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Fair majorities in proportional voting

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FAIR MAJORITIES IN PROPORTIONAL VOTING

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In parliaments elected by proportional systems the seats are allocated to the elected political parties roughly proportionally to the shares of votes for the party lists. Assuming that members of the parliament representing the same party are voting together, it has sense to require that distribution of the influence of the parties in parliamentary decision making is proportional to the distribution of seats. There exist measures (so called voting power indices) reflecting an ability of each party to influence outcome of voting. Power indices are functions of distribution of seats and voting quota (where voting quota means a minimal number of votes required to pass a proposal). By a fair voting rule we call such a quota that leads to proportionality of relative influence to relative representation. Usually simple majority is not a fair voting rule. That is the reason why so called qualified or constitutional majority is being used in voting about important issues requiring higher level of consensus. Qualified majority is usually fixed (60% or 66.67%) independently on the structure of political representation. In the paper we use game-theoretical model of voting to find a quota that defines the fair voting rule as a function of the structure of political representation. Such a quota we call a fair majority. Fair majorities can differ for different structures of the parliament. Concept of a fair majority is applied on the Lower House of the Czech Parliament elected in 2010.

1. FAIRNESS IN VOTING

A qualified majority in committee voting is a requirement for a proposal to gain a specified level or type of support which exceeds a simple majority (over 50 %). In some jurisdictions, for example, parliamentary procedure requires that any action that may alter the rights of the minority has to have a qualified majority support. Particular designs of qualified majority (such as 60 % or two-thirds majority) are selected "ad hoc", without quantitative justification. In this paper we try to provide such a justification, defining qualified majority by a "fair quota", providing each legislator with (approximately) the same influence, measured as an a priori voting power.

Let us consider a committee with n members. Each member has some voting weight (number of votes, shares etc.) and a voting rule is defined by a minimal sum of weights required for passing a proposal (model of such a committee is called a weighted voting committee). Given a voting rule, voting weights provide committee members with voting

power. Voting power means an ability to influence the outcome of voting. Voting power indices are used to quantify the voting power.

Voting power is not directly observable: as a proxy for it voting weights are used. Therefore, fairness is usually defined in terms of voting weights (e. g. voting weights are proportional to the results of an election or to the shares in a share-holding company). Assuming that a principle of fair distribution of voting weights is selected, we are addressing the question of how to achieve equality of relative voting power (at least approximately) to relative voting weights. For evaluation of voting power we are using concepts of a priori power indices (a comprehensive survey of power indices theory see in Felsenthal and Machover [2]). The concepts of optimal quota, introduced by Słomczyński and Życzkowski [9, 10] for the EU Council of Ministers distribution of national voting weights (weights equal to square roots of population and quota that provides each citizen of the EU with the same indirect voting power measured by Penrose–Banzhaf index independently on her national affiliation), and of intervals of stable power (Turnovec [11]) are used to find, given voting weights, a fair voting rule minimizing the distance between actors' relative voting weights and their relative voting power.

In the second section we provide definitions, the third section presents a short overview of the results from Turnovec [11] on concepts of quota intervals of stable power and the fair quota, and the fourth section applies the concept of the fair quota for the Lower House of the Czech Parliament elected in 2010.

2. COMMITTEES AND VOTING POWER

A simple weighted committee is a pair $[N, \boldsymbol{w}]$, where N be a finite set of n committee members $i = 1, 2, \ldots, n$, and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ be a nonnegative vector of committee members' voting weights (e. g. votes or shares). By 2^N we denote the power set of N (set of all subsets of N). By voting coalition we mean an element $S \in 2^N$, the subset of committee members voting uniformly (YES or NO), and $w(S) = \sum_{i \in S} w_i$ denotes the voting weight of coalition S. The voting rule is usually defined by majority quota q satisfying $\frac{w(N)}{2} < q \leq w(N)$, where q represents the minimal total weight necessary to approve the proposal. Triple $[N, q, \boldsymbol{w}]$ we call a simple quota weighted committee. The voting coalition S in committee $[N, q, \boldsymbol{w}]$ is called a winning one if $w(S) \geq q$ and a losing one in the opposite case. The winning voting coalition S is called critical if for at least one member $k \in S$ it holds $w(S \setminus k) < q$ (we say that k is critical in S). Critical winning voting coalition S is called minimal if any of its members is critical in S.

A priori voting power analysis seeks an answer to the following question: Given a simple quota weighted committee $[N, q, \boldsymbol{w}]$, what is an influence of its members over the outcome of voting? The absolute voting power of a member is defined as a probability $\Pi_i[N, q, \boldsymbol{w}]$ that i will be decisive in the sense that such a situation appears in which she would be able to decide the outcome of voting by her vote (Nurmi [5] and Turnovec [12]), and a relative voting power as $\pi_i[N, q, \boldsymbol{w}] = \frac{\Pi_i[N, q, \boldsymbol{w}]}{\sum_{k \in N} \Pi_k[N, q, \boldsymbol{w}]}$.

Two basic concepts of decisiveness are used: swing position and pivotal position. The swing position is an ability of an individual voter to change the outcome of voting by a unilateral switch from YES to NO (if member j is critical with respect to a coalition S, we say that he has a swing in S). The pivotal position is such a position of an individual

voter in a permutation of voters expressing a ranking of attitudes of members to the voted issue (from the most preferable to the least preferable) and the corresponding order of forming of the winning coalition, in which her vote YES means a YES outcome of voting and her vote NO means a NO outcome of voting (we say that j is pivotal in the permutation considered).

Let us denote by i the member of the simple quota weighted committee $[N, q, \boldsymbol{w}]$, by $W(N, q, \boldsymbol{w})$ the set of all winning coalitions and by $W_i(N, q, \boldsymbol{w})$ the set of all winning coalitions with i, by $C(N, q, \boldsymbol{w})$ the set of all critical winning coalitions, and by $C_i(N, q, \boldsymbol{w})$ the set of all critical winning coalitions i has the swing in, by $P(N)$ the set of all permutations of N and by $P_i(N, q, \boldsymbol{w})$ the set of all permutations i is pivotal in. By $\mathrm{card}(S)$ we denote the cardinality of S, $\mathrm{card}(\emptyset) = 0$.

Assuming many voting acts and all coalitions equally likely, it makes sense to evaluate the a priori voting power of each member of the committee by the probability to have a swing, measured by the absolute Penrose–Banzhaf (PB) power index (Penrose [7], Banzhaf [1]) $\Pi_i^{\mathrm{PB}}(N, q, \boldsymbol{w}) = \frac{\mathrm{card}(C_i)}{2^{n-1}}$, where $\mathrm{card}(C_i)$ is the number of all winning coalitions the member i has the swing in, 2^{n-1} is the number of all possible coalitions with i. To compare the relative power of different committee members, the relative form of the PB power index $\pi_i^{\mathrm{PB}}(N, q, \boldsymbol{w}) = \frac{\mathrm{card}(C_i)}{\sum_{k \in N} \mathrm{card}(C_k)}$ is used.

While the absolute PB is based on a well-established probability model (see e. g. Owen [6]), its normalization (relative PB index) destroys this probabilistic interpretation, the relative PB index simply answers the question of what is the voter i's share in all possible swings.

Assuming many voting acts and all possible permutations equally likely, it makes sense to evaluate an a priori voting power of each committee member by the probability of being in pivotal situation, measured by the Shapley–Shubik (SS) power index (Shapley and Shubik [8]): $\Pi_i^{\mathrm{SS}}(N, q, \boldsymbol{w}) = \frac{\mathrm{card}(P_i)}{n!}$, where $\mathrm{card}(P_i)$ is the number of all permutations in which the committee member i is pivotal, and $n!$ is the number of all possible permutations of committee members. Since $\sum_{i \in N} \mathrm{card}(P_i) = n!$ it holds that $\pi_i^{\mathrm{SS}}(N, q, \boldsymbol{w}) = \frac{\mathrm{card}(P_i)}{\sum_{k \in N} \mathrm{card}(P_k)} = \frac{\mathrm{card}(P_i)}{n!}$, i. e. the absolute and relative form of the SS-power index is the same.

It can be easily seen that for any $\alpha > 0$ and any power index based on swings or pivots positions it holds that $\Pi_i(N, \alpha q, \alpha \boldsymbol{w}) = \Pi_i(N, q, \boldsymbol{w})$. Therefore, without the loss of generality, we shall assume throughout the text that $\sum_{i \in N} w_i = 1$ and $0.5 < q \leq 1$, using only relative weights and relative quotas in the analysis.

3. QUOTA INTERVALS OF STABLE POWER AND THE FAIR QUOTA

Let us formally define a few concepts we shall use later in this paper:

Definition 1. Let $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ be a fair distribution of voting weights (with whatever principle is used to justify it) in a simple weighted committee $[N, \boldsymbol{w}]$, $\boldsymbol{\pi}$ is a relative power index, $\boldsymbol{\pi}(N, q, \boldsymbol{w})$ is a vector valued function of q, and d is a distance function, then the voting rule q_1 is said to be *at least as fair* as voting rule q_2 with respect to the selected $\boldsymbol{\pi}$ and distance d if $d(\boldsymbol{w}, \boldsymbol{\pi}(N, q_1, \boldsymbol{w})) \leq d(\boldsymbol{w}, \boldsymbol{\pi}(N, q_2, \boldsymbol{w}))$.

Intuitively, given $\boldsymbol{w}$, the voting rule q_1 is preferred to the voting rule q_2 if q_1 generates a distribution of power closer to the distribution of weights than the voting rule q_2.

Definition 2. The voting rule q^* minimizing a distance d between $\boldsymbol{\pi}(N, q, \boldsymbol{w})$ and $\boldsymbol{w}$ is called a fair voting rule (*fair quota*) for the power index $\boldsymbol{\pi}$ with respect to the distance d.

In Turnovec [11] we provided proofs of the following statements:

Proposition 1. Let $[N, q, \boldsymbol{w}]$ be a simple quota weighted committee and C_{is} be the set of critical winning coalitions of the size s in which i has a swing, then $\mathrm{card}(P_i) = \sum_{s \in N} \mathrm{card}(C_{is})(s-1)!\,(n-s)!$ is the number of permutations with the pivotal position of i in $[N, q, \boldsymbol{w}]$.

From Proposition 1 it follows that the number of pivotal positions corresponds to the number and structure of swings. If in two different committees sets of swing coalitions are identical, then the sets of pivotal positions are also the same.

Proposition 2. Let $[N, q_1, \boldsymbol{w}]$ and $[N, q_2, \boldsymbol{w}]$, $q_1 \neq q_2$, be two simple quota weighted committees such that $W(N, q_1, \boldsymbol{w}) = W(N, q_2, \boldsymbol{w})$, then $C_i(N, q_1, \boldsymbol{w}) = C_i(N, q_2, \boldsymbol{w})$ and $P_i(N, q_1, \boldsymbol{w}) = P_i(N, q_2, \boldsymbol{w})$ for all $i \in N$.

From Proposition 2 it follows that in two different committees with the same set of members, the same weights and the same sets of winning coalitions, the PB-power indices and SS-power indices are the same in both committees, independently of quotas.

Proposition 3. Let $[N, q, \boldsymbol{w}]$ be a simple quota weighted committee with a quota q, $\mu^+(q) = \min_{S \in W(N,q,w)}(w(S) - q)$ and $\mu^-(q) = \min_{S \in 2^N \setminus W(N,q,w)}(q - w(S))$. Then for any particular quota q we have $W(n, q, \boldsymbol{w}) = W(N, \gamma, \boldsymbol{w})$ for all $\gamma \in (q - \mu^-(q), q + \mu^+(q)]$.

From Propositions 2 and 3 it follows that swing and pivot based power indices are the same for all quotas $\gamma \in (q - \mu^-(q), q + \mu^+(q)]$. Therefore the interval of quotas $(q - \mu^-(q), q + \mu^+(q)]$ we call *an interval of stable power* for quota q. Quota $\gamma^* \in (q - \mu^-(q), q + \mu^+(q)]$ is called the marginal quota for q if $\mu^+(\gamma^*) = 0$.

Now let us define a partition of the power set 2^N into equal weight classes $\Omega^{(0)}, \Omega^{(1)}, \dots$ $\dots, \Omega^{(r)}$ (such that the weight of different coalitions from the same class is the same and the weights of different coalitions from different classes are different). For the completeness set $w(\emptyset) = 0$. Consider the weight-increasing ordering of equal weight classes $\Omega^{(0)}, \Omega^{(1)}, \dots, \Omega^{(r)}$ such that for any $t < k$ and $S \in \Omega^{(t)}$, $R \in \Omega^{(k)}$ it holds that $w(S) < w(R)$. Denote $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 1, 2, \dots, r$.

Proposition 4. Let $\Omega^{(0)}, \Omega^{(1)}, \dots, \Omega^{(r)}$ be the weight-increasing ordering of the equal weight partition of 2^N. Set $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 1, 2, \dots, r$. Then there is a finite number $r \leq 2^{n-1}$ of marginal quotas and corresponding intervals of stable power $(q_{t-1}, q_t]$ such that $W(N, q_t, \boldsymbol{w}) \subset W(N, q_{t-1}, \boldsymbol{w})$.

From Proposition 4 it follows that there exist at most r distinct voting situations generating r vectors of power indices.

Proposition 5. Let $[N, q, \boldsymbol{w}]$ be a simple quota weighted committee and $(q_{t-1}, q_t]$ is the interval of stable power for quota q. Then $\mathrm{card}(C_i(N, q, \boldsymbol{w})) = \mathrm{card}(C_i(N, \gamma, \boldsymbol{w}))$ and $\mathrm{card}(P_i(N, q, \boldsymbol{w})) = \mathrm{card}(P_i(N, \gamma, \boldsymbol{w}))$ for any $\gamma = 1 - q_t + \varepsilon$, where $\varepsilon \in (0, q_t - q_{t-1}]$ and for all $i \in N$.

While in $[N, q, \boldsymbol{w}]$ the quota q means the total weight necessary to pass a proposal (and therefore we can call it a *winning quota*), the *blocking quota* means the total weight necessary to block a proposal. If q is a winning quota and $(q_t - 1, q_t]$ is a quota interval of stable power for q, then any voting quota $1 - q_{t-1} + \varepsilon$ (where $0 < \varepsilon \leq q_t - q_{t-1}$), is a blocking quota. From Proposition 5 it follows that the blocking power of the committee members, measured by swing and pivot-based power indices, is equal to their voting power. Let r be the number of marginal quotas, then from Proposition 4 it follows that for power indices based on swings and pivots the number of majority power indices does not exceed $\lceil \frac{r}{2} \rceil$ (smallest integer greater or equal to $\frac{r}{2}$).

Proposition 6. Let $[N, q, \boldsymbol{w}]$ be a simple quota weighted committee, d be a distance function and $\pi_i(N, q, \boldsymbol{w})$ be relative power indices for marginal quotas q_t, and q_{t^*} be the majority marginal quota minimizing the distance $d[\boldsymbol{\pi}(N, q_j, \boldsymbol{w}), \boldsymbol{w}]$ where $j = 1, 2, \ldots, r$, r is the number of intervals of stable power such that q_j are marginal majority quotas, then the fair quota for a particular power index used with respect to distance d is any $\gamma \in (q_{t^*-1}, q_{t^*}]$ from the quota interval of stable power for q_{t^*}.

From Proposition 6 it follows that the voting rule based on quota q_{t^*} minimizes selected distance between the vector of relative voting weights and the corresponding vector of relative voting power. The problem of fair quota has an exact solution via the finite number of majority marginal quotas.

4. FAIR QUOTA IN THE LOWER HOUSE OF THE CZECH PARLIAMENT

The Lower House of the parliament has 200 seats. Members of the Lower House are elected in 14 electoral districts from party lists by proportional system with 5 % threshold. Seats are allocated to the political parties that obtained not less than 5 % of total valid votes roughly proportionally to fractions of obtained votes (votes for parties not achieving the required threshold are redistributed among the successful parties roughly proportionally to the shares of obtained votes). Five political parties qualified to the Lower House: left centre Czech Social Democratic Party (Česká strana sociálně demokratická, ČSSD), right centre Civic Democratic Party (Občanská demokratická strana, ODS), right TOP09 (Tradice, Odpovědnost, Prosperita – Traditions, Responsibility, Prosperity 2009), left Communist Party of Bohemia and Moravia (Komunistická strana Čech a Moravy, KSČM) and supposedly centre (but not very clearly located on left-right political dimension) Public Issues (Věci veřejné, VV).

In Table 1 we provide results of the 2010 Czech parliamentary election (by relative voting weights we mean fractions of seats of each political party, by relative electoral support fractions of votes for political parties that qualified to the Lower House, counted from votes that were considered in allocation of seats). Three parties, ODS, TOP09 and VV, formed right-centre government coalition with 118 seats in the Lower House.

	Seats	Votes in % if valid votes	Relative voting weight	Relative electoral support
ČSSD	56	22.08	0.28	0.273098
ODS	53	20.22	0.265	0.250093
TOP09	41	16.7	0.205	0.206555
KSČM	26	11.27	0.13	0.139394
VV	24	10.58	0.12	0.13086
$\sum$	200	80.85	1	1

Tab. 1. Results of 2010 election to the Lower House of the Czech Parliament.[1]

We assume that all Lower House members of the same party are voting together and all of them are participating in each voting act. Two voting rules are used: simple majority (more than 100 votes) and qualified majority (at least 120 votes). There exist 16 possible winning coalitions for simple majority voting (12 of them are winning coalitions for qualified majority), 16 marginal majority quotas and 16 majority quota intervals of stable power (see Table 2). For analysis of fair voting rule we selected Shapley–Shubik power index and Euclidean distance function. In Table 3 we provide Shapley–Shubik power indices (distribution of relative voting power) for all of marginal majority quotas.

Parties of possible winning coalitions	Absolute marginal majority quota	Relative marginal majority quota	Interval of stable power
ODS+KSČM+VV	103	0.515	(0.485, 0.515]
ČSSD+KSČM+VV	106	0.53	(0.515, 0.53]
ČSSD+ODS	109	0.545	(0.53, 0.545]
ODS+TOP09+VV	118	0.59	(0.545, 0.59]
ODS+TOP09+KSČM	120	0.6	(0.59, 0.6]
ČSSD+TOP09+VV	121	0.605	(0.6, 0.605]
ČSSD+TOP09+KSČM	123	0.615	(0.605, 0.615]
ČSSD+ODS+VV	133	0.665	(0.615, 0.665]
ČSSD+ODS+KSČM	135	0.675	(0.665, 0.675]
ODS+TOP09+KSČM+VV	144	0.72	(0.675, 0.72]
ČSSD+TOP09+KSČM+VV	147	0.735	(0.72, 0.735]
ČSSD+ODS+TOP09	150	0.75	(0.735, 0.75]
ČSSD+ODS+KSČM+VV	159	0.795	(0.75, 0.795]
ČSSD+ODS+TOP09+VV	174	0.87	(0.795, 0.87]
ČSSD+ODS+TOP09+KSČM	176	0.88	(0.87, 0.88]
ČSSD+ODS+TOP09+KSČM+VV	200	1	(0.88, 1]

Tab. 2. Possible winning coalitions in the Lower House of the Czech Parliament (own calculations).

For any quota from each of intervals of stable power is Shapley–Shubik relative power identical with relative power in corresponding marginal majority quota. Entries in the row "distance" give Euclidean distance between vector of relative voting weights and

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[1]Source: `http://www.volby.cz/pls/ps2010/ps?xjazyk=CZ`

Party	Seats	Relative voting weight	SS power for							
			$q = 0.515$	$q = 0.53$	$q = 0.545$	$q = 0.59$	$q = 0.6$	$q = 0.605$	$q = 0.615$	$q = 0.665$
ČSSD	56	0.28	0.3	0.35	0.3167	0.2667	0.3167	0.3667	0.3333	0.3
ODS	53	0.265	0.3	0.2667	0.3167	0.2667	0.2333	0.2	0.25	0.3
TOP09	41	0.205	0.1333	0.1833	0.2333	0.2667	0.2333	0.2	0.1667	0.1333
KSČM	26	0.13	0.1333	0.1	0.0667	0.1	0.15	0.1167	0.1667	0.1333
VV	24	0.12	0.1333	0.1	0.0667	0.1	0.0667	0.1167	0.0833	0.1333
$\sum$	200	1	0.9999	1	1.0001	1.0001	1	1.0001	1	0.9999
distance			0.08339	0.08169	0.10802	0.07271	0.07996	0.10968	0.08501	0.08339
Party	Seats	Relative voting weight	SS power for							
			$q = 0.675$	$q = 0.72$	$q = 0.735$	$q = 0.75$	$q = 0.795$	$q = 0.0.87$	$q = 0.88$	$q = 1$
ČSSD	56	0.28	0.2667	0.2333	0.4333	0.3833	0.35	0.3	0.25	0.2
ODS	53	0.265	0.2667	0.2333	0.1833	0.3833	0.35	0.3	0.25	0.2
TOP09	41	0.205	0.1833	0.2333	0.1833	0.1333	0.1	0.3	0.25	0.2
KSČM	26	0.13	0.1833	0.15	0.1	0.05	0.1	0.05	0.25	0.2
VV	24	0.12	0.1	0.15	0.1	0.05	0.1	0.05	0	0.2
$\sum$	200	1	1	0.9999	0.9999	0.9999	1	1	1	1
distance			0.06238	0.07271	0.17874	0.20275	0.15637	0.14816	0.17875	0.14816

Tab. 3. Shapley–Shubik power of political parties for majority
marginal quotas (own calculations).

relative power for each quota interval of stable power.

The fair relative majority quota in our case is $q = 0.675$ (with respect to Euclidean distance between relative voting weights and relative voting power equal to 0.06238), or any quota from interval of stable power (0.665, 0.675]. It means that minimal number of votes to approve a proposal is 135 (in contrast to 101 votes required by simple majority and 120 votes required by qualified majority). Voting rule defined by this quota minimizes Euclidean distance between relative voting weights and relative voting power (measured by Shapley–Shubik power index) and approximately equalizes the voting power (influence) of the members of the Lower House independently on their political affiliation.

The distance measure of fairness follows the same logic as measures of deviation from proportionality used in political science, evaluating the difference between results of an election and the composition of an elected body – e. g. Gallagher [3] based on the Euclidean distance, or Loosemore and Hanby [4] based on the absolute values distance. Using in our particular case the absolute values distance we shall get the same fair quota.

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