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# DERIVATIONS OF HOMOTOPY ALGEBRAS

Tom Lada and Melissa Tolley

Abstract. We recall the definition of strong homotopy derivations of $A_\infty$ algebras and introduce the corresponding definition for $L_\infty$ algebras. We define strong homotopy inner derivations for both algebras and exhibit explicit examples of both.

## 1. Introduction

The concept of a strong homotopy derivation of an $A_\infty$ algebra was introduced by Kajiura and Stasheff in [2]. In this note we will introduce the corresponding concept for $L_\infty$ algebras. We will discuss several concrete examples of such algebras and strong homotopy inner derivations on them.

In Section 2, we recall the definitions of $A_\infty$ and $L_\infty$ algebras and discuss an explicit example of each. We will use these examples to exhibit examples of strong homotopy derivations.

In Section 3, we review the definition of a strong homotopy derivation of an $A_\infty$ algebra. We introduce the concept of an inner such derivation of these algebras and present an explicit example of this concept by using the $A_\infty$ algebra in the previous section.

The next section contains our definition of a strong homotopy derivation of an $L_\infty$ algebra. We discuss the concept of inner derivation and present a concrete example of such a derivation on the $L_\infty$ algebra in Section 2.

In the final section we will discuss the relationship between the $A_\infty$ data and the $L_\infty$ data using symmetrization.

We work in the setting of $\mathbb{Z}$ graded vector spaces and will occasionally use the notation $|x|$ to denote the degree of an element $x$.

## 2. $A_\infty$ and $L_\infty$ algebras

**Definition 1.** An $A_\infty$ algebra [6] structure on a $\mathbb{Z}$ graded vector space $V$ is a collection of degree one linear maps $m_n : V^{\otimes n} \to V$ that satisfy the relations

$$(2.1)\qquad \sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^\alpha m_k\big(v_1, \dots, v_{i-1}, m_l(v_i, \dots, v_{i+l-1}), v_{i+l}, \dots, v_n\big) = 0$$

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for $n \geq 1$ and $\alpha$ is the sum of the degrees of the elements $v_1, \ldots, v_{i-1}$.

We remark that this definition differs from but is equivalent to the original definition [6] in which the maps $m_n$ have degree $n-2$ and the signs are adjusted accordingly. It is well known [6] that the structure maps $m_n$'s may be extended to a degree $+1$ coderivation $\boldsymbol{m}$ on the tensor coalgebra $T^c(V)$ of $V$, and that the relations are equivalent to the equation $\boldsymbol{m}^2 = 0$.

**Example 2** ([1])**.** Consider the graded vector space in which $V_{-1}$ has basis $\langle x_1, x_2\rangle$, $V_0$ has basis $\langle y\rangle$, and $V_n = 0$ otherwise. Define degree one maps

$$m_1(x_1) = m_1(x_2) = y$$

$$m_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}) = x_1\,, \qquad 0 \leq k \leq n-2$$

$$m_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = x_1$$

$$m_n(x_1 \otimes y^{\otimes n-1}) = y$$

and $m_n = 0$ on the remaining elements of $V$. This determines an $A_\infty$ algebra structure on $V$.

We next recall the definition of $L_\infty$ algebras.

**Definition 3** ([5])**.** An $L_\infty$ algebra structure on a $\mathbb{Z}$ graded vector space $V$ is a collection of degree one graded symmetric linear maps $l_n \colon V^{\otimes n} \to V$, $n \geq 1$, that satisfy the relations (higher order Jacobi relations)

$$\sum_{j+k=n} \sum_{\sigma} (-1)^{e(\sigma)} l_{1+j}\big(l_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}\big) = 0$$

where $\sigma$ runs over all $(k, n-k)$ unshuffle permutations. The exponent $e(\sigma)$ is the sum of the products of the degrees of the elements that are permuted, sometimes known as the Koszul sign.

Again, we remark that this definition differs from but is equivalent to the original definition [5] in which the maps $l_n$ have degree $n-2$ and are graded skew symmetric with the signs adjusted. Also, the structure maps may be extended to a degree $+1$ coderivation $\boldsymbol{l}$ on the symmetric coalgebra $S^c(V)$ on $V$, and the relations are equivalent to $\boldsymbol{l}^2 = 0$, [4],[5].

It is well known that skew symmetrization of an $A_\infty$ algebra structure yields an $L_\infty$ algebra structure [4] when one utilizes the original definitions. However, with the definitions that we use here, we symmetrize the data in the example above to obtain

**Example 4.** Consider the graded vector space in which $V_{-1}$ has basis $\langle x_1, x_2\rangle$, $V_0$ has basis $\langle y\rangle$, and $V_n = 0$ otherwise. Define degree one symmetric maps

$$l_1(x_1) = l_1(x_2) = y$$

$$l_n(x_1 \otimes y^{\otimes n-1}) = (n-1)!\, y$$

$$l_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n-2)!\, x_1$$

and extend the maps using symmetry. This yields an $L_\infty$ algebra structure on $V$.

We will use these examples above to illustrate examples of strong homotopy derivations which we will define in the next two sections.

## 3. Strong homotopy derivations of $A_\infty$ algebras

Kajiura and Stasheff [2] have formulated the following definition:

**Definition 5.** A strong homotopy derivation of degree one of an $A_\infty$ algebra $(V, \{m_n\})$ is a collection of degree one linear maps $\theta_q\colon V^{\otimes q} \to V$, $q \geq 1$, that satisfy the relations

$$0 = \sum_{r+s=q+1} \sum_{i=0}^{r-1} (-1)^{\beta(s,i)} \theta_r\big(v_1, \dots, v_i, m_s(v_{i+1}, \dots, v_{i+s}), \dots, v_q\big)$$
$$+ (-1)^{\beta(s,i)} m_r\big(v_1, \dots, v_i, \theta_s(v_{i+1}, \dots, v_{i+s}), \dots, v_q\big)\,. \tag{3.1}$$

The exponent $\beta(s,i)$ results from moving the degree one maps $m_s$ and $\theta_s$ past $(v_1, \dots, v_i)$. The $\theta_q$'s may be extended to a degree $+1$ coderivation $\boldsymbol{\theta}$ on $T^c(V)$ and the relations then can be described by the equation $[\boldsymbol{m}, \boldsymbol{\theta}] = 0$.

As an example of such a structure, we can define a strong homotopy inner derivation of an $A_\infty$ algebra.

**Proposition 6.** *Let $(V, \{m_n\})$ be an $A_\infty$ algebra and let $a \in V$ have the property that $m_1(a) = 0$ and the degree of $a$ is even. Then the maps*

$$\theta_n(v_1, \dots, v_n) = m_{n+1}(a, v_1, \dots, v_n) + \cdots + m_{n+1}(v_1, \dots, v_i, a, v_{i+1}, \dots, v_n)$$
$$+ \cdots + m_{n+1}(v_1, \dots, v_n, a) \tag{3.2}$$

*define a strong homotopy derivation of $V$. We call such a derivation inner.*

**Proof.** It can be calculated that the defining relations for a strong homotopy derivation in this case result in $n+1$ copies of the defining relations for an $A_\infty$ algebra except for terms that involve $m_1(a)$. Because of our requirement that $m_1(a) = 0$, we may add in the missing terms and utilize the $A_\infty$ algebra relations $n+1$ times to obtain the result. □

Recall the example of an $A_\infty$ algebra in Section 2. There we had the following data. Consider the graded vector space in which $V_{-1}$ has basis $\langle x_1, x_2\rangle$, $V_0$ has basis $\langle y\rangle$, and $V_n = 0$ otherwise. We may construct a strong homotopy inner derivation on $V$ by letting $a = y$. One may then calculate resulting $\theta_n$'s to be

$$\theta_1(x_1) = y$$
$$\theta_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}) = n x_1$$
$$\theta_n(x_1 \otimes y^{\otimes n-1}) = n y$$
$$\theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n-1) x_1$$

and $\theta_n = 0$ on the terms not mentioned.

## 4. Strong homotopy derivations of $L_\infty$ algebras

We now turn our attention to $L_\infty$ algebras.

**Definition 7.** A strong homotopy derivation of degree one of an $L_\infty$ algebra $(V, \{l_n\})$ is a collection of degree one graded symmetric linear maps $\theta_q \colon V^{\otimes q} \to V$, $q \geq 1$, that satisfy the relations

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{\sigma}(-1)^{e(\sigma)}\theta_{n-j+1}\big(l_j(v_{\sigma(1)},\ldots,v_{\sigma(j)}),v_{\sigma(j+1)},\ldots,v_{\sigma(n)}\big)\\
&\qquad+(-1)^{e(\sigma)}l_{n-j+1}\big(\theta_j(v_{\sigma(1)},\ldots,v_{\sigma(j)}),v_{\sigma(j+1)},\ldots,v_{\sigma(n)}\big)=0
\end{aligned}\tag{4.1}$$

where $\sigma$ runs over all $(j, n-j)$ unshuffle permutations.

The exponent $e(\sigma)$ is the sum of the products of the degrees of the permuted elements. As we saw for $A_\infty$ derivations, we may express the defining relations for strong homotopy derivations on $L_\infty$ algebras by the equation $[\boldsymbol{l}, \boldsymbol{\theta}] = 0$ where $\boldsymbol{\theta}$ is the degree $+1$ coderivation on $S^c(V)$ induced by the $\theta_n$'s. See [7] for details.

As an example, we define a strong homotopy inner derivation of an $L_\infty$ algebra.

**Proposition 8.** *Let $(V, \{l_n\})$ be an $L_\infty$ algebra and let $a \in V$ have the property that $l_1(a) = 0$ and the degree of $a$ is even. Then the maps*

$$\theta_n(v_1,\ldots,v_n) = l_{n+1}(v_1,\ldots,v_n,a)\tag{4.2}$$

*define a strong homotopy derivation of $V$.*

**Proof.** We compute

$$\begin{aligned}
&\sum_{\substack{j=1\\ \sigma}}^{n}(-1)^{e(\sigma)}\theta_{n-j+1}\big(l_j(v_{\sigma(1)},\ldots,v_{\sigma(j)}),v_{\sigma(j+1)},\ldots,v_{\sigma(n)}\big)\\
&\qquad+(-1)^{e(\sigma)}l_{n-j+1}\big(\theta_j(v_{\sigma(1)},\ldots,v_{\sigma(j)}),v_{\sigma(j+1)},\ldots,v_{\sigma(n)}\big)\\
&\quad=\sum_{\substack{j=1\\ \sigma}}^{n}(-1)^{e(\sigma)}l_{n-j+2}\big(l_j(v_{\sigma(1)},\ldots,v_{\sigma(j)}),v_{\sigma(j+1)},\ldots,v_{\sigma(n)},a\big)\\
&\qquad+(-1)^{e(\sigma)}l_{n-j+1}\big(l_{j+1}(v_{\sigma(1)},\ldots,v_{\sigma(j)},a),v_{\sigma(j+1)},\ldots,v_{\sigma(n)}\big)\\
&\quad=\sum_{\substack{j=1\\ \sigma}}^{n}(-1)^{e(\sigma)}l_{n-j+2}\big(l_j(v_{\sigma(1)},\ldots,v_{\sigma(j)}),v_{\sigma(j+1)},\ldots,v_{\sigma(n)},a\big)\\
&\qquad+(-1)^{e(\sigma)}(-1)^{\alpha}l_{n-j+1}\big(l_{j+1}(v_{\sigma(1)}\ldots,v_{\sigma(j)},a),v_{\sigma(j+1)},\ldots,v_{\sigma(n)}\big)\\
&\qquad+(-1)^{\beta}l_{n+1}\big(l_1(a),v_1,\ldots,v_n\big)=0
\end{aligned}$$

because these are precisely the $L_\infty$ algebra relations on $(v_1, \ldots, v_n, a)$. Note that it is necessary to add the last line, where $\beta = |a| \sum_{i=1}^{n} |v_i|$, to the homotopy derivation relations to obtain the $L_\infty$ algebra relations; this term, however, is zero because of our assumptions on the element $a$. The sign in the next to last line reduces to the required $(-1)^{e(\sigma)}$ because $\alpha = |a| \sum_{i=j+1}^{n} |v_{\sigma(i)}|$ is even. □

Recall the example of an $L_\infty$ algebra in Section 2. There, $V = V_{-1} \oplus V_0$ with basis for $V_{-1} = \langle x_1, x_2 \rangle$ and basis for $V_0 = \langle y \rangle$ and the degree one graded symmetric maps given by

$$l_1(x_1) = l_1(x_2) = y$$
$$l_n(x_1 \otimes y^{\otimes n-1}) = (n-1)!\, y$$
$$l_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n-2)!\, x_1\,.$$

We construct a strong homotopy derivation of $V$ by letting $a = y$ and then calculate the resulting $\theta_n$'s to be

$$\theta_1(x_1) = y$$
$$\theta_n(x_1 \otimes y^{\otimes n-1}) = n!\, y$$
$$\theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n-1)!\, x_1$$

and $\theta_n$ is zero on the elements not listed.

For example,

$$\begin{aligned}\theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &:= l_{n+1}(x_1 \otimes y^{\otimes n-2} \otimes x_2 \otimes y)\\ &= l_{n+1}(x_1 \otimes y^{\otimes n-1} \otimes x_2) = (n-1)!\, x_1\,.\end{aligned}$$

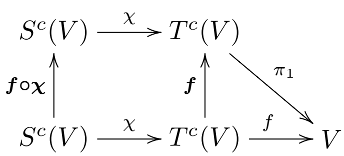
## 5. Symmetrization of $A_\infty$ derivations

We recall that there is a well known injective coalgebra map $\chi\colon S^c(V) \longrightarrow T^c(V)$ given by

$$\chi(v_1, \dots, v_n) = \sum_{\sigma \in S_n} (-1)^{e(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

where $(-1)^{e(\sigma)}$ is the Koszul sign.

Suppose that $f\colon T^c(V) \longrightarrow V$ is a linear map which extends to the coderivation $\boldsymbol{f}\colon T^c(V) \longrightarrow T^c(V)$ such that $\pi_1 \circ \boldsymbol{f} = f$, where $\pi_1\colon T^c(V) \longrightarrow V$ is projection. Then the linear map $f \circ \chi\colon S^c(V) \longrightarrow V$ extends to the coderivation $\boldsymbol{f \circ \chi}\colon S^c(V) \longrightarrow S^c(V)$ and the following diagram commutes ([3, Prop. 5])




The symmetrization of an $A_\infty$ algebra structure that was mentioned in Section 2 may then be described by the commutative diagram

$$
\begin{array}{ccccc}
S^c(V) & \xrightarrow{\chi} & T^c(V) & & \\
\Big\uparrow \boldsymbol{l} & & \Big\uparrow \boldsymbol{m} & \searrow^{\pi_1} & \\
S^c(V) & \xrightarrow{\chi} & T^c(V) & \xrightarrow{m} & V
\end{array}
$$

where $m = \sum m_n \colon T^c(V) \longrightarrow V$ is the collection of the $A_\infty$ algebra structure maps, $\boldsymbol{m}$ is the lift of $m$ to a coderivation on $T^c(V)$ with $\boldsymbol{m}^2 = 0$, and the $L_\infty$ algebra structure $\boldsymbol{l}$ is the lift of the map $m \circ \chi \colon S^c(V) \longrightarrow V$ to a coderivation on $S^c(V)$.

We now address the issue of symmetrization of strong homotopy derivations of $A_\infty$ algebras.

**Proposition 9.** *Let $\theta = \{\theta_n\}$ denote the the collection of maps giving a strong homotopy derivation on the $A_\infty$ algebra $(V, m)$. Regard $\theta$ as a map $T^c(V) \longrightarrow V$ and lift it to the coderivation $\boldsymbol{\theta}$ on $T^c(V)$. Then the extension of the map $\theta \circ \chi \colon S^c(V) \longrightarrow V$ to the coderivation $\boldsymbol{\theta'}$ on $S^c(V)$ is a strong homotopy derivation on the $L_\infty$ algebra $V$ with algebra structure given by $m \circ \chi$.*

**Proof.** We claim that $[\boldsymbol{l}, \boldsymbol{\theta'}] = 0$. We have the commutative diagram

$$
\begin{array}{ccccc}
S^c(V) & \xrightarrow{\chi} & T^c(V) & & \\
\Big\uparrow \boldsymbol{\theta'} & & \Big\uparrow \boldsymbol{\theta} & \searrow^{\pi_1} & \\
S^c(V) & \xrightarrow{\chi} & T^c(V) & \xrightarrow{\theta} & V
\end{array}
$$

and we calculate

$$
\begin{aligned}
\chi[\boldsymbol{l}, \boldsymbol{\theta'}] &= \chi(\boldsymbol{l}\boldsymbol{\theta'} + \boldsymbol{\theta'}\boldsymbol{l}) \\
&= (\chi\boldsymbol{l})\boldsymbol{\theta'} + (\chi\boldsymbol{\theta'})\boldsymbol{l} \\
&= \boldsymbol{m}(\chi\boldsymbol{\theta'}) + \boldsymbol{\theta}(\chi\boldsymbol{l}) \\
&= \boldsymbol{m}\boldsymbol{\theta}\chi + \boldsymbol{\theta}\boldsymbol{m}\chi \\
&= [\boldsymbol{m}, \boldsymbol{\theta}]\chi = 0
\end{aligned}
$$

because $\chi \circ \boldsymbol{l} = \boldsymbol{m} \circ \chi$ from the commutative diagram and $[\boldsymbol{m}, \boldsymbol{\theta}] = 0$ because $\boldsymbol{\theta}$ is a strong homotopy derivation of an $A_\infty$ algebra. Because $\chi$ is injective, it follows that $[\boldsymbol{l}, \boldsymbol{\theta'}] = 0$. $\square$

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