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ANOTHER PROOF OF A RESULT OF JECH AND SHELAH

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Abstract. Shelah's pcf theory describes a certain structure which must exist if $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$ holds. Jech and Shelah proved the surprising result that this structure exists in ZFC. They first give a forcing extension in which the structure exists then argue that by some absoluteness results it must exist anyway. We reformulate the statement to the existence of a certain partially ordered set, and then we show by a straightforward, elementary (i.e., non-metamathematical) argument that such partially ordered sets exist.

Keywords: partially ordered set; pcf theory

MSC 2010: 03E05

Using Shelah's pcf theory, Jech and Shelah described in [1] a certain structure that must be present on ω_1 if $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$ (the consistency of the latter statement is one of the major problems of the set theory). They proved the surprising fact that such a structure exists in ZFC. The original proof was given first by a forcing argument then arguing that structures supplemented by forcing notions with certain properties exist outright in ZFC (the main result of [2]). Here we offer a more direct, forcing-free argument.

Notation and definitions. If f is a function, A is a subset of its domain, then we denote $\{f(x)\colon x \in A\}$ by $f[A]$.

In what follows we construct partially ordered sets of the following type. The underlying set is $T = \bigcup\{T_\alpha\colon \alpha < \omega_1\}$ where each $T_\alpha = \{t_i^\alpha\colon i < \omega\}$ is countable. Set $T_{<\alpha} = \bigcup\{T_\beta\colon \beta < \alpha\}$, $T_{>\alpha} = \bigcup\{T_\beta\colon \beta > \alpha\}$, $T(\beta,\alpha) = \bigcup\{T_\gamma\colon \beta < \gamma < \alpha\}$, $T(\beta,\alpha] = \bigcup\{T_\gamma\colon \beta < \gamma \leqslant \alpha\}$. Further, define functions $h\colon T \to \omega_1$, $d\colon T \to \omega$ by $h(t_i^\alpha) = \alpha$, $d(t_i^\alpha) = i$.

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We are going to construct a partial ordering $<$ of the following type: if $y < x$, $y \in T_\beta$, $x \in T_\alpha$ then $\beta < \alpha$. The construction of the partial order requires that we specify for every element $x \in T$ the set $\{y\colon\ y < x\}$. For technical reasons we construct a function f such that $f(x) \subseteq T_{<\alpha}$ if $x \in T_\alpha$ and then set $y \in f^*(x)$ if there is a sequence $y = x_n$, x_{n-1}, …, $x_0 = x$ such that $x_{i+1} \in f(x_i)$ $(i < n)$. Specifically, $x \in f^*(x)$. That is, f^* is the transitive closure of f, it is the set of all elements y for which $y \leqslant x$ holds.

We are going to construct partially ordered sets which are sufficiently "random" in the sense that for any finite subset there are points being in a predetermined position, assuming that some trivial conditions hold.

Theorem 1. *There is a function f as above such that*

(1) *if $y \in f(x)$, then $d(y) > d(x)$;*
(2) *if $y \neq y' \in f(x)$, then $d(y) \neq d(y')$;*
(3) *if $\beta < \omega_1$, $W, Z \subseteq T_{>\beta}$, $|W|$, $|Z| < \omega$, $f^*[Z] \cap W = \emptyset$, then there are infinitely many $r \in T_\beta$ such that $r \in f(w)$ $(w \in W)$, $r \notin f^*(z)$ $(z \in Z)$.*

P r o o f. We construct $f(x)$ for all $x \in T_\alpha$, by transfinite recursion on α. Assume that we are at stage α and $f(y)$ is determined for all $y \in T_{<\alpha}$.

At step $i = 0, 1, \ldots$ we construct finite sets $f_i(x)$, $g_i(x)$ for $x \in T_\alpha$ such that $f_i(x), g_i(x) \subseteq T_{<\alpha}$, $\emptyset = f_0(x) \subseteq f_1(x) \subseteq \ldots$, $\emptyset = g_0(x) \subseteq g_1(x) \subseteq \ldots$, and $g_i(x) \cap f^*[f_i(x)] = \emptyset$ (specifically $f_i(x) \cap g_i(x) = \emptyset$) will always hold. After ω steps we define $f(x) = \bigcup\{f_i(x)\colon\ i < \omega\}$ for $x \in T_\alpha$. Our sets $f_i(x)$, $g_i(x)$ are approximations: at step i; we determine that the elements of $f_i(x)$ will be in $f(x)$, and that the elements of $g_i(x)$ will not be in $f^*(x)$.

We fix an enumeration $\{(\beta_0, W_0, Z_0), (\beta_1, W_1, Z_1), \ldots\}$ of all triples (β, W, Z) where $\beta < \alpha$, W, Z are finite subsets of $T(\beta, \alpha)$ such that each triple occurs infinitely often. At step $i < \omega$ we either determine that (β_i, W_i, Z_i) is such that it cannot occur in (3) of the theorem or we construct f_{i+1}, g_{i+1} such that it will guarantee the existence of an element $r \in T_\beta$ to satisfy (3) of the theorem.

Assume that we have arrived at step i and we are given $(\beta, W, Z) = (\beta_i, W_i, Z_i)$. Set $W^+ = W \cap T_\alpha$, $W^- = W \cap T_{<\alpha}$, $Z^+ = Z \cap T_\alpha$, $Z^- = Z \cap T_{<\alpha}$. We have to treat the triple (β, W, Z) only if $W \cap f^*[Z] = \emptyset$ holds after the construction is finished. This implies that $W^- \cap f^*[Z^-] = \emptyset$ and

$$W^- \cap (f_i(x) \cup f^*[f_i(x)]) = \emptyset$$

holds for $x \in Z^+$. If either of them does not hold, then we let $f_{i+1}(x) = f_i(x)$, $g_{i+1}(x) = g_i(x)$ for $x \in T_\alpha$.

We therefore assume that the above two equalities do hold. Set

$$Z^* = Z^- \cup (f_i[Z^+] \cap T(\beta,\alpha)) \cup \bigcup\{f^*[f_i(x)] \cap T(\beta,\alpha)\colon\ x \in Z^+\}.$$

As $W \cap f^*[Z^*] = \emptyset$, we can find $r \in T_\beta$ which is appropriate for (W, Z^*), $r \notin f_i(x) \cap T_\beta$ $(x \in Z^+)$, and $d(r) > d(x)$ for every $x \in W^+ \cup f_i[W^+]$.

Define

$$f_{i+1}(x) = \begin{cases} f_i(x) \cup \{r\} & x \in W^+, \\ f_i(x) & x \in T_\alpha - W, \end{cases}$$

and

$$g_{i+1}(x) = \begin{cases} g_i(x) \cup \{r\} & x \in Z^+, \\ g_i(x) & x \in T_\alpha - Z. \end{cases}$$

We have to show that $f^*[f_{i+1}(x)] \cap g_{i+1}(x) = \emptyset$ still holds for $x \in T_\alpha$. If $x \in W^+$, then we have to show that $f^*(r) \cap g_i(x) = \emptyset$. But this is true, as for every element u of $f^*(r)$ and every element v of $g_i(x)$ we have $d(u) \geqslant d(r) > d(v)$ by our choice. If $x \in Z^+$, we have to show that $r \notin f^*[f_i(x)]$. Indeed, if $u \in f_i(x)$, then this holds for $u \in T(\beta,\alpha)$ by our definition of Z^*; if $u \in T_\beta$, then it holds as $r \neq u$ by our choice of r, and finally, if $u \in T_{<\beta}$, then it trivially holds, as $f^*(u) \subseteq T_{<\beta}$.

We claim that r will be as required for the triple (β, W, Z). Indeed, $r \in f(w)$ for $w \in W^-$ as r is good for (W, Z^*). If $w \in W^+$, then $r \in f(w)$, as $r \in f_{i+1}(w) \subseteq f(w)$. If $z \in Z^-$, then $r \notin f^*(z)$, as r was appropriate for (W, Z^*) and $Z^* \supseteq Z^-$. Finally, if $z \in Z^+$, then $r \in g_{i+1}(z)$, therefore r will not be an element of $f^*(z)$. □

Lemma 2. *If f is as in Theorem* 1, *then for every $x \in T$, $i < \omega$, the set $A(x,i) = \{\beta\colon\ t_i^\beta \in f^*(x)\}$ is finite.*

P r o o f. We prove this by induction on α, where $x \in T_\alpha$. By our construction, we either have

$$A(x,i) = A(y_0,i) \cup \ldots \cup A(y_m,i)$$

or

$$A(x,i) = \{x\} \cup A(y_0,i) \cup \ldots \cup A(y_m,i)$$

where $\{y \in f(x)\colon\ d(x) < d(y) \leqslant i\} = \{y_0,\ldots,y_m\}$. As $h(y) < h(x)$ whenever $y \in f(x)$, the sets on the right hand side are finite, and then so is $A(x,i)$. □

Corollary 3 (Jech-Shelah)**.** *There exist a partition* $\{A_n\colon n<\omega\}$ *of* ω_1 *and a family* $\{X_\alpha\colon \alpha<\omega_1\}$ *of subsets of* ω_1 *such that*

(1) $\max(X_\alpha)=\alpha$ $(\alpha<\omega_1)$;

(2) *if* $\beta\in X_\alpha$ *then* $X_\beta\subseteq X_\alpha$;

(3) $|X_\alpha\cap A_n|<\aleph_0$ $(\alpha<\omega_1, n<\omega)$;

(4) *if* $\lambda<\omega_1$ *is limit,* $\lambda\leqslant\alpha<\omega_1$, $\alpha_1,\dots,\alpha_k<\alpha$, $\gamma<\lambda$ *then*

$$X_\alpha\cap\lambda\not\subseteq\gamma\cup X_{\alpha_1}\cup\ldots\cup X_{\alpha_k}.$$

P r o o f. Let f be a function as in Theorem 1. Define

$$X_{\alpha\omega+i}=\{\beta\omega+j\colon\ t_j^\beta\in f^*(t_i^\alpha)\}$$

for $\alpha<\omega_1$, $i<\omega$, that is, identify T_α with the α-th interval of type ω, $[\omega\alpha,\omega(\alpha+1))$. Set $A_n=\{\alpha\omega+n\colon \alpha<\omega_1\}$ for $n<\omega$. Then (1) is obvious. (2) follows as $f^*(y)\subseteq f^*(x)$ for $y\in f^*(x)$. (3) holds by the Lemma 2. Finally, for (4) we have to show that

$$[\beta\omega,(\beta+1)\omega)\cap X_{\alpha\omega+m}-(X_{\alpha\omega}\cup\ldots\cup X_{\alpha\omega+m-1}\cup X_{\beta_1\omega+i_1}\cup\ldots\cup X_{\beta_k\omega+i_k})$$

is infinite when $\beta<\alpha$, $m<\omega$, $\beta_1,\dots,\beta_k<\alpha$, and $i_1,\dots,i_k<\omega$. Translating this back to our construction, we have to prove that

$$(T_\beta\cap f^*(t_m^\alpha))-(f^*(t_0^\alpha)\cup\ldots\cup f^*(t_{m-1}^\alpha)\cup f^*(t_{i_1}^{\beta_1})\cup\ldots\cup f^*(t_{i_k}^{\beta_k}))$$

is infinite. This follows from Theorem 1 if we set $W=\{t_m^\alpha\}$,

$$Z=\{t_0^\alpha,\dots,t_{m-1}^\alpha,t_{i_1}^{\beta_1},\dots,t_{i_k}^{\beta_k}\}$$

and notice that $f^*[Z]\cap W=\emptyset$ as $f^*[Z]\cap T_\alpha=\{t_0^\alpha,\dots,t_{m-1}^\alpha\}$. □

Theorem 4. *There is a function* f *such that if* $\beta<\omega_1$, $U,V\subseteq T_{<\beta}$, $W,Z\subseteq T_{>\beta}$ *are finite sets such that*

(1) $U\subseteq f^*(w)$ $(w\in W)$,

(2) $f^*[U]\cap V=\emptyset$,

(3) $f^*[Z]\cap W=\emptyset$, *then there are infinitely many* $r\in T_\beta$ *such that*

(a) $U\subseteq f^*(r)$;

(b) $V\cap f^*(r)=\emptyset$;

(c) $r\in f^*(w)$ $(w\in W)$;

(d) $r\notin f^*(z)$ $(z\in Z)$.

P r o o f. Similarly to the proof of Theorem 1, we determine $f(x)$, hence $f^*(x)$ for every $x \in T_\alpha$, by transfinite recursion on α. Assume, therefore, that we are at stage α, and so $f(x)$ is already defined for $x \in T_{<\alpha}$.

We call a quintuple (β, U, V, W, Z) satisfying (1)–(3) *consistent*. If $r \in T_\beta$ satisfies (a)–(d), we say that r *is good for* (β, U, V, W, Z). Enumerate all quintuples (β, U, V, W, Z) with $\beta \leqslant \alpha$, $U, V \subseteq T_{<\beta}$, $W, Z \subseteq T(\beta, \alpha]$ such that U, V, W, Z are finite, either $\beta = \alpha$ or else $(W \cup Z) \cap T_\alpha \neq \emptyset$ as $\{(\beta_i, U_i, V_i, W_i, Z_i)\colon\ i < \omega\}$ so that each such quintuple occurs infinitely many times.

We are going to define finite sets $f_i(x)$, $g_i(x) \subseteq T_{<\alpha}$ for all $x \in T_\alpha$ and $i < \omega$ such that

(1) $\emptyset = f_0(x) \subseteq f_1(x) \subseteq \ldots$,
(2) $\emptyset = g_0(x) \subseteq g_1(x) \subseteq \ldots$,
(3) for every $i < \omega$, $f_i(x) = g_i(x) = \emptyset$ holds for all but finitely many $x \in T_\alpha$,
(4) $g_i(x) \cap f^*[f_i(x)] = \emptyset$ $(i < \omega,\ x \in T_\alpha)$.

After ω steps we let $f(x) = \bigcup\{f_i(x)\colon\ i < \omega\}$ for $x \in T_\alpha$. This means that at step $i < \omega$, for every x we have finitely many commitments: if $y \in f_i(x)$ we promise that $y \in f(x)$, if $y \in g_i(x)$, we promise that $y \notin f^*(x)$ will hold.

Assume first that we are at step $i < \omega$ and we have to treat a quintuple $(\beta_i, U_i, V_i, W_i, Z_i)$ where $\beta_i = \alpha$, and consequently $W_i = Z_i = \emptyset$. If $f^*[U_i] \cap V_i \neq \emptyset$, we do nothing, i.e., leave $f_{i+1}(x) = f_i(x)$, $g_{i+1}(x) = g_i(x)$ $(x \in T_\alpha)$ as this quintuple will not occur among those for which the theorem applies. If, however, $f^*[U_i] \cap V_i = \emptyset$, then select an $x \in T_\alpha$ such that $f_i(x) = g_i(x) = \emptyset$ and set $f_{i+1}(x) = U_i$, $g_{i+1}(x) = V_i$, and $f_{i+1}(y) = f_i(y)$, $g_{i+1}(y) = g_i(y)$ $(y \in T_\alpha - \{x\})$.

Assume next that we are at step $i < \omega$ and we are to handle $(\beta, U, V, W, Z) = (\beta_i, U_i, V_i, W_i, Z_i)$ with $\beta < \alpha$. Set $W^+ = W \cap T_\alpha$, $W^- = W \cap T(\beta, \alpha)$, $Z^+ = Z \cap T_\alpha$, $Z^- = Z \cap T(\beta, \alpha)$.

We do nothing, if either (β, U, V, W^-, Z^-) is inconsistent, or there is an $x \in W^+$ such that $g_i(x) \cap f^*[U] \neq \emptyset$, or there is an $x \in Z^+$ such that $f^*[f_i(x)] \cap W^- \neq \emptyset$. In these cases we set $f_{i+1}(x) = f_i(x)$, $g_{i+1}(x) = g_i(x)$ for every $x \in T_\alpha$. After finishing the construction of f on T_α, we will have that (β, U, V, W, Z) is inconsistent.

We can, therefore, assume that (β, U, V, W^-, Z^-) is consistent, $g_i(x) \cap f^*[U] = \emptyset$ $(x \in W^+)$ and $f^*[f_i(x)] \cap W^- = \emptyset$ $(x \in Z^+)$.

Define

$$V^* = V \cup \bigcup\{g_i(x) \cap T_{<\beta}\colon\ x \in W^+\}$$

and

$$Z^* = Z^- \cup \bigcup\{f_i(x) \cap T(\beta, \alpha)\colon\ x \in Z^+\}.$$

Claim 1. $(\beta, U, V^*, W^-, Z^*)$ *is consistent.*

Proof. We have to check (1)–(3) for $(\beta, U, V^*, W^-, Z^*)$.

(1) holds as (β, U, V, W^-, Z^-) is consistent.

For (2), we have to show that $f^*[U] \cap V^* = \emptyset$. On the one hand, $f^*[U] \cap V = \emptyset$, as (β, U, V, W^-, Z^-) is consistent, on the other hand, $f^*[U] \cap g_i(x) = \emptyset$ holds for every $x \in W^+$ by our assumptions.

For (3), we have to show that $f^*[Z^*] \cap W^- = \emptyset$. This holds as, on the one hand, $f^*[Z^-] \cap W^- = \emptyset$, as (β, U, V, W^-, Z^-) is consistent, on the other hand, $f^*[f_i(x)] \cap W = \emptyset$ holds by our assumptions $(x \in Z^+)$. □

By Claim 1 and by the inductive hypothesis, we can find an $r \in T_\beta$ which satisfies (a)–(d) for $(\beta, U, V^*, W^-, Z^*)$ and $r \notin \bigcup\{g_i(x)\colon\ x \in W^+\}$.

Now define

$$f_{i+1}(x) = \begin{cases} f_i(x) \cup \{r\} & x \in W^+, \\ f_i(x) & x \in T_\alpha - W, \end{cases}$$

and

$$g_{i+1}(x) = \begin{cases} g_i(x) \cup \{r\} & x \in Z^+, \\ g_i(x) & x \in T_\alpha - Z. \end{cases}$$

Claim 2. $g_{i+1}(x) \cap f^*[f_{i+1}(x)] = \emptyset$ $(x \in T_\alpha)$.

Proof. As $g_i(x) \cap f^*[f_i(x)] = \emptyset$ holds by the inductive hypothesis, we have to show that $r \notin f^*[f_i(x)]$ $(x \in Z^+)$, and $f^*(r) \cap g_i(x) = \emptyset$ $(x \in W^+)$. The former holds, as r is good for $(\beta, U, V^*, W^-, Z^*)$ and $f_i(x) \cap T_{>\beta} \subseteq Z^*$. The latter holds, as $r \notin g_i(x)$ by our choice and r is good for $(\beta, U, V^*, W^-, Z^*)$ and $g_i(x) \subseteq V^*$ and therefore $f^*(r) \cap T_{<\beta} \cap g_i(x) = \emptyset$. □

Finally we claim that r will be good for (β, U, V, W, Z). Indeed, if $x \in W^+$, then $r \in f_{i+1}(x) \subseteq f(x)$, if $x \in Z^+$, then $r \in g_{i+1}(x)$, so $r \notin f^*(x)$. All other statements are obvious. □

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