omega$. Then*

$$
(1.7)\qquad \int (-h)\,\mathrm{d}\tilde{\mu}_k[u_1, \ldots, u_k] \leqslant \Big(\int (-h)\,\mathrm{d}\mu_k[u_1] \Big)^{1/k} \cdots \Big(\int (-h)\,\mathrm{d}\mu_k[u_k] \Big)^{1/k}.
$$

Theorem 1.4. *Suppose that u, v are bounded k-convex functions satisfying $\lim\limits_{x\to a\in\partial\Omega} u(x) = \lim\limits_{x\to a\in\partial\Omega} v(x) = 0$. Then for $p \geqslant 1$, $0 \leqslant j \leqslant k$,*

$$
\begin{aligned}
(1.8)\qquad & \int (-u)^p\,\mathrm{d}\tilde{\mu}_k[\underbrace{u, \ldots, u}_{j}, v, \ldots, v] \\
& \leqslant C_{p,j} \Big(\int (-u)^p\,\mathrm{d}\mu_k[u] \Big)^{(p+j)/(k+p)} \Big(\int (-v)^p\,\mathrm{d}\mu_k[v] \Big)^{(k-j)/(k+p)}
\end{aligned}
$$

where $C_{p,j} = p^{(p+j)(k-j)/(p-1)}$ for $p > 1$ and $C_{1,j} = \exp[(1+j)(k-j)]$.

For $a \in \Omega$, $k < n/2$, the k-*Green function with a pole at* a is defined as $g_\Omega(x,a) = \sup\{u(x)\colon\ u \in \Phi^k(\Omega, [-\infty, 0));\ u \text{ has a pole at } a \text{ of order } (2-n/k)\}$. Here we say a k-convex function u *has a pole at* a *of order* $(2-n/k)$, if u satisfies $u(x) + \|x-a\|^{2-n/k} \leqslant O(1)$ as $x \to a$. For $k = n/2$, the k-Green function with a pole at a is defined as $g_\Omega(x,a) = \sup\{u(x)\colon\ u \in \Phi^k(\Omega, [-\infty, 0));\ u(x) - \log\|x-a\| \leqslant O(1) \text{ as } x \to a\}$. There is no k-Green function defined in the case of $k > n/2$, since the fundamental solution to the k-Hessian operator F_k in this case is given by $\|x-a\|^{2-n/k}$ which is bounded (cf. [9]). See [12, Appendix A] for more information about k-Green functions.

We proved in [12] that $g_\Omega(x,a) \to 0$ as $x \to \partial\Omega$. We now consider the following question: given a point $y \in \Omega$ and a point $x_0 \in \partial\Omega$, is it true that $\lim_{n\to\infty} g_\Omega(y,x_n) = 0$ for every sequence $\{x_n\}$ tending to x_0? The corresponding question for the classical Green function G is of no interest, since G is symmetrical, that is, $G(x,y) = G(y,x)$ for every x and y. However, it is unknown whether the k-Green function is symmetrical or not. We can, however, prove the following weaker results.

The *k-Hessian capacity* of the Borel set $E \Subset \Omega$ in $\mathbb{R}^n$ is defined by

$$\operatorname{Cap}_k(E) = \operatorname{Cap}_k(E,\Omega) = \sup\{\mu_k[u](E)\colon\ u \in \Phi^k(\Omega),\ -1 \leqslant u \leqslant 0\} \tag{1.9}$$

(cf. [10]). For $u_j, u \in \Phi^k(\Omega)$, we say that a sequence $\{u_j\}$ *converges to u in capacity* if for any $\varepsilon > 0$ and relatively compact subset $K \Subset \Omega$,

$$\lim_{j\to\infty} \operatorname{Cap}_k(K \cap \{|u_j - u| > \varepsilon\}) = 0. \tag{1.10}$$

Proposition 1.1. *Let $k < n/2$ and let $\{x_j\}$ be a sequence of points in Ω tending to $\partial\Omega$ as $j \to \infty$. Then $g_\Omega(\cdot, x_j)$ converges to zero in capacity as $j \to \infty$.*

Proposition 1.2. *Suppose that $k < n/2$, and $\{x_n\}$ is a sequence of points in Ω tending to $x_0 \in \partial\Omega$ as $n \to \infty$. Then there exists a k-polar set $E \subseteq \Omega$ (depending on the sequence) such that*

$$\varlimsup_{n\to\infty} g_\Omega(w,x_n) = 0 \quad \textit{for every } w \in \Omega \setminus E.$$

The above two propositions for the complex Monge-Ampère operator were proved by R. Czyż [5], M. Carlehed, U. Cegrell and F. Wikström [2], respectively. Although we use methods from pluripotential theory, our results are completely new for the k-Hessian operator and some of them are even stronger. The complex Monge-Ampère operator can be written as $(dd^c u)^n$ in terms of the operators d and d^c. This is very important in establishing the properties of the complex Monge-Ampère operator since it allows us to integrate by parts. Although we cannot write the k-Hessian operator in the form of exterior differentials, the k-Hessian operator is an operator of divergence form, and so we can also integrate by parts. This is an essential reason why we can prove some k-Hessian versions of results in the complex pluripotential theory.

The paper is organized as follows. In Section 2, we establish the approximation theorem. The relative extremal function plays a crucial role in our proof. The results about the k-Hessian operator and the mixed k-Hessian operator we need are collected in Section 3, and Theorems 1.2, 1.3, 1.4 and Corollary 1.1 are proved in Section 3. In the last Section, our estimates are applied to the k-Green function. Proposition 1.1 and 1.2 are proved.

2. Approximation of k-convex functions

The purpose of this section is to prove Theorem 1.1. The construction of the approximation sequence follows the process in [3]. The main difference between our proof and the proof in the plurisubharmonic case in [3] lies in the difficulty of finding a continuous k-convex function which vanishes on $\partial\Omega$ and has bounded k-Hessian mass. In the case of plurisubharmonic functions, this is a classical result due to J. B. Walsh [11].

Analogously to the potential theory for subharmonic functions, D. Labutin introduced the regularized relative extremal function for a k-convex function in [8]. For a relatively compact set E in Ω, we denote by $h^*_{E,\Omega}$ its regularized relative extremal function, which was defined as the upper semi-continuous regularization of the relative extremal function $h_{E,\Omega}$:

$$h_{E,\Omega}(x) = \sup\{u(x)\colon\ u \in \Phi^k(\Omega),\ u \leqslant 0,\ u|_E \leqslant -1\}, \quad x \in \Omega. \tag{2.1}$$

If $E_1 \subseteq E_2 \subseteq \Omega_1 \subseteq \Omega_2$, then

$$h_{E_1,\Omega_1} \geqslant h_{E_2,\Omega_1} \geqslant h_{E_2,\Omega_2}. \tag{2.2}$$

The following lemmas concern some properties of k-convex functions, which are used throughout the paper (cf. [12] for the proof).

Lemma 2.1. (1) *Suppose $\{u_\alpha\}_{\alpha\in A}$ is a sequence of k-convex functions in Ω such that its upper envelope $u = \sup\limits_{\alpha\in A} u_\alpha$ is locally bounded from above. Then the upper semi-continuous regularization u^* is k-convex in Ω.*

(2) *If u, v are k-convex in Ω, then $\max\{u, v\}$ is also k-convex.*

(3) *Suppose that Ω is an open set in $\mathbb{R}^n$ and $\tilde\Omega$ is a non-empty proper open subset of Ω. If $u \in \Phi^k(\Omega)$, $v \in \Phi^k(\tilde\Omega)$, and $\limsup\limits_{x\to y} v(x) \leqslant u(y)$ for $y \in \partial\tilde\Omega \cap \Omega$, then*

$$\omega = \begin{cases} \max\{u, v\}, & \text{on } \tilde\Omega,\\ u, & \text{on } \Omega\setminus\tilde\Omega, \end{cases}$$

is a k-convex function in Ω.

Lemma 2.2. *Suppose that $K \subseteq \Omega$ is a compact set such that $h^*_{K,\Omega} \equiv -1$ on K. Then $h_{K,\Omega}$ is continuous in Ω.*

P r o o f. Denote $h = h_{K,\Omega}$. Assume that ϱ is an exhaustion function of Ω such that $\varrho < -1$ on K, then $\varrho \leqslant h$ in Ω. For each $\varepsilon \in (0,1)$, there exists $\eta > 0$ such that $h - \varepsilon/2 \leqslant -\varepsilon/2 < \varrho$ in $\Omega \setminus \Omega_\eta$ and $K \subseteq \Omega_\eta$, where $\Omega_\eta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \eta\}$. By Lemma 2.1 (1), h^* is k-convex, thus it is upper semi-continuous. Denote by h_δ the smooth regularization of h, defined on Ω_δ, then $(h^*)_\delta$ decreasingly converges to h^* as $\delta \to 0$. Note that $h^* = h$ a.e. in Ω, hence we have $(h^*)_\delta = h_\delta$. By Dini's theorem, h_δ converges to h^* uniformly on any compact set. Therefore we can find a uniform $\delta > 0$ such that $h_\delta - \varepsilon < h^* - \varepsilon/2 < \varrho$ on $\partial\Omega_\eta$ and $h_\delta - \varepsilon < -1$ on K. Define

$$g_\varepsilon = \begin{cases} \varrho, & \text{in } \Omega \setminus \Omega_\eta, \\ \max\{h_\delta - \varepsilon, \varrho\}, & \text{in } \Omega_\eta. \end{cases}$$

Then $g_\varepsilon \in \Phi^k \cap C(\Omega)$ by Lemma 2.1 (3) and $g_\varepsilon < -1$ on K. Thus $g_\varepsilon \leqslant h$ in Ω. On the other hand, $h - \varepsilon \leqslant \max\{h_\delta - \varepsilon, \varrho\} \leqslant g_\varepsilon$ at each point in Ω. It follows that h is continuous in Ω. □

Corollary 2.1. *Suppose that $K \subseteq \Omega$ is compact and it is the union of a family of closed balls. Then $h_{K,\Omega} = h^*_{K,\Omega}$ is continuous in Ω.*

P r o o f. By Lemma 2.2, it suffices to prove that $h_{K,\Omega}$ is continuous at ∂K. By hypothesis, let $b \in \partial K$ and choose $a \in K$ and $R > r > 0$ such that $b \in \overline{B}(a,r) \subseteq K$ and $B(a,R) \subseteq \Omega$. Then by (2.2) we have $h_{K,\Omega}(x) \leqslant h_{\overline{B}(a,r),\Omega}(x) \leqslant h_{\overline{B}(a,r),B(a,R)}(x)$ for $x \in B(a,R)$. It follows from the fact shown in [8],

$$h_{\overline{B}(a,r),B(a,R)}(x) = \begin{cases} \max\Big\{-1, \dfrac{R^{2-n/k} - \|x-a\|^{2-n/k}}{r^{2-n/k} - R^{2-n/k}}\Big\}, & \text{for } 1 \leqslant k < n/2, \\ \max\Big\{-1, \dfrac{\log(\|x-a\|/R)}{\log(R/r)}\Big\}, & \text{for } k = n/2, \end{cases}$$

that $\lim\limits_{x \to b} h_{K,\Omega}(x) = -1$. □

P r o o f of Theorem 1.1. If $E = \overline{B}$ is a ball, we have $\mu_k[h_{\overline{B},\Omega}] = 0$ in $\Omega \setminus \overline{B}$, thus $\operatorname{supp}\mu_k[h_{\overline{B},\Omega}] \Subset \overline{B}$ (cf. [8]). By Corollary 2.1, $h_{\overline{B},\Omega}$ is continuous in Ω. Let ϱ be an exhaustion function of Ω such that $C\varrho \leqslant -1$ in $\overline{B}$. Then $C\varrho \leqslant h_{\overline{B},\Omega} \leqslant 0$. Thus $h_{\overline{B},\Omega}(x) \to 0$ as $x \to \partial\Omega$. Therefore, there exists a function $v \in \Phi^k(\Omega) \cap C(\bar{\Omega})$ with $\mu_k[v](\Omega) < +\infty$ and $v|_{\partial\Omega} = 0$. Then follow the process in [3] to construct the approximation sequence. □

3. Main estimates for k-Hessian operator

Before proving our main estimates, we present some properties of the k-Hessian operator (see [10] for details).

The mixed k-Hessian operator $\tilde{F}_k$ defined by (1.2) is linear in each $u^s \in C^2(\Omega)$, $s = 1, \dots, k$, invariant under permutations and $\tilde{F}_k[u, \dots, u] = F_k[u]$ for any $u \in C^2(\Omega)$. Moreover, we have the explicit representation

$$\tilde{F}_k[u^1, \dots, u^k] = \frac{1}{k!} \sum \delta^{i_1 \dots i_k}_{j_1 \dots j_k} u^1_{i_1 j_1} \dots u^k_{i_k j_k}, \tag{3.1}$$

where $u^s_{ij} = D_{ij} u^s$, $i, j = 1, \dots, n$, $s = 1, \dots, k$, the summation takes over all multiindexes $\{i_1 \dots i_k\}$ and $\{j_1 \dots j_k\}$ and $\delta^{i_1 \dots i_k}_{j_1 \dots j_k}$ denotes the generalized Kronecker delta, which vanishes if $\{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\}$ and otherwise it is the sign of the permutation from $(i_1, \dots, i_k)$ to $(j_1, \dots, j_k)$. An inequality of Gårding [7],

$$\tilde{F}_k[u^1, \dots, u^k] \geqslant \prod_{s=1}^{k} (F_k[u^s])^{1/k} \tag{3.2}$$

for $u^1, \dots, u^k \in \Phi^k(\Omega)$, guarantees $\tilde{F}_k \geqslant 0$ on $(\Phi^k)^k$. Let

$$\tilde{F}^{ij}_k[u^1, \dots, u^{k-1}] = \frac{\partial}{\partial u^k_{ij}} \tilde{S}_k(\lambda(D^2 u^1), \dots, \lambda(D^2 u^k)) \tag{3.3}$$

be the coefficient of u^k_{ij} in (3.1). Then we have

$$D_i \tilde{F}^{ij}_k[u^1, \dots, u^{k-1}] = 0, \quad j = 1, \dots, n, \tag{3.4}$$

and

$$\tilde{F}^{ij}_k[u, \dots, u] = \frac{1}{k} F^{ij}_k[u], \quad \text{where} \quad F^{ij}_k[u] = \frac{\partial}{\partial u_{ij}} S_k[\lambda(D^2 u]. \tag{3.5}$$

The matrix $[\tilde{F}^{ij}_k]_{n \times n}$ is positive definite. We can write $\tilde{F}_k$ in the divergence form

$$\tilde{F}_k[u^1, \dots, u^k] = \sum u^k_{ij} \tilde{F}^{ij}_k[u^1, \dots, u^{k-1}] = \sum D_i \{\tilde{F}^{ij}_k[u^1, \dots, u^{k-1}] D_j u^k\}. \tag{3.6}$$

Now we are ready to prove Theorem 1.2. Here we use ideas from Z. Blocki [1].

P r o o f of Theorem 1.2. Let $\tilde{h} = \max\{u, h - \varepsilon\}$. Then $\tilde{h} = u$ in a neighborhood of $\partial\Omega$ and $\tilde{h} - u$ converges to $h - u$ as $\varepsilon \to 0$. We can therefore assume that $h = u$

in a neighborhood of $\partial\Omega$. First assume that h, u and v_i are smooth. By hypothesis, we can let $\|v_i\|_\Omega = 1$. We want to show that for $p = 2, \dots, k$,

$$\begin{aligned}\int_\Omega (h-u)^p \tilde{F}_k[u, \dots, u, v_{k-p+1}, \dots, v_k] \\ \leqslant p \int_\Omega (h-u)^{p-1} \tilde{F}_k[u, \dots, u, v_{k-p+2}, \dots, v_k].\end{aligned} \tag{3.7}$$

By (3.6) and Stokes' theorem, the left-hand side of (3.7) is equal to

$$\begin{aligned}&\int_\Omega (h-u)^p \tilde{F}_k^{ij}[u, \dots, u, v_{k-p+2}, \dots, v_k] D_{ij} v_{k-p+1} \\ &\qquad = -\int_\Omega D_i[(h-u)^p] \tilde{F}_k^{ij}[u, \dots, u, v_{k-p+2}, \dots, v_k] D_j v_{k-p+1} \\ &\qquad = \int_\Omega v_{k-p+1} \tilde{F}_k^{ij}[u, \dots, u, v_{k-p+2}, \dots, v_k] D_{ij}[(h-u)^p] \\ &\qquad = \int_\Omega v_{k-p+1} \tilde{F}_k[u, \dots, u, (h-u)^p, v_{k-p+2}, \dots, v_k].\end{aligned}$$

And it follows from $-v_{k-p+1} = \|v_{k-p+1}\| \leqslant 1$ that the right-hand side of (3.7) is greater than or equal to $p\int_\Omega (-v_{k-p+1})(h-u)^{p-1} \tilde{F}_k[u, \dots, u, v_{k-p+2}, \dots, v_k]$. For $p = 2, \dots, k$, by (3.6) we have

$$\begin{aligned}&\tilde{F}_k[u, \dots, u, (h-u)^p, v_{k-p+2}, \dots, v_k] \\ &\quad = \tilde{F}_k^{ij}[u, \dots, u, v_{k-p+2}, \dots, v_k] D_{ij}[(h-u)^p] \\ &\quad = \tilde{F}_k^{ij}[u, \dots, u, v_{k-p+2}, \dots, v_k] p(p-1)(h-u)^{p-2} D_i(h-u) D_j(h-u) \\ &\qquad + \tilde{F}_k^{ij}[u, \dots, u, v_{k-p+2}, \dots, v_k] p(h-u)^{p-1} D_{ij}(h-u) \\ &\quad \geqslant p(h-u)^{p-1}(\tilde{F}_k[u, \dots, u, v_{k-p+2}, \dots, v_k, h] - \tilde{F}_k[u, \dots, u, v_{k-p+2}, \dots, v_k, u]) \\ &\quad \geqslant -p(h-u)^{p-1} \tilde{F}_k[u, \dots, u, v_{k-p+2}, \dots, v_k],\end{aligned}$$

where the first inequality follows from the positivity of the Hessian matrix $[F_k^{ij}]_{n\times n}$ and the last inequality follows from (3.2). Then we get (3.7). Now by (3.2), (3.7) and Stokes' theorem,

$$\begin{aligned}\int_\Omega (h-u)^k \tilde{F}_k[v_1, \dots, v_k] &\leqslant k! \int_\Omega (h-u) \tilde{F}_k[u, \dots, u, v_k] \\ &= k! \int_\Omega (-v_k) \tilde{F}_k[u, \dots, u, u-h] \\ &\leqslant k! \int_\Omega |v_k| \tilde{F}_k[u, \dots, u] = k! \int_\Omega |v_k| F_k[u].\end{aligned}$$

Now assume u and h are arbitrary and v_i still smooth. Let u_ε and h_ε be the smooth regularizations of u, h respectively, such that u_ε and h_ε decreasingly converge to u and h respectively as ε tends to 0. Then $u_\varepsilon \leqslant h_\varepsilon$. If we shrink Ω a little, then $u_\varepsilon = h_\varepsilon$ in a neighborhood of $\partial\Omega$. So by the proof above, we have

$$\int_\Omega (h_\varepsilon - u_\varepsilon)^k \tilde{F}_k[v_1, \ldots, v_k] \leqslant k! \|v_1\|_\Omega \ldots \|v_{k-1}\|_\Omega \int_\Omega |v_k| F_k[u_\varepsilon]. \tag{3.8}$$

By Lebesgue's dominated convergence theorem, the left-hand side of (3.8) tends to $\int_\Omega (h - u)\tilde{F}_k[v_1, \ldots, v_k]$, and by (1.3) the right-hand side of (3.8) tends to $k! \|v_1\|_\Omega \ldots \|v_{k-1}\|_\Omega \int_\Omega |v_k| \,\mathrm{d}\mu_k[u]$.

Finally, let $v_1, \ldots, v_k$ be arbitrary. Denote by $(v_i)_\varepsilon$ the smooth regularization of v_i, then $(v_i)_\varepsilon$ decreasingly converges to v_i as ε tends to 0. Note that

$$\begin{aligned}\int_\Omega (h-u)^k \tilde{F}_k[(v_1)_\varepsilon, \ldots, (v_k)_\varepsilon] &\leqslant k! \|(v_1)_\varepsilon\|_\Omega \ldots \|(v_{k-1})_\varepsilon\|_\Omega \int_\Omega |(v_k)_\varepsilon| \,\mathrm{d}\mu_k[u] \\ &\leqslant k! \|v_1\|_\Omega \ldots \|v_{k-1}\|_\Omega \int_\Omega |v_k| \,\mathrm{d}\mu_k[u],\end{aligned}$$

so that we have

$$\begin{aligned}\int_\Omega (h-u)^k \,\mathrm{d}\tilde{\mu}_k[v_1, \ldots, v_k] &\leqslant \liminf_{\varepsilon \to 0} \int_\Omega (h-u)^k \tilde{F}_k[(v_1)_\varepsilon, \ldots, (v_k)_\varepsilon] \\ &\leqslant k! \|v_1\|_\Omega \ldots \|v_{k-1}\|_\Omega \int_\Omega |v_k| \,\mathrm{d}\mu_k[u].\end{aligned}$$

□

Let $h = 0$ in Theorem 1.2. Then we have the following result.

Corollary 3.1. *Suppose that u is a nonpositive k-convex function satisfying $\lim\limits_{x \to \partial\Omega} u(x) = 0$, and $v_1, \ldots, v_k$ are nonpositive bounded k-convex functions in Ω. Then*

$$\int_\Omega |u|^k \,\mathrm{d}\tilde{\mu}_k[v_1, \ldots, v_k] \leqslant k! \|v_1\|_\Omega \ldots \|v_{k-1}\|_\Omega \int_\Omega |v_k| \,\mathrm{d}\mu_k[u].$$

Now we use the induction method in [3] to prove Theorem 1.3.

P r o o f of Theorem 1.3. First assume that all the functions are smooth. Let us prove the statement for $p=q=1$. By Stokes' theorem,

$$\begin{aligned}
\int (-h)\tilde{F}_k[u_1,u_2,v_1,\dots,v_{k-2}] &= \int (-h)\tilde{F}_k^{ij}[u_2,v_1,\dots,v_{k-2}]D_{ij}u_1 \\
&= \int D_ihD_ju_1\tilde{F}_k^{ij}[u_2,v_1,\dots,v_{k-2}] = \int (-u_1)\tilde{F}_k^{ij}[u_2,v_1,\dots,v_{k-2}]D_{ij}h \\
&= \int (-u_1)\tilde{F}_k[h,u_2,v_1,\dots,v_{k-2}] = \int D_iu_1D_ju_2\tilde{F}_k^{ij}[h,v_1,\dots,v_{k-2}] \\
&\leqslant \Big(\int D_iu_1D_ju_1\tilde{F}_k^{ij}[h,v_1,\dots,v_{k-2}]\Big)^{1/2}\Big(\int D_iu_2D_ju_2\tilde{F}_k^{ij}[h,v_1,\dots,v_{k-2}]\Big)^{1/2} \\
&= \Big(\int (-u_1)\tilde{F}_k^{ij}[h,v_1,\dots,v_{k-2}]D_{ij}u_1\Big)^{1/2}\Big(\int (-u_2)\tilde{F}_k^{ij}[h,v_1,\dots,v_{k-2}]D_{ij}u_2\Big)^{1/2} \\
&= \Big(\int (-u_1)\tilde{F}_k[h,u_1,v_1,\dots,v_{k-2}]\Big)^{1/2}\Big(\int (-u_2)\tilde{F}_k[h,u_2,v_1,\dots,v_{k-2}]\Big)^{1/2} \\
&= \Big(\int (-h)\tilde{F}_k[u_1,u_1,v_1,\dots,v_{k-2}]\Big)^{1/2}\Big(\int (-h)\tilde{F}_k[u_2,u_2,v_1,\dots,v_{k-2}]\Big)^{1/2}.
\end{aligned}$$

Assume that the theorem is proved for $p+q\leqslant m$. We now prove it for $p+q\leqslant m+1$. We first show that the following inequality holds:

$$\begin{aligned}
&\int (-h)\tilde{F}_k[u_1,\underbrace{u_2,\dots,u_2}_{p+q},v_1,\dots,v_{k-p-q-1}] \\
&\qquad\leqslant \Big(\int (-h)\tilde{F}_k[\underbrace{u_1,\dots,u_1}_{p+q+1},v_1,\dots,v_{k-p-q-1}]\Big)^{1/(p+q+1)} \\
&\qquad\quad\times\Big(\int (-h)\tilde{F}_k[\underbrace{u_2,\dots,u_2}_{p+q+1},v_1,\dots,v_{k-p-q-1}]\Big)^{(p+q)/(p+q+1)}.
\end{aligned}\tag{3.9}$$

By the assumption above,

$$\begin{aligned}
&\int (-h)\tilde{F}_k[u_1,\underbrace{u_2,\dots,u_2}_{p+q},v_1,\dots,v_{k-p-q-1}] \\
&\qquad\leqslant \Big(\int (-h)\tilde{F}_k[\underbrace{u_1,\dots,u_1}_{p+q},u_2,v_1,\dots,v_{k-p-q-1}]\Big)^{1/(p+q)} \\
&\qquad\quad\times\Big(\int (-h)\tilde{F}_k[\underbrace{u_2,\dots,u_2}_{p+q},u_2,v_1,\dots,v_{k-p-q-1}]\Big)^{(p+q-1)/(p+q)}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\int (-h) \tilde{F}_k[\underbrace{u_1, \dots, u_1}_{p+q-1}, u_2, u_1, v_1, \dots, v_{k-p-q-1}] \Big)^{1/(p+q)} \\
&\quad \times \Big(\int (-h) \tilde{F}_k[\underbrace{u_2, \dots, u_2}_{p+q+1}, v_1, \dots, v_{k-p-q-1}] \Big)^{(p+q-1)/(p+q)} \\
&\leqslant \Big[\Big(\int (-h) \tilde{F}_k[\underbrace{u_1, \dots, u_1}_{p+q}, u_1, v_1, \dots, v_{k-p-q-1}] \Big)^{(p+q-1)/(p+q)} \\
&\quad \times \Big(\int (-h) \tilde{F}_k[\underbrace{u_2, \dots, u_2}_{p+q}, u_1, v_1, \dots, v_{k-p-q-1}] \Big)^{1/(p+q)} \Big]^{1/(p+q)} \\
&\quad \times \Big(\int (-h) \tilde{F}_k[\underbrace{u_2, \dots, u_2}_{p+q+1}, v_1, \dots, v_{k-p-q-1}] \Big)^{(p+q-1)/(p+q)} .
\end{aligned}
$$

Therefore (3.9) follows. Using (3.9) to complete our induction we arrive at

$$
\begin{aligned}
&\int (-h) \tilde{F}_k[\underbrace{u_1, \dots, u_1}_{p+1}, \underbrace{u_2, \dots, u_2}_{q}, v_1, \dots, v_{k-p-q-1}] \\
&\quad \leqslant \Big(\int (-h) \tilde{F}_k[\underbrace{u_1, \dots, u_1}_{p+q}, u_1, v_1, \dots, v_{k-p-q-1}] \Big)^{p/(p+q)} \\
&\qquad \times \Big(\int (-h) \tilde{F}_k[\underbrace{u_2, \dots, u_2}_{p+q}, u_1, v_1, \dots, v_{k-p-q-1}] \Big)^{q/(p+q)} \\
&\quad \leqslant \Big(\int (-h) \tilde{F}_k[\underbrace{u_1, \dots, u_1}_{p+q+1}, v_1, \dots, v_{k-p-q-1}] \Big)^{p/(p+q)} \\
&\qquad \times \Big[\Big(\int (-h) \tilde{F}_k[\underbrace{u_2, \dots, u_2}_{p+q+1}, v_1, \dots, v_{k-p-q-1}] \Big)^{(p+q)/(p+q+1)} \\
&\qquad \times \Big(\int (-h) \tilde{F}_k[\underbrace{u_1, \dots, u_1}_{p+q+1}, v_1, \dots, v_{k-p-q-1}] \Big)^{1/(p+q+1)} \Big]^{q/(p+q)} \\
&\quad = \Big(\int (-h) \tilde{F}_k[\underbrace{u_1, \dots, u_1}_{p+q+1}, v_1, \dots, v_{k-p-q-1}] \Big)^{(p+1)/(p+q+1)} \\
&\qquad \times \Big(\int (-h) \tilde{F}_k[\underbrace{u_2, \dots, u_2}_{p+q+1}, v_1, \dots, v_{k-p-q-1}] \Big)^{q/(p+q+1)} .
\end{aligned}
$$

Let $\tilde{h} = \max\{h, -\varepsilon\}$, then $\tilde{h} = h$ in a neighborhood of $\partial\Omega$ and $\tilde{h} - h$ converges to $-h$ as $\varepsilon \to 0$. We can therefore assume that h vanishes in a neighborhood of $\partial\Omega$. For arbitrary h, denote by h_ε the smooth regularization of h such that h_ε decreasingly converges to h as ε tends to 0. By shrinking Ω a little, h_ε vanishes in a neighborhood of $\partial\Omega$. So the inequality (1.6) with h replaced by h_ε holds. And by the monotone convergence theorem, each integral $\int -h_\varepsilon \tilde{F}_k$ converges to $\int -h\tilde{F}_k$. Thus we get (1.6) for nonsmooth h. For the same reason, the formula (1.6) with u_i, v_j replaced by $(u_i)_\varepsilon, (v_j)_\varepsilon$ respectively holds. By Theorem 2.4 in [10] and Theorem 1.1,

$$\begin{aligned}
&\int (-h)\,\mathrm{d}\tilde{\mu}_k[\underbrace{(u_1)_\varepsilon, \ldots, (u_1)_\varepsilon}_{p}, \underbrace{(u_2)_\varepsilon, \ldots, (u_2)_\varepsilon}_{q}, (v_1)_\varepsilon, \ldots, (v_{k-p-q})_\varepsilon] \\
&\qquad \longrightarrow \int (-h)\,\mathrm{d}\tilde{\mu}_k[\underbrace{u_1, \ldots, u_1}_{p}, \underbrace{u_2, \ldots, u_2}_{q}, v_1, \ldots, v_{k-p-q}], \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

The theorem is finally proved. $\square$

P r o o f of Corollary 1.1. It follows from Theorem 1.3 that

$$\begin{aligned}
&\int (-h)\,\mathrm{d}\tilde{\mu}_k[u_1, \underbrace{u_2, \ldots, u_2}_{k-1}] \\
&\qquad \leqslant \Big(\int (-h)\,\mathrm{d}\tilde{\mu}_k[u_1, \ldots, u_1]\Big)^{1/k} \Big(\int (-h)\,\mathrm{d}\tilde{\mu}_k[u_2, \ldots, u_2]\Big)^{(k-1)/k} \\
&\qquad = \Big(\int (-h)\,\mathrm{d}\mu_k[u_1]\Big)^{1/k} \Big(\int (-h)\,\mathrm{d}\mu_k[u_2]\Big)^{(k-1)/k}.
\end{aligned}$$

This is the case of Corollary 1.1 for $u_2 = \ldots = u_k = u$. Suppose that the corollary is proved for $u_{p+1} = \ldots = u_k = u$. We now prove it for $u_{p+2} = \ldots = u_k = u$. By Theorem 1.3,

$$\begin{aligned}
&\int (-h)\,\mathrm{d}\tilde{\mu}_k[u_1, \ldots, u_p, u_{p+1}, u, \ldots, u] \\
&\leqslant \Big(\int (-h)\,\mathrm{d}\tilde{\mu}_k[u_1, \ldots, u_p, u_{p+1}, \ldots, u_{p+1}]\Big)^{1/(k-p)} \\
&\quad \times \Big(\int (-h)\,\mathrm{d}\tilde{\mu}_k[u_1, \ldots, u_p, u, \ldots, u]\Big)^{(k-p-1)/(k-p)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \Big[\Big(\int(-h)\,\mathrm{d}\mu_k[u_1]\Big)^{1/k}\dots\Big(\int(-h)\,\mathrm{d}\mu_k[u_p]\Big)^{1/k}\\
&\quad\times\Big(\int(-h)\,\mathrm{d}\mu_k[u_{p+1}]\Big)^{(k-p)/k}\Big]^{1/(k-p)}\\
&\quad\times\Big[\Big(\int(-h)\,\mathrm{d}\mu_k[u_1]\Big)^{1/k}\dots\Big(\int(-h)\,\mathrm{d}\mu_k[u_p]\Big)^{1/k}\\
&\quad\times\Big(\int(-h)\,\mathrm{d}\mu_k[u]\Big)^{(k-p)/k}\Big]^{(k-p-1)/(k-p)}\\
&=\Big(\int(-h)\,\mathrm{d}\mu_k[u_1]\Big)^{1/k}\dots\Big(\int(-h)\,\mathrm{d}\mu_k[u_{p+1}]\Big)^{1/k}\Big(\int(-h)\,\mathrm{d}\mu_k[u]\Big)^{(k-p-1)/k}.
\end{aligned}
$$

$\square$

Corollary 3.2. *Let $u_1,\dots,u_k$ be nonpositive k-convex functions of x in Ω satisfying $\mu_k[u_i](\Omega)<+\infty$, $i=1,\dots,k$. Suppose that each function converges to zero as x tends to $\partial\Omega$, then*

$$\tilde\mu_k[u_1,\dots,u_k](\Omega)\leqslant(\mu_k[u_1](\Omega))^{1/k}\dots(\mu_k[u_k](\Omega))^{1/k}.$$

P r o o f of Theorem 1.4. First we assume that both u,v are smooth. By Hölder's inequality and using Stokes' theorem twice we get

(3.10)

$$
\begin{aligned}
&\int(-u)^p\,\mathrm{d}\tilde\mu_k[\underbrace{u,\dots,u}_{j},v,\dots,v]=\int(-u)^p\tilde F_k^{ij}[\underbrace{u,\dots,u}_{j},v,\dots,v]D_{ij}v\\
&=-\int D_i[(-u)^p]\tilde F_k^{ij}[\underbrace{u,\dots,u}_{j},v,\dots,v]D_jv=\int v\tilde F_k^{ij}[\underbrace{u,\dots,u}_{j},v,\dots,v]D_{ij}(-u)^p\\
&=p(p-1)\int v(-u)^{p-2}D_iuD_ju\tilde F_k^{ij}[\underbrace{u,\dots,u}_{j},v,\dots,v]\\
&\quad+p\int(-v)(-u)^{p-1}\tilde F_k[\underbrace{u,\dots,u}_{j+1},v,\dots,v]\\
&\leqslant p\int(-v)(-u)^{p-1}\tilde F_k[\underbrace{u,\dots,u}_{j+1},v,\dots,v]\\
&\leqslant\Big(p\int(-v)^p\tilde F_k[\underbrace{u,\dots,u}_{j+1},v,\dots,v]\Big)^{1/p}\times\Big(p\int(-u)^p\tilde F_k[\underbrace{u,\dots,u}_{j+1},v,\dots,v]\Big)^{(p-1)/p},
\end{aligned}
$$

where the first inequality follows from the positivity of the Hessian matrix $[F_k^{ij}]_{n\times n}$.

Let

$$x_j = \log \int (-u)^p \tilde{F}_k[\underbrace{u,\ldots,u}_{j}, v,\ldots,v],\ y_j = \log \int (-v)^p \tilde{F}_k[v,\ldots,v,\underbrace{u,\ldots,u}_{k-j}].$$

By using the method in [4, p. 97–99], we then get the conclusion when $u, v \in C^\infty$.

Let $\tilde{u} = \max\{u, -\varepsilon\}$, then $\tilde{u} = u$ in a neighborhood of $\partial\Omega$ and $\tilde{u} - u$ converges to $-u$ as $\varepsilon \to 0$. We can therefore assume that u and v vanish in a neighborhood of $\partial\Omega$. For nonsmooth u and v, denote by u_ε and v_ε the smooth regularizations of u and v respectively. By shrinking Ω a little, u_ε and v_ε vanish in a neighborhood of $\partial\Omega$. So the inequality (1.8) with u, v replaced by $u_\varepsilon, v_\varepsilon$ respectively is valid. By the approximation theorem in [10, Theorem 2.6], as $\varepsilon \to 0$, we have

$$\int (-u_\varepsilon)^p \,\mathrm{d}\tilde{\mu}_k[\underbrace{u_\varepsilon,\ldots,u_\varepsilon}_{j}, v_\varepsilon,\ldots,v_\varepsilon] \longrightarrow \int (-u)^p \,\mathrm{d}\tilde{\mu}_k[\underbrace{u,\ldots,u}_{j}, v,\ldots,v],$$

$$\int (-u_\varepsilon)^p \,\mathrm{d}\mu_k[u_\varepsilon] \longrightarrow \int (-u)^p \,\mathrm{d}\mu_k[u] \quad \text{and} \quad \int (-v_\varepsilon)^p \,\mathrm{d}\mu_k[v_\varepsilon] \longrightarrow \int (-v)^p \,\mathrm{d}\mu_k[v].$$

□

4. Applications to k-Green functions

For $k < n/2$, $a \in \Omega$, we proved in [12] that $u(x) = g_\Omega(x, a)$ is the unique solution to the Dirichlet problem

$$\begin{cases} u \in \Phi^k(\Omega) \cap C(\Omega \setminus \{a\}), & \\ \mu_k[u] = C_{n,k}\delta_a, & \text{on } \Omega, \\ u(x) + \|x-a\|^{2-n/k} = O(1), & \text{as } x \to a, \\ u(x) \to 0, & \text{as } x \to \partial\Omega, \end{cases} \tag{4.1}$$

where $C_{n,k} = \omega_n \binom{n}{k}(n/k-2)^k$.

P r o o f of Proposition 1.1. With help of (4.1), Lemma 2.1 and Corollary 3.1, we can get the conclusion by repeating the process in [5, Theorem 1]. □

D. Labutin defined two types of exceptional sets in [8]. A set $E \subseteq \mathbb{R}^n$ is said to be *k-polar* if for each point $a \in E$ there are a neighborhood $B(a, r)$ and a function $u \in \Phi^k(B(a, r))$ such that $u|_{E\cap B(a,r)} = -\infty$. Let $\mathscr{U} \subseteq \Phi^k(\Omega)$ be a family of functions which are locally bounded from above. Define $U(x) = \sup\{u(x)\colon u \in \mathscr{U}\}$ for $x \in \Omega$. Denote by U^* the upper semicontinuous regularization of U. A set N is called *k-negligible* if $N \subseteq \{x \in \Omega\colon U(x) < U^*(x)\}$ for some family $\mathscr{U}$ as above. D. Labutin proved in [8, Theorem 4.2] that the following three statements are equivalent for a set $E \Subset B_R$: (1) E is k-polar; (2) E is k-negligible; (3) $\mathrm{Cap}_k(E, B_R) = 0$. D. Labutin also proved that compact sets with finite $(n-2k)$-Hausdorff measure have zero capacity, thus being k-polar sets (see [8, Section 5] for details).

Now we proceed with the proof of Proposition 1.2. Our method is due to M. Carlehed, U. Cegrell and F. Wikström [2]. We need the following lemma. Denote by $\Phi_0^k(\Omega)$ the convex cone of proper k-convex functions φ with $\lim\limits_{x\to\partial\Omega}\varphi(x) = 0$ and $\mu_k[\varphi](\Omega) < +\infty$.

Lemma 4.1. $C_0^\infty(\Omega) \subseteq \Phi_0^k \cap C(\bar\Omega) - \Phi_0^k \cap C(\bar\Omega)$.

Proof. Let $\chi \in C_0^\infty(\Omega)$. Then $\chi + m\|x\|^2$ is positive and k-convex if the constant $m > 0$ is sufficiently large. Let $a < \inf\chi < \sup\limits_\Omega(\chi + m\|x\|^2) < b$. Choose a negative function $f \in \Phi_0^k \cap C(\bar\Omega)$. Define $g_1 = \max(\chi + m\|x\|^2 - b, Cf)$, where C is sufficiently large such that $Cf < a - b$ on the support of χ. Then by Lemma 2.1 (2), $g_1 \in \Phi_0^k \cap C(\bar\Omega)$ and so $g_2 = \max(m\|x\|^2 - b, Cf)$. Then $\chi = g_1 - g_2 \in \Phi_0^k \cap C(\bar\Omega) - \Phi_0^k \cap C(\bar\Omega)$. $\square$

Proof of Proposition 1.2. Define $u_j(w) := \sup\limits_{n\geqslant j} g_\Omega(w, x_n)$. By Lemma 2.1 (1), u_j^* is a bounded k-convex function and converges to 0 as w tends to $\partial\Omega$. Let h be a function in $\Phi_0^k \cap C(\bar\Omega)$ and $h(x_n) \to 0$ as $n \to \infty$. Denote by $(u_j^*)_\varepsilon$ the smooth regularization of u_j^*. By (1.3), (3.6) and Stokes' theorem, we have

$$
\begin{aligned}
0 &\leqslant \int_\Omega (-h)\,\mathrm{d}\mu_k[u_j^*] = \lim_{\varepsilon\to 0}\int_\Omega (-h)F_k[(u_j^*)_\varepsilon] \\
&= \lim_{\varepsilon\to 0}\int_\Omega (-h)\tilde F_k^{ij}[(u_j^*)_\varepsilon, \ldots, (u_j^*)_\varepsilon]D_{ij}(u_j^*)_\varepsilon \\
&= \lim_{\varepsilon\to 0}\int_\Omega D_i h\tilde F_k^{ij}[(u_j^*)_\varepsilon, \ldots, (u_j^*)_\varepsilon]D_j(u_j^*)_\varepsilon \\
&= \lim_{\varepsilon\to 0}\int_\Omega -(u_j^*)_\varepsilon\tilde F_k^{ij}[(u_j^*)_\varepsilon, \ldots, (u_j^*)_\varepsilon]D_{ij}h \\
&\leqslant \lim_{\varepsilon\to 0}\int_\Omega -(g_\Omega(\cdot, x_n))_\varepsilon\tilde F_k^{ij}[(u_j^*)_\varepsilon, \ldots, (u_j^*)_\varepsilon]D_{ij}h \\
&= \lim_{\varepsilon\to 0}\int_\Omega (-h)\tilde F_k^{ij}[(u_j^*)_\varepsilon, \ldots, (u_j^*)_\varepsilon]D_{ij}(g_\Omega(\cdot, x_n))_\varepsilon \\
&= \lim_{\varepsilon\to 0}\int_\Omega (-h)\tilde F_k[(u_j^*)_\varepsilon, \ldots, (u_j^*)_\varepsilon, (g_\Omega(\cdot, x_n))_\varepsilon] \leqslant \ldots \\
&\leqslant \lim_{\varepsilon\to 0}\int_\Omega (-h)\tilde F_k[(g_\Omega(\cdot, x_n))_\varepsilon, \ldots, (g_\Omega(\cdot, x_n))_\varepsilon] = \lim_{\varepsilon\to 0}\int_\Omega (-h)F_k[(g_\Omega(\cdot, x_n))_\varepsilon] \\
&= \int_\Omega (-h)\,\mathrm{d}\mu_k[(g_\Omega(\cdot, x_n))] = -C_{n,k}h(x_n) \to 0
\end{aligned}
$$

as $n \to \infty$, where the last identity follows from (4.1). It follows from Lemma 4.1 that the measures $\mu_k[u_j^*]$ converge weakly to 0. By the continuity of the k-Hessian operator on the sequence $\{u_j^*\} \subseteq \Phi^k(\Omega)$ (cf. [9, Theorem 1.1]), the limit function $u := \lim_j u_j^*$ satisfies $\mu_k[u] = 0$. Thus u is k-maximal (cf. [12, Theorem A.1]) and equals 0 on $\partial\Omega$. It follows that $u \equiv 0$. Since $\{u_j^*\}$ is a decreasing sequence, each u_j^* must vanish identically in Ω. For each j, there is a k-polar set E_j such that $u_j \equiv 0$ on $\Omega \setminus E_j$. Let $E = E_1$. Since $\{u_j\}$ is decreasing, the k-polar sets E_j form an increasing sequence. On the other hand, if $w \in E_j \setminus E$, then $g_\Omega(w, x_l) = 0$ for some l, $1 \leqslant l \leqslant j$, which is impossible. So we have $E_j = E$ for all j. Thus $u_j \equiv 0$ on $\Omega \setminus E$. $\square$

The exceptional set may depend on the sequence $\{x_n\}$. We can get a weaker result which is independent of the sequence. Let S denote the set of all sequences in Ω tending to x_0. Take $\tilde{E} = \bigcap_{s \in S} E_s$, where E_s denotes the exceptional set corresponding to the sequence s, then we get the following corollary.

Corollary 4.1. *Let $x_0 \in \partial\Omega$, $k < n/2$. Then there exists a k-polar set $\tilde{E} \subseteq \Omega$ such that*

$$\overline{\lim_{x \to x_0}}\, g_\Omega(w, x) = 0 \quad \text{for every } w \in \Omega \setminus \tilde{E}.$$

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