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Applications of Mathematics, Vol. 58 (2013), No. 4, 423–437

Persistent URL: `http://dml.cz/dmlcz/143339`

58 (2013) APPLICATIONS OF MATHEMATICS No. 4, 423–437

STATISTICAL CONVERGENCE OF A SEQUENCE OF RANDOM VARIABLES AND LIMIT THEOREMS

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(Received July 29, 2011)

Abstract. In this paper the ideas of three types of statistical convergence of a sequence of random variables, namely, statistical convergence in probability, statistical convergence in mean of order r and statistical convergence in distribution are introduced and the interrelation among them is investigated. Also their certain basic properties are studied.

Keywords: asymptotic density, random variable, statistical convergence, statistical convergence in probability, statistical convergence in mean of order r, statistical convergence in distribution

MSC 2010: 40Axx, 40Cxx, 60Fxx, 60Gxx

1. Introduction and background

The idea of statistical convergence was known to A. Zygmund as early as 1935 and in particular after 1951 when Steinhaus [22] and Fast [4] reintroduced statistical convergence for sequences of real numbers, several generalizations and applications of this notion have been investigated (see [1], [5], [6], [8], [10], [12], [19], [21] where many more references can be found). Recall that a subset A of the set $\mathbb{N}$ of natural numbers is said to have 'asymptotic density' $d(A)$ if

$$d(A) = \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ k \in A\}|,$$

where the vertical bars denote the cardinality of the enclosed set. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be statistically convergent to a real number x if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ |x_k - x| \geqslant \varepsilon\}| = 0;$$

and we write $x_n \xrightarrow{\text{st}} x$ or $\text{st-}\lim\limits_{n\to\infty} x_n = x$.

Now statistical convergence has turned out to be one of the most active areas of research in summability theory and has several applications in different fields of mathematics: number theory [3], trigonometric series [23], probability theory [7], measure theory [15], optimization [16], approximation theory [9], Hopfield neural network [14] (in this paper a geometrical vision of the dynamic states is explained). Statistical convergence has also been discussed in more general abstract spaces such as the fuzzy number space [20], locally convex space [13] and Banach space [11].

On the other hand, if for each positive integer n a random variable X_n is defined on a given event space S (the same for all n) with respect to a given class of events Δ and a probability function $P\colon \Delta \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of all real numbers) then we say that $X_1, X_2, X_3, \dots, X_n, \dots$ is a sequence of random variables and as in analysis we denoted the sequence by $\{X_n\}_{n\in\mathbb{N}}$.

From the practical point of view the discussion of a random variable X will be highly significant if it is known that there exists a real constant c for which $P(|X - c| < \varepsilon) \simeq 1$, where $\varepsilon > 0$ is sufficiently small, that is, it is nearly certain that values of X lie in a very small neighbourhood of c.

For a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$, each X_n may not have the above property but it may happen that the aforesaid property (with respect to a real constant c) becomes more and more distinguished as n gradually increases and the question of existence of such a real constant c will be answered by a concept of convergence in probability of the sequence $\{X_n\}_{n\in\mathbb{N}}$.

Again the sequence $\{X_n\}_{n\in\mathbb{N}}$ may be such that as n gradually increases the distribution function $F_n(x)$ of X_n may more and more resemble the distribution function of a particular random variable and the question of existence of such a distribution function is related to the concept of 'convergence in distribution' of the sequence $\{X_n\}_{n\in\mathbb{N}}$.

Besides the above mentioned two modes of convergence, there are other modes of convergence of the sequence $\{X_n\}_{n\in\mathbb{N}}$. In this paper, we will limit our discussion to three types of statistical convergence of a sequence of random variables, namely,

(a) statistical convergence in probability,
(b) statistical convergence in mean of order r,
(c) statistical convergence in distribution.

We will conclude the paper discussing some fundamental limit theorems related to the modes of convergence (a), (b), (c) which effectively extend and improve all the existing results in this direction [18].

Now probability convergence has several applications in different fields of mathematics: graph theory (minimal spanning tree) [17], measure theory [15], Lorentz gas [2].

2. Statistical convergence in probability

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, where each X_n is defined on the same event space S with respect to a given class of subsets (of S) as the class Δ of events and a given probability function $P\colon \Delta \to \mathbb{R}$. The sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be statistically convergent in probability to a random variable X (where $X\colon S \to \mathbb{R}$) if for any $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ P(|X_k - X| \geqslant \varepsilon) \geqslant \delta\}| = 0,$$

$$\text{or equivalently,}\ \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ 1 - P(|X_k - X| < \varepsilon) \geqslant \delta\}| = 0$$

and we write $\operatorname{st-}\lim\limits_{n\to\infty} P(|X_n - X| \geqslant \varepsilon) = 0$ or $\operatorname{st-}\lim\limits_{n\to\infty} P(|X_n - X| < \varepsilon) = 1$ or $X_n \overset{\mathrm{st_p}}{\to} X$.

Theorem 2.1. *If a sequence of constants $x_n \overset{\mathrm{st}}{\to} x$ then regarding a constant as a random variable having a one-point distribution at that point, we may also write $x_n \overset{\mathrm{st_p}}{\to} x$.*

Proof. Let ε be any positive real number. Then $d(M) = 1$, where $M = \{n \in \mathbb{N}\colon\ |x_n - x| < \varepsilon\}$. Thus for any $\delta > 0$, $B = \{n \in \mathbb{N}\colon\ 1 - P(|x_n - x| < \varepsilon) \geqslant \delta\} \subseteq \mathbb{N} \setminus M$ implies $d(B) = 0$. This implies $x_n \overset{\mathrm{st_p}}{\to} x$. □

The following example shows that in general the converse of Theorem 2.1 is not true.

Example 2.1.1. Let the probability density function X_n be

$$f_n(x) = \begin{cases} 1, & \text{where } 0 < x < 1;\quad 0 \text{ otherwise, if } n = 2^m \text{ where } \forall m \in \mathbb{N}; \\ \dfrac{nx^{n-1}}{2^n}, & \text{where } 0 < x < 2;\quad 0 \text{ otherwise, if } n \neq 2^m \text{ where } \forall m \in \mathbb{N}. \end{cases}$$

Let $0 < \varepsilon, \delta < 1$. Then

$$P(|X_n - 2| \geqslant \varepsilon) = \begin{cases} 1 \quad \text{if } n = 2^m \text{ where } \forall m \in \mathbb{N}, \\ 1 - P(|X_n - 2| < \varepsilon) = 1 - \Big\{1 - \Big(\dfrac{2-\varepsilon}{2}\Big)^n\Big\} = \Big(1 - \dfrac{\varepsilon}{2}\Big)^n \\ \quad \text{if } n \neq 2^m \text{ where } \forall m \in \mathbb{N} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ P(|X_k - 2| \geqslant \varepsilon) \geqslant \delta\}| \leqslant \lim_{n\to\infty} \frac{1}{n}|\{2^0, 2^1, 2^2, \ldots\} \cup A| = 0,$$

(where A is a finite subset of $\mathbb{N}$)

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ P(|X_k - 2| \geqslant \varepsilon) \geqslant \delta\}| = 0.$$

However, it is not ordinary statistical convergence of a sequence of numbers.

Although the concept of statistical convergence in probability is basically different from that of ordinary statistical convergence of sequence of numbers, the following simple results hold for statistical convergence in probability as well.

Theorem 2.2 (Elementary Properties)**.**

(i) $X_n \xrightarrow{\mathrm{st_p}} X$ *and* $X_n \xrightarrow{\mathrm{st_p}} Y \Rightarrow P\{X = Y\} = 1$,

(ii) $X_n \xrightarrow{\mathrm{st_p}} X \Leftrightarrow X_n - X \xrightarrow{\mathrm{st_p}} 0$,

(iii) $X_n \xrightarrow{\mathrm{st_p}} X \Rightarrow cX_n \xrightarrow{\mathrm{st_p}} cX$ *where* $c \in \mathbb{R}$,

(iv) $X_n \xrightarrow{\mathrm{st_p}} X$ *and* $Y_n \xrightarrow{\mathrm{st_p}} Y \Rightarrow X_n + Y_n \xrightarrow{\mathrm{st_p}} X + Y$,

(v) $X_n \xrightarrow{\mathrm{st_p}} X$ *and* $Y_n \xrightarrow{\mathrm{st_p}} Y \Rightarrow X_n - Y_n \xrightarrow{\mathrm{st_p}} X - Y$,

(vi) $X_n \xrightarrow{\mathrm{st_p}} x \Rightarrow {X_n}^2 \xrightarrow{\mathrm{st_p}} x^2$,

(vii) $X_n \xrightarrow{\mathrm{st_p}} x$ *and* $Y_n \xrightarrow{\mathrm{st_p}} y \Rightarrow X_n \cdot Y_n \xrightarrow{\mathrm{st_p}} x \cdot y$,

(viii) $X_n \xrightarrow{\mathrm{st_p}} x$ *and* $Y_n \xrightarrow{\mathrm{st_p}} y \Rightarrow X_n/Y_n \xrightarrow{\mathrm{st_p}} x/y$ *provided* $y \neq 0$,

(ix) $X_n \xrightarrow{\mathrm{st_p}} X$ *and* $Y_n \xrightarrow{\mathrm{st_p}} Y \Rightarrow X_n \cdot Y_n \xrightarrow{\mathrm{st_p}} X \cdot Y$,

(x) *if* $0 \leqslant X_n \leqslant Y_n$ *and* $Y_n \xrightarrow{\mathrm{st_p}} 0 \Rightarrow X_n \xrightarrow{\mathrm{st_p}} 0$,

(xi) *if* $X_n \xrightarrow{\mathrm{st_p}} X$, *then for each* $\varepsilon, \delta > 0$ *there exists* $k \in \mathbb{N}$ *such that*

$$d(\{n \in \mathbb{N}\colon\ P(|X_n - X_k| \geqslant \varepsilon) \geqslant \delta\}) = 0$$

(this is called the statistical Cauchy condition in probability).

P r o o f. Let ε, δ be any positive real numbers.

(i) Let $k \in \{n \in \mathbb{N}\colon\ P(|X_n - X| \geqslant \frac{1}{2}\varepsilon) < \frac{1}{2}\delta\} \cap \{n \in \mathbb{N}\colon\ P(|X_n - Y| \geqslant \frac{1}{2}\varepsilon) < \frac{1}{2}\delta\}$ (the existence of k is granted, since the asymptotic density of both the sets is 1). Then $P(|X - Y| \geqslant \varepsilon) \leqslant P(|X_k - X| \geqslant \frac{1}{2}\varepsilon) + P(|X_k - Y| \geqslant \frac{1}{2}\varepsilon) < \delta$. This implies $P\{X = Y\} = 1$.

For (ii), (iii), (iv), (v) the proofs are straightforward and hence omitted.

(vi) If $Z_n \xrightarrow{\mathrm{st_p}} 0$ then ${Z_n}^2 \xrightarrow{\mathrm{st_p}} 0$ for $\{n \in \mathbb{N}\colon\ P(|{Z_n}^2 - 0| \geqslant \varepsilon) \geqslant \delta\} = \{n \in \mathbb{N}\colon\ P(|Z_n - 0| \geqslant \sqrt{\varepsilon}) \geqslant \delta\}$. Now ${X_n}^2 = (X_n - x)^2 + 2x(X_n - x) + x^2 \xrightarrow{\mathrm{st_p}} x^2$.

(vii) We have $X_n Y_n = \frac{1}{4}\{(X_n + Y_n)^2 - (X_n - Y_n)^2\} \xrightarrow{\mathrm{st_p}} \frac{1}{4}\{(x+y)^2 - (x-y)^2\} = xy$.

(viii) Let A, B represent the events '$|Y_n - y| < |y|$', '$|1/Y_n - 1/y| \geqslant \varepsilon$', respectively. Now

$$\Big|\frac{1}{Y_n} - \frac{1}{y}\Big| = \frac{|Y_n - y|}{|yY_n|} = \frac{|Y_n - y|}{|y| \cdot |y + (Y_n - y)|} \leqslant \frac{|Y_n - y|}{|y| \cdot |(|y| - |Y_n - y|)|}.$$

If A, B occur simultaneously, then

$$|Y_n - y| \geqslant \frac{\varepsilon |y|^2}{1 + \varepsilon |y|} \quad \text{(by the above inequality).}$$

Let $\varepsilon_0 = \varepsilon|y|^2/(1+\varepsilon|y|)$ and let C be the event '$|Y_n - y| \geqslant \varepsilon_0$'. This implies $AB \subseteq C \Rightarrow P(B) \leqslant P(C) + P(\overline{A})$, where bar denotes the set complement. This implies $\{n \in \mathbb{N}\colon\ P(|1/Y_n - 1/y| \geqslant \varepsilon) \geqslant \delta\} \subseteq \{n \in \mathbb{N}\colon\ P(|Y_n - y| \geqslant \varepsilon_0) \geqslant \frac{1}{2}\delta\} \cup \{n \in \mathbb{N}\colon\ P(|Y_n - y| \geqslant |y|) \geqslant \frac{1}{2}\delta\}$. Hence, $1/Y_n \overset{\mathrm{st_p}}{\to} 1/y$ provided $y \neq 0$. Finally, $X_n/Y_n \overset{\mathrm{st_p}}{\to} x/y$ provided $y \neq 0$ (by Theorem 2.2 (vii)).

(ix) First prove that if $X_n \overset{\mathrm{st_p}}{\to} X$ and Z is a random variable then $X_nZ \overset{\mathrm{st_p}}{\to} XZ$. Since Z is a random variable, so given $\delta > 0$, there exists an $\alpha > 0$ such that $P(|Z| > \alpha) \leqslant \frac{1}{2}\delta$. Then for any $\varepsilon > 0$, $P(|X_nZ - XZ| \geqslant \varepsilon) = P(|X_n - X||Z| \geqslant \varepsilon, |Z| > \alpha) + P(|X_n - X||Z| \geqslant \varepsilon, |Z| \leqslant \alpha) \leqslant \frac{1}{2}\delta + P(|X_n - X| \geqslant \varepsilon/\alpha)$. This implies $\{n \in \mathbb{N}\colon\ P(|X_nZ - XZ| \geqslant \varepsilon) \geqslant \delta\} \subseteq \big\{n \in \mathbb{N}\colon\ P\big(|X_n - x| \geqslant \varepsilon/\alpha\big) \geqslant \frac{1}{2}\delta\big\}$. Next $(X_n - X)(Y_n - Y) \overset{\mathrm{st_p}}{\to} 0$. This implies $X_nY_n \overset{\mathrm{st_p}}{\to} XY$.

(x) Proof is straightforward and hence omitted.

(xi) Choose $k \in \mathbb{N}$ such that $P\big(|X_k - X| \geqslant \frac{1}{2}\varepsilon\big) < \frac{1}{2}\delta$ (k exists since $d\big(\big\{n \in \mathbb{N}\colon\ P\big(|X_n - X| \geqslant \frac{1}{2}\varepsilon\big) < \frac{1}{2}\delta\big\}\big) = 1$). Then $\{n \in \mathbb{N}\colon\ P(|X_n - X_k| \geqslant \varepsilon) \geqslant \delta\} \subseteq \big\{n \in \mathbb{N}\colon\ P\big(|X_n - X| \geqslant \frac{1}{2}\varepsilon\big) \geqslant \frac{1}{2}\delta\big\}$. □

Theorem 2.3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that there exists a sequence of constants $\{x_n\}_{n\in\mathbb{N}}$ with the property that*

$$X_n - x_n \overset{\mathrm{st_p}}{\to} 0.$$

If $m(X_n)$ is a median of X_n then

$$X_n - m(X_n) \overset{\mathrm{st_p}}{\to} 0 \quad \text{and} \quad x_n - m(X_n) \overset{\mathrm{st}}{\to} 0.$$

P r o o f. Proof is straightforward and hence omitted. □

Theorem 2.4. *Let $X_n \overset{\mathrm{st_p}}{\to} X$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous on $\mathbb{R}$. Then $g(X_n) \overset{\mathrm{st_p}}{\to} g(X)$.*

P r o o f. Since X is a random variable, hence for each $\delta > 0$ there exists α such that $P(|X| > \alpha) \leqslant \frac{1}{2}\delta$. Since g is uniformly continuous on $[-\alpha, \alpha]$, hence for each $\varepsilon > 0$ there exists δ_0 such that

$$|g(x_n) - g(x)| < \varepsilon \quad \text{whenever } |x| \leqslant \alpha \text{ and } |x_n - x| < \delta_0.$$

It follows that

$$P(|g(X_n) - g(X)| \geqslant \varepsilon) \leqslant P(|X_n - X| \geqslant \delta_0) + P(|X| > \alpha) \leqslant P(|X_n - X| \geqslant \delta_0) + \frac{1}{2}\delta.$$

This implies $\{n \in \mathbb{N}\colon\ P(|g(X_n) - g(X)| \geqslant \varepsilon) \geqslant \delta\} \subseteq \{n \in \mathbb{N}\colon\ P(|X_n - X| \geqslant \delta_0) \geqslant \frac{1}{2}\delta\}$. Hence the result. □

Corollary 2.4.1. *If* $X_n \overset{\mathrm{st_p}}{\to} x$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *is a continuous function, then* $g(X_n) \overset{\mathrm{st_p}}{\to} g(x)$.

Proposition 2.1. *Let* $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $\{c_n\}_{n\in\mathbb{N}}$ *be three sequences of real numbers such that* $d(\{n \in \mathbb{N}\colon a_n \leqslant b_n \leqslant c_n\}) = 1$ *and* $\operatorname{st-}\lim\limits_{n\to\infty} a_n = \operatorname{st-}\lim\limits_{n\to\infty} c_n = x$. *Then* $\operatorname{st-}\lim\limits_{n\to\infty} b_n = x$.

The next theorem is the generalization of Tchebycheff's theorem.

Theorem 2.5. *If* $\{X_n\}_{n\in\mathbb{N}}$ *is a sequence of random variables such that for any* n, X_n *has a finite mean* m_n, *then*

$$X_n - m_n \overset{\mathrm{st_p}}{\to} 0 \quad \textit{provided } \operatorname{st-}\lim_{n\to\infty} \sigma_n = 0.$$

P r o o f. The proof is straightforward and hence omitted. □

Corollary 2.5.1. *If moreover* $\operatorname{st-}\lim\limits_{n\to\infty} m_n = m$, *then by Theorem* 2.2, (v) *together with the preceding remark* $X_n \overset{\mathrm{st_p}}{\to} m$.

E x a m p l e 2.5.1. (i) The following example shows that if $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables such that for any n, X_n has a finite mean m_n and finite standard deviation σ_n then

$$X_n - m_n \overset{\mathrm{st_p}}{\to} 0, \quad \text{but} \quad \lim_{n\to\infty} \sigma_n \neq 0.$$

$$\text{Let } X_n = \begin{cases} \pm n & \text{if } n = m^2 \quad \forall m \in \mathbb{N} \text{ with probability } \frac{1}{2}, \\ 0 & \text{if } n \neq m^2 \quad \forall m \in \mathbb{N} \text{ with probability } 1. \end{cases}$$

Then $m_n = E(X_n) = 0\ \forall n \in \mathbb{N}$ and

$$\sigma_n = \begin{cases} n & \text{if } n = m^2 \quad \forall m \in \mathbb{N}, \\ 0 & \text{if } n \neq m^2 \quad \forall m \in \mathbb{N} \end{cases}$$

$\Rightarrow \lim\limits_{n\to\infty} \sigma_n \neq 0$ (in fact it is an unbounded sequence) but $\operatorname{st-}\lim\limits_{n\to\infty} \sigma_n = 0$. Then by Tchebycheff's inequality we get

$$P(|X_n - 0| \geqslant \varepsilon) \leqslant \frac{{\sigma_n}^2}{\varepsilon^2} \quad \forall n \in \mathbb{N} \text{ and } \varepsilon > 0.$$

Since $\operatorname{st-}\lim\limits_{n\to\infty} \sigma_n = 0$, by Proposition 2.1 we get $\operatorname{st-}\lim\limits_{n\to\infty} P(|X_n - 0| \geqslant \varepsilon) = 0$, i.e., $X_n - 0 \overset{\mathrm{st_p}}{\to} 0$.

(ii) Again let

$$X_n = \begin{cases} \pm 1 & \text{if } n = 2^m \quad \forall m \in \mathbb{N} \text{ with probability } \frac{1}{2}, \\ 0 & \text{if } n \neq 2^m \quad \forall m \in \mathbb{N} \text{ with probability } 1. \end{cases}$$

Then $m_n = E(X_n) = 0\ \forall n \in \mathbb{N}$ and

$$\sigma_n = \begin{cases} 1 & \text{if } n = 2^m \quad \forall m \in \mathbb{N}, \\ 0 & \text{if } n \neq 2^m \quad \forall m \in \mathbb{N} \end{cases}$$

$\Rightarrow \lim\limits_{n\to\infty} \sigma_n \neq 0$ but $\sigma_n \overset{\mathrm{st_p}}{\to} 0$. Then by Tchebycheff's inequality we get

$$P(|X_n - 0| \geqslant \varepsilon) \leqslant \frac{{\sigma_n}^2}{\varepsilon^2} \quad \forall n \in \mathbb{N} \text{ and } \varepsilon > 0.$$

Since $\sigma_n \overset{\mathrm{st_p}}{\to} 0$, by Proposition 2.1 we get $\operatorname{st-}\lim\limits_{n\to\infty} P(|X_n - 0| \geqslant \varepsilon) = 0$, i.e., $X_n - 0 \overset{\mathrm{st_p}}{\to} 0$.

Theorem 2.6. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function such that $g(x) > 0$ for all $x \in \mathbb{R}$. If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables such that for any n, X_n has a finite mean m_n, then*

$$X_n - m_n \overset{\mathrm{st_p}}{\to} 0, \ \textit{provided} \operatorname{st-}\lim_{n\to\infty} E\{g(|X_n - m_n|)\} = 0.$$

P r o o f. Proof is easily done by the inequality $P(|X-m| \geqslant \varepsilon) \leqslant E\{g(|X-m|)\}/g(\varepsilon)$. □

Theorem 2.7. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that $S_n = X_1 + X_2 + \ldots + X_n$ has a finite mean M_n and a finite variance Σ_n for all n. Then $(S_n - M_n)/n \overset{\mathrm{st_p}}{\to} 0$ provided $\operatorname{st-}\lim\limits_{n\to\infty} \Sigma_n/n^2 = 0$.*

P r o o f. The proof is straightforward and hence omitted. □

Theorem 2.8. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of identically and independently distributed random variables and $Y_n = (S_n - E(S_n))/n$, where $S_n = X_1 + X_2 + \ldots + X_n$. Then $Y_n \overset{\mathrm{st_p}}{\to} 0$ if and only if $\operatorname{st-}\lim\limits_{n\to\infty} E\{Y_n^2/(1+Y_n^2)\} = 0$.*

P r o o f. Let $\operatorname{st-}\lim\limits_{n\to\infty} E\{Y_n^2/(1+Y_n^2)\} = 0$ and $\varepsilon > 0$, then $|Y_n| \geqslant \varepsilon \Rightarrow |Y_n|^2 \geqslant \varepsilon^2 \Rightarrow Y_n^2 + Y_n^2\varepsilon^2 \geqslant \varepsilon^2 + Y_n^2\varepsilon^2 \Rightarrow (Y_n^2/(1+Y_n^2))/(\varepsilon^2/(1+\varepsilon^2)) \geqslant 1$. This implies

$$P(|Y_n| \geqslant \varepsilon) \leqslant P\Big\{\frac{Y_n^2/(1+Y_n^2)}{\varepsilon^2/(1+\varepsilon^2)}\Big\} \leqslant E\Big\{\frac{Y_n^2/(1+Y_n^2)}{\varepsilon^2/(1+\varepsilon^2)}\Big\} \quad \text{(by Markov's Inequality)}$$

$\Rightarrow Y_n \overset{\mathrm{st_p}}{\to} 0$.

Conversely, let us assume that X_i's are continuous and let Y_n have pdf $f_n(y)$. Then

$$\begin{aligned}
&E\Big\{\frac{Y_n^2}{1+Y_n^2}\Big\} = \int_{-\infty}^{\infty} \frac{y_n^2}{1+y_n^2} f_n(y)\,\mathrm{d}y \\
\Rightarrow\ &E\Big\{\frac{Y_n^2}{1+Y_n^2}\Big\} = \int_{|Y_n|\geqslant\varepsilon} \frac{y_n^2}{1+y_n^2} f_n(y)\,\mathrm{d}y + \int_{|Y_n|<\varepsilon} \frac{y_n^2}{1+y_n^2} f_n(y)\,\mathrm{d}y \\
&\qquad\leqslant \int_{|Y_n|\geqslant\varepsilon} f_n(y)\,\mathrm{d}y + \int_{|Y_n|<\varepsilon} y_n^2 f_n(y)\,\mathrm{d}y \\
&\qquad\quad \Big(\text{since } \frac{y_n^2}{1+y_n^2} < 1 \text{ and } \frac{y_n^2}{1+y_n^2} < y^2\Big) \\
&\qquad\leqslant P(|Y_n|\geqslant\varepsilon) + \varepsilon^2 \int_{|Y_n|<\varepsilon} f_n(y)\,\mathrm{d}y = P(|Y_n|\geqslant\varepsilon) + \varepsilon^2 P(|Y_n|<\varepsilon) \\
&\qquad\leqslant P(|Y_n|\geqslant\varepsilon) + \varepsilon^2 \quad (\text{since } P(|Y_n|<\varepsilon)\leqslant 1).
\end{aligned}$$

Since $Y_n \xrightarrow{\mathrm{st_p}} 0$ and ε^2 is an arbitrarily small positive real number, we get $\operatorname{st-lim}_{n\to\infty} E\{Y_n^2/1+Y_n^2\} = 0$. □

N o t e 2.8.1. The result of Theorem 2.8 holds even if $E(X_i)$ does not exist. In this case we simply define $Y_n = S_n/n$ rather than $(S_n - E(S_n))/n$.

A slightly stronger concept of statistical convergence in probability is defined by statistical convergence in mean of order r, which is shown by Theorem 3.1.

3. Statistical convergence in mean of order r

A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be statistically convergent in the r^{th}-mean (where $r>0$) to a random variable X (where $X\colon S\to\mathbb{R}$) if for any $\delta>0$

$$d(\{n\in\mathbb{N}\colon\ E(|X_n - X|^r)\geqslant\delta\}) = 0$$

provided $E(|X_n|^r)$ exists for every $n\in\mathbb{N}$ and $E(|X|^r)$ exists and we write $\operatorname{st-lim}_{n\to\infty} E(|X_n - X|^r) = 0$ or $X_n \xrightarrow{\mathrm{st_{rm}}} X$.

Statistical convergences in mean of orders one and two are called statistical convergence in mean and quadratic mean (or mean square), respectively.

In this section we observe the above form of statistical convergence and a notion of distance, which is widely used in statistics and time series analysis. It is based on the $L_p (p\geqslant 1)$ metric ϱ which is defined by $\varrho(X,Y) = E\{|X-Y|^p\}^{1/p}$.

Theorem 3.1. *Let $X_n \overset{\mathrm{st_{rm}}}{\to} X$ (any $r > 0$). Then $X_n \overset{\mathrm{st_p}}{\to} X$, i.e., statistical convergence in r^{th}-mean implies statistical convergence in probability.*

Proof. Proof is easily done by Bienayme-Tchebycheff's inequality, i.e., $P(|X - m|^r \geqslant \varepsilon) \leqslant E\{|X - m|^r\}/\varepsilon^r$. □

The following example shows that in general the converse of Theorem 3.1 is not true.

Example 3.1.1. We consider the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ defined by

$$X_n \in \{0, n\} \quad \text{with } P(X_n = 0) = 1 - \frac{1}{n^r}$$

and

$$P(X_n = n) = \frac{1}{n^r}, \quad \text{where } r > 0,\ n \in \mathbb{N}.$$

For any $\varepsilon > 0$, $P(|X_n - 0| \geqslant \varepsilon) = P(X_n = n)$ if $0 < \varepsilon \leqslant n$ and $P(|X_n - 0| \geqslant \varepsilon) = 0$ if $\varepsilon > n$, hence for any $\delta > 0$, $\{n \in \mathbb{N}\colon P(|X_n - 0| \geqslant \varepsilon) \geqslant \delta\} =$ finite set $\Rightarrow$ $\operatorname{st-}\lim_{n\to\infty} P(|X_n - 0| \geqslant \varepsilon) = 0$. But $E(|X_n|^r) = 1$ for all $n \in \mathbb{N} \Rightarrow \big\{n \in \mathbb{N}\colon E(|X_n - 0|^r) \geqslant \frac{1}{2}\big\} = \mathbb{N}$. This implies $\{E(|X_n - 0|^r)\}_{n\in\mathbb{N}}$ is not statistically convergent to 0. Hence the result.

Theorem 3.2. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that $P(|X_n| \leqslant M) = 1$ for all n and some constant $M > 0$. Suppose that $X_n \overset{\mathrm{st_p}}{\to} X$. Then $X_n \overset{\mathrm{st_{rm}}}{\to} X$ for any $r > 0$.*

Proof. Proof is obvious and hence omitted. □

Theorem 3.3. *Let $X_n \overset{\mathrm{st_{rm}}}{\to} X$ and $Y_n \overset{\mathrm{st_{rm}}}{\to} Y$ (any $r > 0$) be such that $X_n - X$, $Y_n - Y \geqslant 0$. Then $X_n + Y_n \overset{\mathrm{st_{rm}}}{\to} X + Y$.*

Proof. Proof is easily done by the inequality $E(X+Y)^r \leqslant 2^r[E(X)^r + E(Y)^r]$, where X and Y are non-negative random variables and $r > 0$. □

Theorem 3.4. (i) $X_n \overset{\mathrm{st_{1m}}}{\to} X \Leftrightarrow \sup\limits_{A\in\Delta} \big| \int_A X_n \,\mathrm{d}P - \int_A X \,\mathrm{d}P \big| \overset{\mathrm{st}}{\to} 0$.

(ii) *If $X_n \overset{\mathrm{st_{1m}}}{\to} X$ then $E(X_n) \overset{\mathrm{st}}{\to} E(X)$.*

Proof. Proof is easily done by 'Scheff's lemma'. □

Proposition 3.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non negative real numbers such that* $\operatorname{st-lim}_{n\to\infty} a_n = a$ *and* $a \geqslant 0$. *Then* $\operatorname{st-lim}_{n\to\infty} (a_n)^q = a^q$, *where* $q \in \mathbb{Q}^+$.

Proof. Let $q = m/r \in \mathbb{Q}^+$. If $a = 0$ then the result is obvious. So choose $a > 0$. It is sufficient to prove that $\operatorname{st-lim}_{n\to\infty} a_n = a$ implies $\operatorname{st-lim}_{n\to\infty} (a_n)^{1/r} = a^{1/r}$, where $r \in \mathbb{N} \setminus \{1\}$. Then $\{n \in \mathbb{N}\colon |a_n - a| < \frac{1}{2}a\} \subseteq \{n \in \mathbb{N}\colon a_n > \frac{1}{2}a\} = M$ (say, then $d(M) = 1$). If $n \in M$ then

$$\begin{aligned} |a_n - a| &= |(a_n)^{1/r} - a^{1/r}|\,|\{(a_n)^{(r-1)/r} + (a_n)^{(r-2)/r}a^{1/r} \\ &\qquad + \ldots + (a_n)^{1/r}a^{(r-2)/r} + a^{(r-1)/r}\}| \\ &\geqslant L|(a_n)^{1/r} - a^{1/r}|, \text{ where } L = \frac{a^{(r-1)/r}}{2\big(1 - 1/\sqrt[r]{2}\big)}, \end{aligned}$$

i.e., $M \subseteq \{n \in \mathbb{N}\colon 0 \leqslant |(a_n)^{1/r} - a^{1/r}| \leqslant L^{-1}|a_n - a|\}$. Then by Proposition 2.1, $\operatorname{st-lim}_{n\to\infty} \sqrt[r]{a_n} = \sqrt[r]{a}$. □

Theorem 3.5. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that* $X_n \overset{\mathrm{st_{2m}}}{\to} X$. *Then* $\operatorname{st-lim}_{n\to\infty} E(X_n) = E(X)$ *and* $\operatorname{st-lim}_{n\to\infty} E(X_n^2) = E(X^2)$.

Proof. The proof is parallel to that of Theorem 8 in [18] (and using the result of the Proposition 3.1) and so it is omitted. □

Theorem 3.6. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables and $r \in \mathbb{Q}^+$ such that* $X_n \overset{\mathrm{st_{rm}}}{\to} X$. *Then* $\operatorname{st-lim}_{n\to\infty} E(|X_n|^s) = E(|X|^s)$ *for* $s \leqslant r$ *and* $s \in \mathbb{Q}^+$.

Proof. The proof is parallel to that of Theorem 9, 10 in [18] (and using the result of the Proposition 3.1) and so it is omitted. □

4. Statistical convergence in distribution

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, where $F_n(x)$ is the distribution function of X_n for $n \in \mathbb{N}$. If there exists a random variable X whose distribution function is $F(x)$ such that $\operatorname{st-lim}_{n\to\infty} F_n(x) = F(x)$ at every point of continuity x of $F(x)$, then $\{X_n\}_{n\in\mathbb{N}}$ is said to be statistically convergent in distribution to X and we write $X_n \overset{\mathrm{st_d}}{\to} X$.

Theorem 4.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Also, let $f_n(x_i) = P(X_n = x_i)$, for all $i \in \mathbb{N}$, be the probability mass function of X_n, for all $n \in \mathbb{N}$, and $f(x_i) = P(X = x_i)$, for all $i \in \mathbb{N}$ be the probability mass function of X. Then*

$$f_n(x) \stackrel{\text{st}}{\to} f(x) \text{ for all } x \Leftrightarrow X_n \stackrel{\text{st}_\text{d}}{\to} X.$$

Proof. Proof is straightforward and hence omitted. □

Proposition 4.1. *Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be two sequences of real numbers such that $a_n \leqslant b_n$ for all $n \in \mathbb{N}$. Then*

$$\text{st-}\underline{\lim}\, a_n \leqslant \text{st-}\underline{\lim}\, b_n \text{ and } \text{st-}\overline{\lim}\, a_n \leqslant \text{st-}\overline{\lim}\, b_n.$$

Proof. The proof is straightforward and hence omitted. □

Theorem 4.2. *Let $X_n \stackrel{\text{st}_\text{p}}{\to} X$. Then $X_n \stackrel{\text{st}_\text{d}}{\to} X$, i.e., statistical convergence in probability implies statistical convergence in distribution.*

Proof. Let $F_n(x)$ and $F(x)$ be the distribution functions of X_n and X, respectively. Now for any two real numbers x and y with $x < y$, we have

$$(X \leqslant x) = (X_n \leqslant y, X \leqslant x) + (X_n > y, X \leqslant x).$$

Since $(X_n \leqslant y, X \leqslant x) \subseteq (X_n \leqslant y)$, we have

$$(X \leqslant x) \subseteq (X_n \leqslant y) + (X_n > y, X \leqslant x).$$

Therefore,

$$\begin{aligned}P(X \leqslant x) &\leqslant P\{(X_n \leqslant y) + (X_n > y, X \leqslant x)\} \\ &\leqslant P(X_n \leqslant y) + P(X_n > y, X \leqslant x) \\ &\Rightarrow F_n(y) \geqslant F(x) - P(X_n > y, X \leqslant x)\end{aligned} \tag{1}$$

Now if $X_n > y, X \leqslant x$ occur simultaneously, then $X_n > y$, $-X \geqslant -x$ and so $X_n - X > y - x$, i.e., $(X_n > y, X \leqslant x) \subseteq (X_n - X > y - x) \subseteq (|X_n - X| > y - x)$ which implies

$$P(X_n > y, X \leqslant x) \leqslant P(|X_n - X| > y - x).$$

Since $x < y$ and $X_n \stackrel{\text{st}_\text{p}}{\to} X$, we get

$$\text{st-}\lim_{n\to\infty} P(X_n > y, X \leqslant x) = 0.$$

Now from (1) we get $\text{st-}\underline{\lim}\, F_n(y) \geqslant F(x)$ (by the above Proposition 4.1).

Similarly, if $y < z$ then

$$(X_n \leqslant y) = (X \leqslant z, X_n \leqslant y) + (X > z, X_n \leqslant y),$$

$$\text{then } F_n(y) \leqslant F(z) + P(X > z, X_n \leqslant y) \text{ and st-}\lim_{n\to\infty} P(X > z, X_n \leqslant y) = 0.$$

Finally, we get st-$\overline{\lim}\, F_n(y) \leqslant F(z)$.

Thus, for $x < y < z$ we have

$$F(x) \leqslant \text{st-}\underline{\lim}\, F_n(y) \leqslant \text{st-}\overline{\lim}\, F_n(y) \leqslant F(z).$$

If F is continuous at y then

$$F(y) = \lim_{x\to y-} F(x) \leqslant \text{st-}\underline{\lim}\, F_n(y) \leqslant \text{st-}\overline{\lim}\, F_n(y) \leqslant \lim_{z\to y+} F(z) = F(y).$$

This implies st-$\underline{\lim}\, F_n(y) =$ st-$\overline{\lim}\, F_n(y) = F(y)$, i.e., $X_n \overset{\mathrm{st_d}}{\to} X$. Hence the result. $\square$

N o t e 4.2.1. The converse of Theorem 4.2 is not true in general, i.e., $X_n \overset{I_d}{\to} X$ does not imply $X_n \overset{I_p}{\to} X$ in general. For, let us consider random variables $X, X_1, X_2, \ldots$ having identical distribution. Let the spectrum of the two dimensional random variable (X_n, X) be $(0,0), (0,1), (1,0), (1,1)$ and

$$\begin{aligned}
P(X_n = 0, X = 0) &= 0 = P(X_n = 1, X = 1),\\
P(X_n = 0, X = 1) &= \frac{1}{2} = P(X_n = 1, X = 0).
\end{aligned}$$

Hence, the marginal distribution of X_n is given by $X_n = i$ $(i = 0, 1)$, with p.m.f $p_{x_n i} = P(X_n = i)$, where $p_{x_n 0} = \frac{1}{2} = p_{x_n 1}$ and that of $X = j$ $(j = 0, 1)$, with p.m.f $p_{xj} = P(X = j)$, where $p_{x0} = \frac{1}{2} = p_{x1}$.

If $F_n(x)$ and $F(x)$ are the distribution functions of X_n and X, respectively, then

$$F(x) = F_n(x) = \begin{cases} 0, & x < 0,\\ \frac{1}{2}, & 0 \leqslant x < 1,\\ 1, & x \geqslant 1. \end{cases}$$

Therefore, st-$\lim_{n\to\infty} F_n(x) = F(x)$ for all $x \in \mathbb{R}$, i.e., $X_n \overset{\mathrm{st_d}}{\to} X$ as $n \to \infty$. But $P\big(|X_n - X| \geqslant \frac{1}{2}\big) = P(|X_n - X| = 1) = P(X_n = 0, X = 1) + P(X_n = 1, X = 0) = 1$. Therefore, st-$\lim_{n\to\infty} P\big(|X_n - X| \geqslant \frac{1}{2}\big) \neq 0$. Then st-$\lim_{n\to\infty} P(|X_n - X| \geqslant \varepsilon) \neq 0$. Hence, the result follows.

Theorem 4.3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that $X_n \xrightarrow{\mathrm{st_d}} X$. Let $\{\zeta_n\}_{n\in\mathbb{N}}$ be a sequence of positive constants such that $\zeta_n \xrightarrow{\mathrm{st}} 0$. Then $\zeta_n X_n \xrightarrow{\mathrm{st_p}} 0$.*

P r o o f. The proof is straightforward and hence omitted. $\square$

The following result is easy to establish.

Theorem 4.4. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that $X_n \xrightarrow{\mathrm{st_d}} X$, and c a constant. Then*
(a) *$X_n + c \xrightarrow{\mathrm{st_d}} X + c$, and*
(b) *$cX_n \xrightarrow{\mathrm{st_d}} cX$, $c \neq 0$.*

Theorem 4.5. *Let c be a constant, then $X_n \xrightarrow{\mathrm{st_d}} c \Rightarrow X_n \xrightarrow{\mathrm{st_p}} c$.*

P r o o f. The proof is straightforward and hence omitted. $\square$

Corollary 4.5.1. *Let c be a constant, then $X_n \xrightarrow{\mathrm{st_d}} c \Leftrightarrow X_n \xrightarrow{\mathrm{st_p}} c$.*

Theorem 4.6. *If $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are sequences of random variables on some probability space with $X_n - Y_n \xrightarrow{\mathrm{st_p}} 0$, and $Y_n \xrightarrow{\mathrm{st_d}} X$, then $X_n \xrightarrow{\mathrm{st_d}} X$.*

P r o o f. Let x, $x \pm \varepsilon$ be a point of continuity of the distribution function F corresponding to the random variable X, where $\varepsilon > 0$. Then $P(X_n \leqslant x) = P(Y_n \leqslant x + Y_n - X_n) = P(Y_n \leqslant x + Y_n - X_n; Y_n - X_n \leqslant \varepsilon) + P(Y_n \leqslant x + Y_n - X_n; Y_n - X_n > \varepsilon) \leqslant P(Y_n \leqslant x + \varepsilon) + P(Y_n - X_n > \varepsilon)$. This implies st-$\overline{\lim}\, F_n(x) \leqslant F(x+\varepsilon)$ and analogously $F(x-\varepsilon) \leqslant$ st-$\underline{\lim}\, F_n(x)$. Since ε is arbitrary, so $F(x) =$ st-$\lim F_n(x)$. $\square$

Theorem 4.7. *If $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are sequences of random variables on some probability space and c is a constant, then*
(a) *$X_n \xrightarrow{\mathrm{st_d}} X$, $Y_n \xrightarrow{\mathrm{st_p}} c \Rightarrow X_n + Y_n \xrightarrow{\mathrm{st_d}} X + c$,*
(b) *$X_n \xrightarrow{\mathrm{st_d}} X$, $Y_n \xrightarrow{\mathrm{st_p}} 0 \Rightarrow X_nY_n \xrightarrow{\mathrm{st_p}} 0$,*
(c) *$X_n \xrightarrow{\mathrm{st_d}} X$, $Y_n \xrightarrow{\mathrm{st_p}} c \Rightarrow X_nY_n \xrightarrow{\mathrm{st_d}} cX$ if $c \neq 0$,*
(d) *$X_n \xrightarrow{\mathrm{st_d}} X$, $Y_n \xrightarrow{\mathrm{st_p}} c \Rightarrow X_n/Y_n \xrightarrow{\mathrm{st_d}} X/c$ if $c \neq 0$.*

P r o o f. (a) Proof is parallel to that of Theorem 15 (Slutsky's theorem) in [18] and so omitted.

(b) For any $\delta > 0$, choose $\pm\alpha \in$ set of points of continuity of the distribution function F of X such that $F(\alpha) - F(-\alpha) \geqslant 1 - \delta$. Any $\varepsilon > 0$, $P(|X_nY_n| \geqslant \varepsilon) = P\big(|X_nY_n| \geqslant \varepsilon, |Y_n| < \varepsilon/\alpha\big) + P\big(|X_nY_n| \geqslant \varepsilon, |Y_n| \geqslant \varepsilon/\alpha\big) \leqslant P(|X_n| > \alpha) + P\big(|Y_n| \geqslant \varepsilon/\alpha\big)$. So st-$\overline{\lim}\, P(|X_nY_n| \geqslant \varepsilon) < \delta$. This implies $X_nY_n \xrightarrow{\mathrm{st_p}} 0$.

Parts (c) and (d) are obvious and hence omitted. $\square$

A p p l i c a t i o n s. (i) Much of classical probability theory and its applications to statistics concerns limit theorems; i.e., the asymptotic behaviour of a sequence of random variables. The sequence could consist of sample averages, cumulative sums, extremes, sample quantiles, sample correlations and so on. Whereas probability theory discusses limit theorems, the theory of statistics is concerned with large sample properties of statistics, where a statistic is just a function of the sample.

(ii) A classical significant statistical application of 'statistical convergence in probability' is to quantile estimation and geometric probability (by using the theorem Weak laws of large numbers).

(iii) Classical application of 'statistical convergence in mean of order r' is to time series analysis.

(iv) In statistical estimation the delta method allows us to take a basic 'statistical convergence in distribution', for instance to a limiting normal distribution, apply smooth functions and conclude that the functions are asymptotically normal as well.

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