Todor Milev; Petr Somberg

The F-method and a branching problem for generalized Verma modules associated to $(\text{Lie } G_2, \text{so}(7))$

Archivum Mathematicum, Vol. 49 (2013), No. 5, 317--332

Persistent URL: `http://dml.cz/dmlcz/143556`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 49 (2013), 317–332

THE F-METHOD AND A BRANCHING PROBLEM FOR GENERALIZED VERMA MODULES ASSOCIATED TO (Lie G_2, so(7))

Todor Milev and Petr Somberg

Abstract. The branching problem for a couple of non-compatible Lie algebras and their parabolic subalgebras applied to generalized Verma modules was recently discussed in [15]. In the present article, we employ the recently developed F-method, [10], [11] to the couple of non-compatible Lie algebras Lie $G_2 \overset{i}{\hookrightarrow}$ so(7), and generalized conformal so(7)-Verma modules of scalar type. As a result, we classify the $i(\text{Lie } G_2) \cap \mathfrak{p}$-singular vectors for this class of so(7)-modules.

1. Introduction and motivation

The subject of our article has its motivation in the Lie theory problem of branching rules for finite dimensional simple Lie algebras and composition structure of generalized Verma modules, and dually in the geometrical problem related to the construction of invariant differential operators in parabolic invariant theories.

We assume that $\mathfrak{g}$, $\mathfrak{g}'$ are complex semisimple Lie algebras and $i\colon \mathfrak{g}' \hookrightarrow \mathfrak{g}$ is an injective homomorphism. Then $i(\mathfrak{g}')$ is (complex) reductive in $\mathfrak{g}$ and we can choose Borel subalgebras $\mathfrak{b}' \subset \mathfrak{g}'$ and $\mathfrak{b} \subset \mathfrak{g}$ such that $i(\mathfrak{b}') \subset \mathfrak{b}$. Let $\mathfrak{p} \supset \mathfrak{b}$ be a parabolic subalgebra of $\mathfrak{g}$. Let $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V}_\lambda)$ be the generalized Verma $\mathfrak{g}$-module induced from the irreducible finite dimensional $\mathfrak{p}$-module $\mathbb{V}_\lambda$ with highest weight λ. We define the branching problem for $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V}_\lambda)$ over $\mathfrak{g}'$ to be the problem of finding all $\mathfrak{b}'$-singular vectors in $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V}_\lambda)$, that is, the set of all vectors annihilated by image of the nilradical of $\mathfrak{b}'$ on which the image of the Cartan subalgebra of $\mathfrak{b}'$ has diagonal action.

In the recent article [15], under certain technical assumptions, we proved that $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V}_\lambda)$ has (finite or infinite) Jordan-Hölder series over $\mathfrak{g}'$, and enumerated the $\mathfrak{b}'$-highest weights μ appearing in the series. We also computed the dimension $m(\mu,\lambda)$ of the vector space of $\mathfrak{b}'$-highest weights of weight μ as a function of μ and λ. Further we gave a procedure for producing explicit formulas for some (but not all) $\mathfrak{b}'$-highest weight vectors.

2010 *Mathematics Subject Classification*: primary 22E47; secondary 17B10, 13C10.
Key words and phrases: generalized Verma modules, conformal geometry in dimension 5, exceptional Lie algebra Lie G_2, F-method, branching problem.
DOI: 10.5817/AM2013-5-317

As an example, we discussed Lie $G_2 \overset{i}{\hookrightarrow} \mathrm{so}(7)$. Restricting our attention to the parabolic subalgebra $\mathfrak{p} \simeq \mathfrak{p}_{(1,0,0)}$ and the 6 infinite families of highest weights $x\varepsilon_1$, $x\varepsilon_1 + \omega_2$, $x\varepsilon_1 + \omega_3$, $x\varepsilon_1 + 2\omega_2$, $x\varepsilon_1 + \omega_1 + \omega_2$, $x\varepsilon_1 + 2\omega_3$ we computed in [15] all $\mathfrak{b}'$-singular vectors with $\mathfrak{b}'$-dominant weights. From the theory of generalized Verma modules we know that, depending on the integrality and dominance of the $\mathfrak{b}$-highest weight, $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{V}_\lambda)$ has $\mathfrak{b}$-singular (and therefore $\mathfrak{b}'$-singular) vectors other than the highest weight vector. Therefore these vectors give additional $\mathfrak{b}'$-singular vectors whose weights are not $\mathfrak{b}'$-dominant (and are not computed in [15]).

Fix the pair Lie $G_2 \overset{i}{\hookrightarrow} \mathrm{so}(7)$ and fix the parabolic subalgebra to be the parabolic subalgebra $\mathfrak{p}_{(1,0,0)} \subset \mathrm{so}(7)$ obtained by crossing out the first (long) root of so(7). Let

$$\mathfrak{p}'_{(1,0)} = i^{-1}(\mathrm{Lie}\ G_2)\,.$$

In the present article, for the family of so(7)-highest weights of the form $x\varepsilon_1$, we prove that if $x \in \{-3/2, -1/2, 1/2, \dots\}$, the module $M^{\mathfrak{g}}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{V}_{x\varepsilon_1})$ has, besides its highest weight vector, exactly one $\mathfrak{p}'_{(1,0)}$-singular vector, and has no $\mathfrak{p}'_{(1,0)}$-singular vectors otherwise. Here we recall that, for an arbitrary parabolic subalgebra $\mathfrak{p}'$, a $\mathfrak{p}'$-singular vector is defined as a vector that is annihilated by all elements of the Levi part of $\mathfrak{p}'$, and therefore has weight that projects to zero onto the Levi part of $\mathfrak{p}'$ ("weight of scalar type"). Our result has a somewhat unusually sounding consequence: the $\mathfrak{p}'_{(1,0)}$-singular vector in $M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{V}_{x\varepsilon_1})$ must automatically be $\mathfrak{p}_{(1,0,0)}$-singular. This fact must necessarily fail to generalize for sufficiently large values of a, b and highest weights of the form $\lambda = x\omega_1 + a\omega_2 + b\omega_3$. Indeed, the number of $\mathfrak{p}_{(1,0,0)}$-singular vectors in $M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{V}_\lambda)$ is uniformly bounded, while the number $m(x\omega_1, x\omega_1 + a\omega_2 + b\omega_3)$ grows as a linear function of a and b.

Finally, we note that the $\mathfrak{p}'$-singular vectors constructed here correspond to non-standard homomorphisms of generalized Verma modules for both so(7) and Lie G_2. We would like to emphasize that our example goes beyond the compatible couples of Lie algebras discussed in [10], [11].

A geometric motivation for the branching problem can be described as follows. Let G, G' be the connected and simply connected Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{g}'$. Let P be the parabolic subgroup of G with Lie algebra $\mathfrak{p}$, and let $L \subset P$ be its Levi factor. Then there is a well-known equivalence between invariant differential operators acting on induced representations and homomorphisms of generalized Verma modules, realized by the natural pairing

$$\mathrm{Ind}_P^G(\mathbb{V}_\lambda(L)^*) \times M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{V}_\lambda) \longrightarrow \mathbb{C}\,, \tag{1}$$

where $\mathbb{V}_\lambda(L)$ denotes the finite-dimensional irreducible L-module, $\mathbb{V}_\lambda(L)^*$ is its dual, and Ind_P^G denotes induction from P to G. As a consequence, the singular vectors constructed in the article determine invariant differential operators acting between induced representations of $i(G')$. It is quite interesting to construct these invariant differential operators, in particular their curved extensions as lifts to homomorphisms of semiholonomic generalized Verma modules.

Our motivation for the particular example Lie $G_2 \overset{i}{\hookrightarrow}$ so(7) comes from a natural problem in conformal geometry of dimension 5 (note that so(7) is the complexification of the conformal Lie algebra in dimension 5), see [4] and references therein. A geometrical characterization of the reduction of the structure group with Lie algebra so(7) down to Lie G_2 for a given inducing representation $\mathbb{V}_\lambda$ is then given by invariant differential operators acting on sections of the associated vector bundles, intertwined by actions of so(7) and Lie G_2.

The structure of the article is as follows. In Section 2 we recall basic conventions on $\mathrm{so}(7), \mathrm{Lie}\ G_2, i(\mathrm{Lie}\ G_2)$ and the structure of their parabolic subalgebras relative to the embedding i. In Section 3, we use (1) to transform the problem of finding differential invariants for $(\mathrm{so}(7), \mathrm{Lie}\ G_2), \mathbb{V}_\lambda$ into an algebraic question about homomorphisms between generalized Verma modules, corresponding to solutions of the branching problem. In Section 4 we fix the conformal parabolic subalgebra to be $\mathfrak{p}_{(1,0,0)} \subset \mathrm{so}(7)$. Therefore by Lemma 2.1 the subalgebra $\mathfrak{p}'$ is given by $i(\mathfrak{p}') = i(\mathfrak{g}') \cap \mathfrak{p}$ and therefore equals the subalgebra $\mathfrak{p}'_{(1,0)}$ obtained by crossing out the first root of Lie G_2. We note that $\mathfrak{p}'_{(1,0)}$ is not compatible $(\mathfrak{g}, \mathfrak{p})$. We further fix the highest weight to be $\lambda\varepsilon_1$ (here we use λ as a scalar). We then apply the distribution Fourier transform (the "F-method") developed in [10], [11] to obtain our main result Theorem 4.2.

2. Branching problem and (non-compatible) parabolic subalgebras for the pair Lie $G_2 \overset{i}{\hookrightarrow}$ so (7)

In the present section we introduce the Lie theoretic conventions for the complex Lie algebra so(7), exceptional Lie algebra Lie G_2, and Levi resp. parabolic subalgebras $\mathfrak{p}$ of so(7) relative to parabolic subalgebras $i(\mathfrak{p}')$ of $i(\mathrm{Lie}\ G_2)$. These will be used in the subsequent Section 3, where we employ the F-method. For more detailed review, cf. [15].

We start by fixing a Chevalley-Weyl basis of the Lie algebra $\mathrm{so}(2n+1)$. Let the defining vector space V of $\mathrm{so}(2n+1)$ have a basis $e_1, \dots e_n, e_0, e_{-1}, \dots e_{-n}$, where the defining symmetric bilinear form B of $\mathrm{so}(2n+1)$ is given by $B(e_i, e_j) := 0, i \neq -j$, $B(e_i, e_{-i}) := 1$, $B(e_i, e_0) := 0$, $B(e_0, e_0) := 1$, or alternatively defined as an element of $S^2(V^*)$,

$$B := \sum_{i=-n}^{n} e_i^* \otimes e_{-i}^* = (e_0^*)^2 + 2\sum_{i=1}^{n} e_i^* e_{-i}^*, \tag{2}$$

under the identification $v^* w^* := \frac{1}{2!}\left(v^* \otimes w^* + w^* \otimes v^*\right)$.

In the basis $e_1, \dots e_n, e_0, e_{-1}, \dots e_{-n}$, the matrices of the elements of $\mathrm{so}(2n+1)$ are of the form

$$\left(\begin{array}{ccc|c|ccc} & & & v_1 & & & \\ & A & & \vdots & & C=-C^T & \\ & & & v_n & & & \\ \hline w_1 & \dots & w_n & 0 & -v_1 & \dots & -v_n \\ \hline & & & -w_1 & & & \\ & D=-D^T & & \vdots & & -A^T & \\ & & & -w_n & & & \end{array}\right),$$

i.e., all matrices $\mathbf{C}$ such that $\mathbf{A}^t\mathbf{B}+\mathbf{BA}=0$. We fix $e_1^*,\dots e_n^*, e_0^*, e_{-1}^*,\dots e_{-n}^*$ to be basis of V^* dual to $e_1,\dots e_n, e_0, e_{-1},\dots e_{-n}$. We identify elements of $\mathrm{End}(V)$ with elements of $V\otimes V^*$. In turn, we identify elements of $\mathrm{End}(V)$ with their matrices in the basis $e_1,\dots,e_n$, e_0, $e_{-1},\dots,e_{-n}$.

Fix the Cartan subalgebra $\mathfrak{h}$ of $\mathrm{so}(2n+1)$ to be the subalgebra of diagonal matrices, i.e., the subalgebra spanned by the vectors $e_i\otimes e_i^*-e_{-i}\otimes e_{-i}^*$. Then the basis vectors $e_1,\dots e_n, e_0, e_{-1},\dots e_{-n}$ are a basis for the $\mathfrak{h}$-weight vector decomposition of V. Let the $\mathfrak{h}$-weight of $e_i, i>0$, be ε_i. Then the $\mathfrak{h}$-weight of $e_{-i}, i>0$ is $-\varepsilon_i$, and an $\mathfrak{h}$-weight decomposition of $\mathrm{so}(2n+1)$ is given by the elements $g_{\varepsilon_i-\varepsilon_j} := e_i\otimes e_j^* - e_{-j}\otimes e_{-i}^*$, $g_{\pm(\varepsilon_i+\varepsilon_j)} := e_{\pm i}\otimes e_{\mp j}^* - e_{\pm j}\otimes e_{\mp i}^*$ and $g_{\pm\varepsilon_i} := \sqrt{2}\left(e_{\pm i}\otimes e_0^* - e_0\otimes e_{\mp i}^*\right)$, where $i,j>0$.

Define the symmetric bilinear form $\langle \bullet,\bullet\rangle_{\mathfrak{g}}$ on $\mathfrak{h}^*$ by $\langle \varepsilon_i,\varepsilon_j\rangle_{\mathfrak{g}}=1$ if $i=j$ and zero otherwise.

The root system of $\mathrm{so}(2n+1)$ with respect to $\mathfrak{h}$ is given by $\Delta(\mathfrak{g}) := \Delta^+(\mathfrak{g})\cup\Delta^-(\mathfrak{g})$, where we define

$$\Delta^+(\mathfrak{g}) := \{\varepsilon_i\pm\varepsilon_j | 1\leq i<j\leq n\}\cup\{\varepsilon_i|1\leq i\leq n\} \tag{3}$$

and $\Delta^-(\mathfrak{g}) := -\Delta^+(\mathfrak{g})$. We fix the Borel subalgebra $\mathfrak{b}$ of $\mathrm{so}(2n+1)$ to be the subalgebra spanned by $\mathfrak{h}$ and the elements $g_\alpha, \alpha\in\Delta^+(\mathfrak{g})$. The simple positive roots corresponding to $\mathfrak{b}$ are then given by

$$\eta_1 := \varepsilon_1-\varepsilon_2,\dots,\eta_{n-1} := \varepsilon_{n-1}-\varepsilon_n, \eta_n := \varepsilon_n\,.$$

For the remainder of this Section we fix the odd orthogonal Lie algebra to be $\mathrm{so}(7)$. We order the 18 roots of $\mathrm{so}(7)$ in graded lexicographic order with respect to their simple basis coordinates. We then label the negative roots by the indices $-9,\dots,-1$ and the positive roots by the indices $1,\dots,9$. Finally, we abbreviate the Chevalley-Weyl generator $g_\alpha\in\mathrm{so}(7)$ by g_i, where i is the label of the corresponding root. For example, $g_{\pm1}=g_{\pm(\varepsilon_1-\varepsilon_2)}$, $g_{\pm2}=g_{\pm(\varepsilon_2-\varepsilon_3)}$, $g_{\pm3}=g_{\pm(\varepsilon_3)}$ are the simple positive and negative generators, the element $g_{-9}=g_{-\varepsilon_1-\varepsilon_2}$ is the Chevalley-Weyl generator corresponding to the lowest root, and so on. We furthermore set $h_1 := [g_1,g_{-1}]$, $h_2 := [g_2,g_{-2}]$, $h_3 := 1/2[g_3,g_{-3}]$.

Let now $\mathfrak{g}'=\mathrm{Lie}\,G_2$. One way of defining the positive root system of $\mathrm{Lie}\,G_2$ is by setting it to be the set of vectors

$$\Delta(\mathfrak{g}') := \{\pm(1,0),\pm(0,1),\pm(1,1),\pm(1,2),\pm(1,3),\pm(2,3)\}\,. \tag{4}$$

We set $\alpha_1 := (1,0)$ and $\alpha_2 := (0,1)$. We fix a bilinear form $\langle \bullet, \bullet \rangle_{\mathfrak{g}'}$ on $\mathfrak{h}'$, proportional to the one induced by Killing form by setting

$$\begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle_{\mathfrak{g}'} & \langle \alpha_1, \alpha_2 \rangle_{\mathfrak{g}'} \\ \langle \alpha_2, \alpha_1 \rangle_{\mathfrak{g}'} & \langle \alpha_2, \alpha_2 \rangle_{\mathfrak{g}'} \end{pmatrix} := \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}. \tag{5}$$

In an $\langle \bullet, \bullet \rangle_{\mathfrak{g}'}$-orthogonal basis the root system of Lie G_2 is drawn in Figure 1.

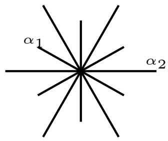

FIG. 1: The root system of Lie G_2

Similarly to the so(7) case, we order the 12 roots of Lie G_2 in the graded lexicographic order with respect to their simple basis coordinates, and label the roots with the indices $-6, \ldots, -1, 1, \ldots, 6$. We fix a basis for the Lie algebra Lie G_2 by giving a set of Chevalley-Weyl generators g'_i, $i \in \{\pm 1, \cdots \pm 6\}$, and by setting $h'_1 := [g'_1, g'_{-1}]$, $h'_2 := 3[g'_2, g'_{-2}]$. Just as in the so(7) case, we ask that the generator $g'_{\pm i}$ correspond to the root space labelled by $\pm i$.

All embeddings Lie $G_2 \overset{i}{\hookrightarrow}$ so(7) are conjugate over $\mathbb{C}$. One such embedding is given via

$$i(g'_{\pm 2}) := g_{\pm 2}, \quad i(g'_{\pm 1}) := g_{\pm 1} + g_{\pm 3}\,.$$

As $g'_{\pm 1}, g'_{\pm 2}$ generate Lie G_2, the preceding data determines the map i and one can directly check it is a Lie algebra homomorphism. Alternatively, we can use $i(g'_{\pm 1}), i(g'_{\pm 2})$ to generate a Lie subalgebra of so(7), verify that this subalgebra is indeed 14-dimensional and simple, and finally use this 14-dimensional image to compute the structure constants of Lie G_2.

We denote by $\omega_1 := \varepsilon_1$, $\omega_2 := \varepsilon_1 + \varepsilon_2$ and $\omega_3 := \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ the fundamental weights of so(7) and by $\psi_1 := 2\alpha_1 + \alpha_2$, $\psi_2 := 3\alpha_1 + 2\alpha_2$ the fundamental weights of Lie G_2.

Let $\mathrm{pr}\colon \mathfrak{h}^* \to \mathfrak{h}'^*$ be the map naturally induced by i. Then

$$\mathrm{pr}(\underbrace{\varepsilon_1 - \varepsilon_2}_{\eta_1}) = \mathrm{pr}(\underbrace{\ \varepsilon_3\ }_{\eta_3}) = \alpha_1, \quad \mathrm{pr}(\underbrace{\varepsilon_2 - \varepsilon_3}_{\eta_2}) = \alpha_2, \tag{6}$$

or equivalently

$$\mathrm{pr}(\omega_1) = \mathrm{pr}(\omega_3) = \psi_1\,, \quad \mathrm{pr}(\omega_2) = \psi_2\,.$$

Conversely, $\iota : \mathfrak{h}'^* \to \mathfrak{h}^*$ is the map

$$\iota(\alpha_2) = 3\eta_2 = 3\varepsilon_2 - 3\varepsilon_3\,, \quad \iota(\alpha_1) = \eta_1 + 2\eta_3 = \varepsilon_1 - \varepsilon_2 + 2\varepsilon_3\,. \tag{7}$$

According to the usual convention, to an arbitrary subset of the simple positive roots of so(7) ("crossed-out" roots) we assign a parabolic subalgebra by requesting that the crossed out root spaces lie outside of the Levi part of $\mathfrak{p}$. In turn, we parametrize the subsets of the simple positive roots of so(7) by triples of 0's and 1's with 1 standing for "crossed-out" root. Finally, we index the parabolic subalgebra by the corresponding triples of 0's and 1's. For example, by $\mathfrak{p}_{(1,1,0)}$ we denote the parabolic subalgebra of so(7) whose Levi part has roots $\pm\varepsilon_3$. Define the four parabolic subalgebra $\mathfrak{b}' \simeq \mathfrak{p}'_{(1,1)}, \mathfrak{p}'_{(1,0)}, \mathfrak{p}'_{(0,1)}, \mathfrak{p}'_{(0,0)} \simeq \text{Lie } G_2$ of Lie G_2 in analogous fashion.

We recall from [15] that the pairwise inclusions between the parabolic subalgebras of so(7) and the embeddings of the parabolic subalgebras of Lie G_2 are given as follows.

Lemma 2.1. *For the pair $G_2 \overset{i}{\hookrightarrow} \mathrm{so}(7)$, let $\mathfrak{h}, \mathfrak{b}, \mathfrak{p}$ denote Cartan, Borel and parabolic subalgebras of* so(7) *and $\mathfrak{h}', \mathfrak{b}', \mathfrak{p}'$ denote Cartan, Borel and parabolic subalgebras of* Lie G_2 *with the assumptions that $i(\mathfrak{h}') \subset \mathfrak{h} \subset \mathfrak{b}$, $i(\mathfrak{b}') \subset \mathfrak{b} \subset \mathfrak{p}$, $\mathfrak{b}' \subset \mathfrak{p}'$. Then we have the following inclusion diagram for the possible values of $\mathfrak{p}, \mathfrak{p}'$.*

$$\begin{tikzcd}
 & \mathfrak{p}_{(0,0,0)} \simeq \mathrm{so}(7) & & & \\
\mathfrak{p}_{(1,0,0)} \arrow[ur] & \mathfrak{p}_{(0,1,0)} \arrow[u] & \mathfrak{p}_{(0,0,1)} \arrow[ul] & \mathfrak{p}'_{(0,0)} \simeq \text{Lie } G_2 \arrow[ull] & \\
\mathfrak{p}_{(1,1,0)} \arrow[u] \arrow[ur] & \mathfrak{p}_{(1,0,1)} \arrow[ul] \arrow[ur] & \mathfrak{p}_{(0,1,1)} \arrow[ul] \arrow[u] & & \mathfrak{p}'_{(0,1)} \arrow[ulll] \arrow[ul] \\
 & \mathfrak{p}_{(1,1,1)} \simeq \mathfrak{b} \arrow[ul] \arrow[u] \arrow[ur] & & \mathfrak{p}'_{(1,0)} \arrow[ull] \arrow[uu] & \\
 & & & \mathfrak{p}'_{(1,1)} \simeq \mathfrak{b}' \arrow[ull] \arrow[u] \arrow[uur] &
\end{tikzcd}$$

The arrows in the diagram indicate the inclusions between the corresponding parabolic subalgebras. In addition, if an arrow is drawn between the parabolic subalgebra $\mathfrak{p}'$ of Lie G_2 *and a parabolic subalgebra $\mathfrak{p}$ of* so(7)*, then $\mathfrak{p}' = i^{-1}(i(\mathfrak{g}') \cap \mathfrak{p})$.*

The structure of so(7) as a module over the Levi part of parabolic subalgebras of Lie G_2 is described in detail in [15, Lemma 5.2] (the lemma is too large to recall here) and we will implicitly use it throughout Section 4.

Note that the conformal parabolic subalgebra $\mathfrak{p}_{(1,0,0)} \subset \mathrm{so}(7)$ and the parabolic subalgebra $\mathfrak{p}'_{(1,0)} \subset \text{Lie } G_2$ are not compatible.

3. Branching problem and the F-method (algebraic distribution Fourier transformation)

In the present section we briefly review the F-method developed in [10], [11]. It is based on the analytical tool of algebraic Fourier transformation on the commutative nilradical $\mathfrak{n}$ of $\mathfrak{p}$, which allows to find singular vectors in generalized Verma modules exploiting the algebraic Fourier transform and classical invariant theory. The method converts a problem in the universal enveloping algebra of a Lie algebra into a system of partial or ordinary special differential equations acting on a polynomial ring. In examples known to us, the conversion to partial differential equations yields a lot more tractable problem than the starting universal enveloping algebra one.

Let $\tilde{G}$ be a connected real reductive Lie group with the Lie algebra $\tilde{\mathfrak{g}}$, $\tilde{P} \subset \tilde{G}$ a parabolic subgroup and $\tilde{\mathfrak{p}}$ its Lie algebra, $\tilde{\mathfrak{p}} = \tilde{\mathfrak{l}} \oplus \tilde{\mathfrak{n}}$ the Levi decomposition of $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{n}}_-$ its opposite nilradical, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{p}}$. The corresponding Lie groups are denoted $\tilde{N}_-, \tilde{L}, \tilde{N}$. Let p denote the fibration $p\colon \tilde{G} \to \tilde{G}/\tilde{P}$ and let $\tilde{M} := p(\tilde{N}_- \cdot \tilde{P})$ denote the big Schubert cell of $\tilde{G}/\tilde{P}$. Then the exponential map

$$\tilde{\mathfrak{n}}_- \to \tilde{M}\,, \quad X \mapsto \exp(X)\cdot o \in \tilde{G}/\tilde{P},\, o := e\cdot \tilde{P} \in \tilde{G}/\tilde{P}\,, \quad e \in \tilde{G}$$

gives the canonical identification of the vector space $\mathfrak{n}_-$ with $\tilde{M}$.

Given a complex finite dimensional $\tilde{P}$-module $\mathbb{V}$ (in the present section we do not indicate explicitly its highest weight), let $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V})$ denote the space of smooth sections of the homogeneous vector bundle $\tilde{G} \times_{\tilde{P}} \mathbb{V} \to \tilde{G}/\tilde{P}$, i.e.,

$$\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V}) = C^\infty(\tilde{G},\mathbb{V})^{\tilde{P}} := \{f \in C^\infty(\tilde{G},\mathbb{V}) \mid f(g\cdot p) = p^{-1}\cdot f(g),\, g \in \tilde{G}, p \in \tilde{P}\}\,.$$

Let $\tilde{\pi}$ denote the induced representation of $\tilde{G}$ on $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V})$.

Let $\mathcal{U}(\tilde{\mathfrak{g}}_{\mathbb{C}})$ denote the universal enveloping algebra of the complexified Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$. Let $\mathbb{V}^\vee$ be the dual (contragredient) representation to $\mathbb{V}$. The generalized Verma module $M_{\tilde{\mathfrak{p}}}^{\tilde{\mathfrak{g}}}(\mathbb{V}^\vee)$ is defined by

$$M_{\tilde{\mathfrak{p}}}^{\tilde{\mathfrak{g}}}(\mathbb{V}^\vee) := \mathcal{U}(\tilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\tilde{\mathfrak{p}})} \mathbb{V}^\vee\,,$$

and there is a $(\tilde{\mathfrak{g}}, \tilde{P})$-invariant natural pairing between $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V})$ and $M_{\tilde{\mathfrak{p}}}^{\tilde{\mathfrak{g}}}(\mathbb{V}^\vee)$, described as follows. Let $\mathcal{D}'(\tilde{G}/\tilde{P}) \otimes \mathbb{V}^\vee$ be the space of all distributions on $\tilde{G}/\tilde{P}$ with values in $\mathbb{V}^\vee$. The evaluation defines a canonical equivariant pairing between $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V})$ and $\mathcal{D}'(\tilde{G}/\tilde{P}) \otimes \mathbb{V}^\vee$, and this restricts to the pairing

$$\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V}) \times \mathcal{D}'_{[o]}(\tilde{G}/\tilde{P}) \otimes \mathbb{V}^\vee \to \mathbb{C}\,, \tag{8}$$

where $\mathcal{D}'(\tilde{G}/\tilde{P})_{[o]} \otimes \mathbb{V}^\vee$ denotes the space of distributions supported at the base point $o \in \tilde{G}/\tilde{P}$. As shown in [1], the space $\mathcal{D}'_{[o]}(\tilde{G}/\tilde{P}) \otimes \mathbb{V}^\vee$ can be identified, as an $\mathcal{U}(\tilde{\mathfrak{g}})$-module, with the generalized Verma module $M_{\tilde{\mathfrak{p}}}^{\tilde{\mathfrak{g}}}(\mathbb{V}^\vee)$.

Moreover, given two inducing representations $\mathbb{V}$ and $\mathbb{V}'$ of $\tilde{P}$, the space of $\tilde{G}$-equivariant differential operators from $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V})$ to $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V}')$ is isomorphic to the space of $(\tilde{\mathfrak{g}}, \tilde{P})$-homomorphisms between $M_{\tilde{\mathfrak{p}}}^{\tilde{\mathfrak{g}}}(\mathbb{V}'^\vee)$ and $M_{\tilde{\mathfrak{p}}}^{\tilde{\mathfrak{g}}}(\mathbb{V}^\vee)$. The homomorphisms of generalized Verma modules are determined by their singular vectors, and the F-method translates the problem of finding singular vectors to the study of distributions on $\tilde{G}/\tilde{P}$ supported at the origin, and consequently to the problem of finding the solution space for a system of partial differential equations acting on polynomials $\mathrm{Pol}(\tilde{\mathfrak{n}})$ on $\tilde{\mathfrak{n}}$.

The representation $\tilde{\pi}$ of $\tilde{G}$ on $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V})$ has the infinitesimal representation $d\tilde{\pi}$ of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. In the non-compact case, $\tilde{\pi}$ acts on functions on the big Schubert cell

$\tilde{\mathfrak{n}}_- \simeq \tilde{M} \subset \tilde{G}/\tilde{P}$ with values in $\mathbb{V}$. The latter representation space can be identified via the exponential map with $C^\infty(\tilde{\mathfrak{n}}_-, \mathbb{V})$. The action $d\tilde{\pi}(Z)$ of elements $Z \in \tilde{\mathfrak{n}}$ on $\mathcal{C}^\infty(\tilde{\mathfrak{n}}_-, V)$ is realized by vector fields on $\tilde{\mathfrak{n}}_-$ with coefficients in $\mathrm{Pol}(\tilde{\mathfrak{n}}_-) \otimes \mathrm{End}\,\mathbb{V}$, see [12].

By the Poincaré-Birkhoff-Witt theorem, the generalized Verma module $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$ is isomorphic to $\mathcal{U}(\tilde{\mathfrak{n}}_-) \otimes \mathbb{V}^\vee \simeq \mathrm{Diff}_{\tilde{N}_-}(\tilde{\mathfrak{n}}_-) \otimes \mathbb{V}^\vee$ as an $\tilde{\mathfrak{l}}$-module. In the special case when $\tilde{\mathfrak{n}}_-$ is commutative, $\mathrm{Diff}_{\tilde{N}_-}(\tilde{\mathfrak{n}}_-)$ is the space of holomorphic differential operators on $\tilde{\mathfrak{n}}_-$ with constant coefficients regarded as a subspace of the Weyl algebra $\mathrm{Diff}(\tilde{\mathfrak{n}}_-)$ of algebraic differential operators on $\tilde{\mathfrak{n}}_-$. Moreover, the operators $d\tilde{\pi}^\vee(X)$, $X \in \tilde{\mathfrak{g}}$, are realized as differential operators on $\tilde{\mathfrak{n}}_-$ with coefficients in $\mathrm{End}(\mathbb{V}^\vee)$. The application of Fourier transform on $\tilde{\mathfrak{n}}_-$ gives the identification of the generalized Verma module $\mathrm{Diff}_{\tilde{N}_-}(\tilde{\mathfrak{n}}_-) \otimes \mathbb{V}^\vee$ with the space $\mathrm{Pol}(\tilde{\mathfrak{n}}) \otimes \mathbb{V}^\vee$, and the action $d\tilde{\pi}^\vee$ of $\tilde{\mathfrak{g}}$ on $\mathrm{Diff}_{\tilde{N}_-}(\tilde{\mathfrak{n}}_-) \otimes \mathbb{V}^\vee$ translates to the action $(d\tilde{\pi}^\vee)^F$ of $\tilde{\mathfrak{g}}$ on $\mathrm{Pol}(\tilde{\mathfrak{n}}) \otimes \mathbb{V}^\vee$ and is realized again by differential operators with values in $\mathrm{End}(\mathbb{V}^\vee)$. The explicit form of $(d\tilde{\pi}^\vee)^F(X)$ is easy to compute by Fourier transform from the explicit form of $d\tilde{\pi}^\vee$.

The previous framework can be applied to any pair of couples $\tilde{P} \subset \tilde{G}$ and $\tilde{P}' \subset \tilde{G}'$ of Lie groups for which $\tilde{G}' \subset \tilde{G}$ is a reductive subgroup of $\tilde{G}$ and $\tilde{P}' = \tilde{P} \cap \tilde{G}'$ is a parabolic subgroup of $\tilde{G}'$. The Lie algebras of $\tilde{G}', \tilde{P}'$ are denoted by $\tilde{\mathfrak{g}}', \tilde{\mathfrak{p}}'$. In this case, $\tilde{\mathfrak{n}}' := \tilde{\mathfrak{n}} \cap \tilde{\mathfrak{g}}'$ is the nilradical of $\tilde{\mathfrak{p}}'$, and $\tilde{L}' = \tilde{L} \cap \tilde{G}'$ is the Levi subgroup of $\tilde{P}'$. We are interested in the branching problem for generalized Verma modules $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$ over, $\tilde{\mathfrak{g}}$, i.e., in the structure of the restriction of $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$ to $\tilde{\mathfrak{g}}'$.

Definition 3.1. Let $\mathbb{V}$ be an irreducible $\tilde{P}$-module. Define the $\tilde{L}'$-module

$$M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'} := \{v \in M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee) |\, d\pi^\vee(Z)v = 0 \text{ for all } Z \in \tilde{\mathfrak{n}}'\}. \tag{9}$$

The set $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ is a completely reducible $\tilde{\mathfrak{l}}'$-module. Note that for $\tilde{G} = \tilde{G}'$, $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ is necessarily finite-dimensional. However for $\tilde{G} \neq \tilde{G}'$, the set $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(V^\vee)^{\mathfrak{n}'}$ will in general (but not necessarily, as illustrated in the next section) be infinite dimensional. An irreducible $\tilde{L}'$-submodule $\mathbb{W}^\vee$ of $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ gives an injective $\mathcal{U}(\tilde{\mathfrak{g}}')$-homomorphism from $M^{\tilde{\mathfrak{g}}'}_{\tilde{\mathfrak{p}}'}(\mathbb{W}^\vee)$ to $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$. Dually, we get an equivariant differential operator acting from $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V})$ to $\mathrm{Ind}^{\tilde{G}'}_{\tilde{P}'}(\mathbb{W})$.

Using the F-method, the space of $\tilde{L}'$-singular vectors $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ is realized in the ring of polynomials on $\tilde{\mathfrak{n}}$ valued in $\mathbb{V}^\vee$ and equipped with the action of the Lie algebra via $(d\tilde{\pi}^\vee)^F$.

Definition 3.2. We define

$$\mathrm{Sol}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}', \mathbb{V}^\vee) := \{f \in \mathrm{Pol}(\tilde{\mathfrak{n}}) \otimes \mathbb{V}^\vee |\, (d\tilde{\pi}^\vee)^F(Z)f = 0 \text{ for all } Z \in \tilde{\mathfrak{n}}'\}\,. \tag{10}$$

Then the inverse Fourier transform gives an $\tilde{L}'$-isomorphism

$$\mathrm{Sol}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}'; \mathbb{V}^\vee) \xrightarrow{\sim} M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}. \tag{11}$$

An explicit form of the action $(d\tilde{\pi}^\vee)^F(Z)$ leads to a system of differential equation for elements in Sol. The transition from $M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ to Sol transforms the problem of computation of singular vectors in generalized Verma modules into a system of partial differential equations.

In the dual language of differential operators acting on principal series representation, the set of $\tilde{G}'$-intertwining differential operators from $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{V})$ to $\mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\mathbb{V}')$ is in bijective correspondence with the space of all $(\tilde{\mathfrak{g}}', \tilde{P}')$-homomorphisms from $M_{\tilde{\mathfrak{p}}'}^{\tilde{\mathfrak{g}}'}(\mathbb{V}'^{\vee})$ to $M_{\tilde{\mathfrak{p}}}^{\tilde{\mathfrak{g}}}(\mathbb{V}^{\vee})$.

4. Lie $G_2 \cap \mathfrak{p}'$-singular vectors in the so$\,(7)$-generalized Verma modules of scalar type for the conformal parabolic subalgebra

In this subsection we determine the $i(\text{Lie } G_2) \cap \mathfrak{p}$-singular vectors in the family of $\tilde{\mathfrak{g}} = \mathrm{so}(7)$ generalized Verma modules $M_{\mathfrak{p}_{(1,0,0)}}^{\mathrm{so}(7)}(\mathbb{C}_\lambda)$ induced from character $\chi_\lambda : \tilde{\mathfrak{p}} \to \mathbb{C}$ of the weight $\lambda\varepsilon_1$ (ε_1 is the first fundamental weight of so(7)). In this way, the results computed in the present section are analytic counterpart realized by F-method of the algebraic results developed in [15].

Denote by v_λ the highest weight vector of the generalized Verma so(7)-module $M_{\mathfrak{p}_{(1,0,0)}}^{\mathrm{so}(7)}(\mathbb{V}_\lambda)$. Note that as $i(h_2') = 3h_2 = 3h_{\varepsilon_2-\varepsilon_3}$, $i(h_1') = h_1 + 2h_2 = h_{\varepsilon_1-\varepsilon_2} + 2h_{\varepsilon_3}$, $\langle \mu, \alpha_1 \rangle = 0$ and $\langle \mu, \alpha_2 \rangle = \lambda$, we have that the $\mathfrak{h}'$-weight of v_λ is $\mu = \lambda(\alpha_1 + 2\alpha_2)$.

Let $\mathfrak{n}_-$ denote the nilradical opposite to the nilradical of the parabolic subalgebra $\mathfrak{p}$. Then $\mathfrak{n}_-$ is commutative,

$$\mathcal{U}(\mathfrak{n}_-) \otimes \mathbb{V}^\vee \simeq \mathrm{Pol}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_5}\right) \otimes \mathbb{C}_\lambda$$

and the variables $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_5}$ denote the following so(7)-root space generators:

$$\begin{aligned} &\frac{\partial}{\partial x_1} := g_{-\varepsilon_1+\varepsilon_2} = g_{-1}\,, \quad \frac{\partial}{\partial x_2} := g_{-\varepsilon_1-\varepsilon_3} = g_{-8}\,, \quad \frac{\partial}{\partial x_3} := g_{-\varepsilon_1} = g_{-6}\,, \\ &\frac{\partial}{\partial x_5} := g_{-\varepsilon_1+\varepsilon_3} = g_{-4}, \quad \frac{\partial}{\partial x_4} := g_{-\varepsilon_1-\varepsilon_2} = g_{-9}\,. \end{aligned}$$

Here, we recall that $[x_i, \frac{\partial}{\partial x_j}] = -[\frac{\partial}{\partial x_j}, x_i] = \begin{cases} 0 & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$ is the adjoint action of the differential operator x_i on the differential operator $\frac{\partial}{\partial x_j}$.

By Lemma 2.1, the simple part of the Levi factor of $i(\mathfrak{p}')$ is isomorphic to sl(2) and its action on $\mathfrak{n}_-$ can be extended to action on $\mathcal{U}(\mathfrak{n}_-) \simeq S^\star(\mathfrak{n}_-)$. The elements $h := h_2, e := g_2, f := g_{-2}$ give the standard h, e, f-basis of sl(2), i.e., $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. Then the action of h on $\mathfrak{n}_-$ is the adjoint action of $x_5\frac{\partial}{\partial x_5} + x_4\frac{\partial}{\partial x_4} - x_2\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_1}$, the action of e is the adjoint action of $x_4\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial x_1}$ and the action of f is the adjoint action of $-x_1\frac{\partial}{\partial x_5} + x_2\frac{\partial}{\partial x_4}$.

We now proceed to generate all $\mathfrak{l}'$-invariant singular vectors in $M_{\mathfrak{p}_{(1,0,0)}}^{\mathrm{so}(7)}(\mathbb{C}_\lambda)$, i.e., the singular vectors that induce $i(\text{Lie } G_2)$-generalized Verma modules induced from character (scalar generalized Verma modules). To do that we need the following lemma from classical invariant theory of reductive Lie algebras.

Lemma 4.1. *Then the* sl(2)*-invariants of* $S^\star(\mathfrak{n}_-)$ *are an associative algebra generated by the elements*

$$
\begin{array}{rcl}
u_1 & := & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_5} = g_{-1}g_{-9} + g_{-8}g_{-4} \\
u_2 & := & \frac{\partial}{\partial x_3} = g_{-6}\,.
\end{array} \tag{12}
$$

Proof. Direct computation shows that u_1, u_2 are invariants. Alternatively, as the direct sum of two two-dimensional sl(2)-modules gives a natural embedding $\mathrm{sl}(2) \hookrightarrow \mathrm{sl}(2) \times \mathrm{sl}(2)$, we can view u_1 as the invariant element induced by the defining symmetric bilinear form of $\mathrm{so}(4) \simeq \mathrm{sl}(2) \times \mathrm{sl}(2)$. Let the positive root of sl(2) be[1] η , and the multiplicity of the sl(2)-module with highest weight $t\frac{\eta}{2}$ in $S^l(\mathfrak{n}_-)$ be $b(l,t)$. Denoting by x, z a couple of formal variables, we have that $\sum_{l\in\mathbb{Z}_{\geq 0}, t\in\mathbb{Z}_{\geq 0}} b(l,t)(z^l x^t + z^l x^{-1-t})$ is the power series expansion of the rational function

$$
(1-x^{-2})\frac{1}{(1-zx)^2}\frac{1}{(1-zx^{-1})^2}\frac{1}{(1-z)}\,.
$$

Direct computation shows that $b(l,t)$ equals $-1/2t^2 + 1 + 1/2tl + 1/2l + 1/2t$ whenever $l+t$ is even and $-1/2t^2 + 1/2 + 1/2tl + 1/2l$ whenever $l+t$ is odd, and l, t satisfy the inequalities $l \geq t \geq 0$. Finally, substituting with $t = 0$, we get $b(l,0) = 1 + l/2$ for even l and $b(l,0) = 1/2 + l/2$. For a fixed l, this is exactly the dimension of the vector space generated by the linearly independent invariants $u_1^q u_2^r \in S^l(\mathfrak{n}_-)$ with $r + 2q = l$, which completes the proof of our Lemma. □

From the definition of embedding map i it follows that

$$
\begin{aligned}
\mathrm{ad}\big(i(g_2')\big) &= -x_2\frac{\partial}{\partial x_4} + x_1\frac{\partial}{\partial x_5}\,,\\
\mathrm{ad}(i(g_{-2}')) &= -x_4\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_1}\,,\\
\frac{1}{3}\mathrm{ad}(i(h_2')) &= \mathrm{ad}(h_2) = [\mathrm{ad}(i(g_2')), \mathrm{ad}(i(g_{-2}'))]\\
&= x_5\frac{\partial}{\partial x_5} + x_4\frac{\partial}{\partial x_4} - x_2\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_1}\,,\\
\mathrm{ad}(i(h_1')) &= -x_5\frac{\partial}{\partial x_5} + x_3\frac{\partial}{\partial x_3} + 3x_2\frac{\partial}{\partial x_2} + 2x_1\frac{\partial}{\partial x_1}\,,
\end{aligned}
$$

and therefore

$$
\mathrm{ad}\big(i(2h_1' + h_2')\big) = x_5\frac{\partial}{\partial x_5} + 3x_4\frac{\partial}{\partial x_4} + 2x_3\frac{\partial}{\partial x_3} + 3x_2\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_1} \tag{13}
$$

[1] η is of course the projection of long Lie G_2-root $\alpha_2 = \mathrm{pr}(\varepsilon_2 - \varepsilon_3)$ from the dual of the two-dimensional Cartan subalgebra of Lie G_2 to the dual of the one-dimensional Cartan subalgebra of a long-root sl(2)-subalgebra of Lie G_2

represents the central element of the Levi factor $i(\mathfrak{l}')$. Its action therefore naturally induces a grading gr on the Weyl algebra of $\mathfrak{n}_-$ in the variables

$$\left\{x_1, x_2, x_3, x_4, x_5, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\right\},$$

via

$$\begin{aligned}(14)\qquad & -\mathrm{gr}\,(x_1) = \mathrm{gr}\left(\frac{\partial}{\partial x_1}\right) = -1\,, \qquad -\mathrm{gr}\,(x_2) = \mathrm{gr}\left(\frac{\partial}{\partial x_2}\right) = -3,\\ & -\mathrm{gr}(x_3) = \mathrm{gr}\left(\frac{\partial}{\partial x_3}\right) = -2\,, \qquad -\mathrm{gr}(x_4) = \mathrm{gr}\left(\frac{\partial}{\partial x_4}\right) = -3,\\ & -\mathrm{gr}\,(x_5) = \mathrm{gr}\left(\frac{\partial}{\partial x_5}\right) = -1\,.\end{aligned}$$

In particular, we get that the invariants $u_1 = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_5}$ and $u_2 = \left(\frac{\partial}{\partial x_3}\right)^2$ are homogeneous with respect to the gr-grading.

Let $\xi_1, \ldots \xi_5$ be formal variables, Fourier-dual with respect to $x_1, \ldots, x_n$. Let

$$\partial_1 := \frac{\partial}{\partial \xi_1}, \ldots, \partial_5 := \frac{\partial}{\partial \xi_5}\,,$$

denote the derivatives in the ξ_i-variables. We recall that the distributive Fourier transform F maps the Weyl algebra generated by $x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_5}$ to the Weyl algebra generated by $\partial_1, \ldots, \partial_5, \xi_1, \ldots, \xi_5$ via

$$F(x_i) := \partial_i \qquad F\left(\frac{\partial}{\partial x_i}\right) := \xi_i\,.$$

As the Fourier transform is a Lie algebra homomorphism, by Lemma 4.1 the subalgebra of $\mathfrak{l}'_s = sl(2)$-invariants with respect to the Fourier dual representation is the polynomial ring $\mathrm{Pol}[\xi_1\xi_4 + \xi_2\xi_5, \xi_3]$.

Theorem 4.2. *Let v_λ be the highest weight vector of the* so(7)*-generalized Verma module $M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$ induced from character χ_λ, $\lambda \in \mathbb{C}$. Let $N \in \mathbb{N}$ be a positive integer and $A_i \in \mathbb{C}$, $i \in \mathbb{N}$ a collection of complex numbers such that at least one of them is non-zero. Let*

$$(15)\qquad u \cdot v_\lambda := \sum_{k=0}^{N} A_k u_1^k u_2^{N-k} \cdot v_\lambda \quad ,$$

where u_1, u_2 are given by (12)*.*

1. *A vector $u \cdot v_\lambda$ is $i(\mathrm{Lie}\ G_2) \cap \mathfrak{p}$-singular ("singular vector of scalar type") of homogeneity $2N$ if and only if $\lambda = N - 5/2$ and $u = (2u_1 + u_2)^N = (2u_1 + u_2)^{\lambda+5/2}$.*
2. *$M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$ has no $i(\mathrm{Lie}\ G_2) \cap \mathfrak{p}$-singular vector of homogeneity $2N+1$.*
3. *A vector $v \in M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$, not proportional to v_λ, is $\mathrm{so}(7) \cap \mathfrak{p}$-singular if and only if $\lambda = N - 5/2$ and $v = u \cdot v_\lambda$ is the vector given in 1.*

Proof. 1. By Lemma 4.1 and Section 2 a $\mathfrak{p}'$-singular vector must be polynomial in u_1 and u_2 and therefore a homogeneous $\mathfrak{p}'$-singular vector of homogeneity $2N$ must be of the form (15).

First we determine the action of the second simple positive root g_2 in the Fourier dual representation $d\tilde{\pi}(\mathrm{ad}(i(g_1')))$, acting on $\mathrm{Pol}[\xi_1,\dots,\xi_5]$.

Let n_i be non-negative integers. Then

$$
\begin{aligned}
i(g_1')\cdot(\xi_1^{n_1}\xi_2^{n_2}\xi_3^{n_3}\xi_4^{n_4}\xi_5^{n_5}\cdot v_\lambda) = \big(&(-n_1^2+n_1)\xi_1^{n_1-1}\xi_2^{n_2}\xi_3^{n_3}\xi_4^{n_4}\xi_5^{n_5}\\
&- n_2\xi_1^{n_1}\xi_2^{n_2-1}\xi_3^{n_3+1}\xi_4^{n_4}\xi_5^{n_5}\\
&+ n_1\lambda\xi_1^{n_1-1}\xi_2^{n_2}\xi_3^{n_3}\xi_4^{n_4}\xi_5^{n_5} + (n_3^2-n_3)\xi_1^{n_1}\xi_2^{n_2}\xi_3^{n_3-2}\xi_4^{n_4+1}\xi_5^{n_5}\\
&+ 2n_3\xi_1^{n_1}\xi_2^{n_2}\xi_3^{n_3-1}\xi_4^{n_4}\xi_5^{n_5+1} - n_1n_5\xi_1^{n_1-1}\xi_2^{n_2}\xi_3^{n_3}\xi_4^{n_4}\xi_5^{n_5}\\
&+ n_2n_5\xi_1^{n_1}\xi_2^{n_2-1}\xi_3^{n_3}\xi_4^{n_4+1}\xi_5^{n_5-1} - n_1n_2\xi_1^{n_1-1}\xi_2^{n_2}\xi_3^{n_3}\xi_4^{n_4}\xi_5^{n_5}\\
&- n_1n_3\xi_1^{n_1-1}\xi_2^{n_2}\xi_3^{n_3}\xi_4^{n_4}\xi_5^{n_5}\big)\cdot v_\lambda\\
= (-\xi_1\partial_1^2 - \xi_3\partial_2 &+ \lambda\partial_1 + \xi_4\partial_3^2 + 2\xi_5\partial_3 - \xi_5\partial_1\partial_5 + \xi_4\partial_2\partial_5\\
&- \xi_2\partial_1\partial_2 - \xi_3\partial_1\partial_3)\cdot(\xi_1^{n_1}\xi_2^{n_2}\xi_3^{n_3}\xi_4^{n_4}\xi_5^{n_5})\cdot v_\lambda\,.
\end{aligned}\tag{16}
$$

Let $P(\lambda)$ denote the differential operator on $\mathbb{C}[\xi_1,\xi_2,\xi_3,\xi_4,\xi_5]$ obtained in the following computation:

$$
\begin{aligned}
(-\xi_1\partial_1^2 - &\xi_3\partial_2 + \lambda\partial_1 + \xi_4\partial_3^2 + 2\xi_5\partial_3 - \xi_5\partial_1\partial_5 + \xi_4\partial_2\partial_5 - \xi_2\partial_1\partial_2 - \xi_3\partial_1\partial_3)\\
&= (-\xi_3\partial_2 + \xi_4\partial_3^2 + 2\xi_5\partial_3 + (-\xi_5\partial_1 + \xi_4\partial_2)\partial_5\\
&\qquad - (\xi_1\partial_1 + \xi_2\partial_2 + \xi_3\partial_3 - \lambda)\partial_1)\\
&= (-\xi_3\partial_2 + \xi_4\partial_3^2 + 2\xi_5\partial_3 + \partial_5(-\xi_5\partial_1 + \xi_4\partial_2)\\
&\qquad - (\xi_1\partial_1 + \xi_2\partial_2 + \xi_3\partial_3 - \lambda - 1)\partial_1)\,.
\end{aligned}\tag{17}
$$

We compute

$$
\begin{aligned}
\partial_1\cdot(u_1^{b_1}u_2^{b_2}) &= b_1\xi_4u_1^{b_1-1}u_2^{b_2}\,,\\
\partial_2\cdot(u_1^{b_1}u_2^{b_2}) &= b_1\xi_5u_1^{b_1-1}u_2^{b_2}\,,\\
(\xi_1\partial_1+\xi_2\partial_2)\cdot(u_1^{b_1}u_2^{b_2}) &= b_1u_1^{b_1}u_2^{b_2}\,,\\
\partial_3\cdot(u_1^{b_1}u_2^{b_2}) &= 2b_2\xi_3u_1^{b_1}u_2^{b_2-1}\,,\\
\partial_3^2\cdot(u_1^{b_1}u_2^{b_2}) &= 2b_2(2b_2-1)u_1^{b_1}u_2^{b_2-1}\,,
\end{aligned}
$$

and so

$$
\begin{aligned}
(-\xi_3\partial_2+ &\xi_4\partial_3^2+2\xi_5\partial_3+\partial_5(-\xi_5\partial_1+\xi_4\partial_2)-(\xi_1\partial_1+\xi_2\partial_2+\xi_3\partial_3-\lambda-1)\partial_1)\cdot(u_1^{b_1}u_2^{b_2})\\
&= (-\xi_3\partial_2+\xi_4\partial_3^2+2\xi_5\partial_3-(\xi_1\partial_1+\xi_2\partial_2+\xi_3\partial_3-\lambda-1)\partial_1)\cdot(u_1^{b_1}u_2^{b_2})\\
&= -b_1\xi_3\xi_5u_1^{b_1-1}u_2^{b_2}+2b_2(2b_2-1)\xi_4u_1^{b_1}u_2^{b_2-1}+4b_2\xi_5\xi_3u_1^{b_1}u_2^{b_2-1}\\
&\quad -(\xi_1\partial_1+\xi_2\partial_2+\xi_3\partial_3-\lambda-1)\cdot(b_1\xi_4u_1^{b_1-1}u_2^{b_2})\\
&\quad - b_1\xi_3\xi_5u_1^{b_1-1}u_2^{b_2}+2b_2(2b_2-1)\xi_4u_1^{b_1}u_2^{b_2-1}+4b_2\xi_5\xi_3u_1^{b_1}u_2^{b_2-1}\\
&\quad +(-b_1+1+\lambda+1-2b_2)b_1\xi_4u_1^{b_1-1}u_2^{b_2} = 2b_2((2b_2-1)\xi_4+2\xi_5\xi_3)u_1^{b_1}u_2^{b_2-1}\\
&\quad + b_1\big((-b_1-2b_2+\lambda+2)\xi_4-\xi_3\xi_5)u_1^{b_1-1}u_2^{b_2}\,. \qquad (18)
\end{aligned}
$$

The operator $P(\lambda)$ is homogeneous with respect to the grading in (14), and its application to a homogeneous polynomial in $u_1=u_1(\xi_1,\dots,\xi_5)$, $u_2=u_2(\xi_1,\dots,\xi_5)$ yields

$$
\begin{aligned}
P(\lambda)(\sum_{k=0}^{N}A_ku_1^ku_2^{N-k}) &= \sum_{k=0}^{N}A_k(2(N-k)((2(N-k)-1)\xi_4+2\xi_5\xi_3)u_1^ku_2^{N-k-1}\\
&\quad +k((-k-2(N-k)+\lambda+2)\xi_4-\xi_3\xi_5)u_1^{k-1}u_2^{N-k})\\
&= \sum_{s=1}^{N+1}2A_{s-1}(N-(s-1))((2(N-(s-1))-1)\xi_4\\
&\quad +2\xi_5\xi_3)u_1^{(s-1)}u_2^{N-(s-1)-1}\\
&\quad +\sum_{k=0}^{N}kA_k((-k-2(N-k)+\lambda+2)\xi_4-\xi_3\xi_5)u_1^{k-1}u_2^{N-k}\\
&= \sum_{s=1}^{N}(2A_{s-1}(N-s+1)((2N-2s+1)\xi_4+2\xi_5\xi_3)\\
&\quad +sA_s((s-2N+\lambda+2)\xi_4-\xi_3\xi_5))u_1^{s-1}u_2^{N-s}).
\end{aligned}
$$

The $2N$ summands of the form $\xi_4u_1^{s-1}u_2^{N-s}$ and $\xi_3\xi_5u_1^{s-1}u_2^{N-s}$ are linearly independent and therefore the above sum is zero if and only if

$$
2A_{s-1}(N-s+1)\big((2N-2s+1)\xi_4+2\xi_5\xi_3\big)+sA_s\big((s-2N+\lambda+2)\xi_4-\xi_3\xi_5\big) \qquad (19)
$$

equals zero for all values of s. When $s=N$, the above sum becomes

$$
2A_{N-1}(\xi_4+2\xi_3\xi_5)+NA_N((-N+\lambda+2)\xi_4-\xi_3\xi_5)\,.
$$

It is a straightforward check that if A_N vanishes, then $A_{N-1},A_{N-2},\dots$ must also vanish; therefore we may assume $A_N\neq 0$. The vanishing of the coefficient in front of ξ_4 implies $A_{N-1}=-\frac{1}{2}NA_N\,(-N+\lambda+2)$ and in turn, the vanishing of the coefficient in front of $\xi_3\xi_5$ implies $-5+2N-2\lambda=0$. Therefore

$$
\lambda = N-5/2\,.
$$

Substituting λ back into (19), we get

$$\begin{aligned}&2A_{s-1}(N-s+1)((2N-2s+1)\xi_4+2\xi_5\xi_3)\\&+sA_s((-N+s-1/2)\xi_4-\xi_3\xi_5)=0.\end{aligned}$$

This implies $A_s=\frac{4(N-s+1)}{s}A_{s-1}=\cdots=4^s\binom{N}{s}A_0$, which completes the proof of 1).

2. A homogeneous $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular vector is, in particular, $\mathrm{sl}(2)\simeq i([\mathfrak{l}',\mathfrak{l}'])$-singular and by Lemma 4.1 must be of the form $u=\xi_3\sum_{k=0}^{N}A_k u_1^k u_2^{N-k}$. The application of $2\xi_5\partial_3$ converts $A_N(\xi_1\xi_4+\xi_2\xi_5)^N\xi_3$ into $2A_N(\xi_1\xi_4+\xi_2\xi_5)^N\xi_5$. Furthermore $2A_N(\xi_1\xi_4+\xi_2\xi_5)^N\xi_5$ contains in its binomial expansion $2A_N(\xi_1\xi_4)^N\xi_5$. Direct check shows that the action of $P(\lambda)$ on $(\xi_1\xi_4+\xi_2\xi_5)^{N-i}\xi_3^{1+2i}$ for $i>0$ does not contain the monomial $(\xi_1\xi_4)^N\xi_5$. This implies that $A_N=0$ and by induction, the polynomial is trivial. Consequently, there is no nontrivial odd homogeneity polynomial solving the differential equation $P(\lambda)$.

As an illustration, for $N=0$ we have $P(\lambda)(A_0\xi_3)=2A_0\xi_5$. This vanishes provided $A_0=0$, which implies the polynomial is trivial.

3. An $\mathrm{so}(7)\cap\mathfrak{p}$-singular vector must be $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular. As the grading element from (13) maps $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular to $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular vectors, it quickly follows that an $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular vector is a linear combination of gr-homogeneous elements (see (14)). From the explicit form of u_1 and u_2 it immediately follows that a homogeneous $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular vector is of the form (15).

From 1) we know that, other than v_λ, there is at most one more homogeneous $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular vector, and thus the vector (15) is the only candidate for a $\mathrm{so}(7)\cap\mathfrak{p}$-singular vector. The simple part of $\mathfrak{l}$ is isomorphic to $\mathrm{so}(5)$ and induces the quadratic form with matrix in the coordinates $\xi_1,\dots,\xi_5$

$$Q=\begin{pmatrix}0&0&0&2&0\\0&0&0&0&2\\0&0&1&0&0\\2&0&0&0&0\\0&2&0&0&0\end{pmatrix},$$

i.e., the metric of the form

$$g(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)=(d\xi_3)^2+2(d\xi_1\otimes d\xi_4+d\xi_4\otimes d\xi_1)+2(d\xi_2\otimes d\xi_5+d\xi_5\otimes d\xi_2).$$

The Fourier transform of the $\mathrm{so}(5)$-invariant Laplace operator associated to Q is

$$\mathcal{F}(\triangle_\xi)=Q(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)=4(\xi_1\xi_4+\xi_2\xi_5)+\xi_3^2.$$

Relying on $\triangle_\xi$ and the binomial formula for $(4(\xi_1\xi_4+\xi_2\xi_5)+\xi_3^2)^s$, we see that the Lie $G_2\cap\mathfrak{p}$-singular vector constructed 1) is indeed $\mathrm{so}(7)\cap\mathfrak{p}$-singular. The proof is complete. □

Remark. As noted in the proof of 3) every $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular is a linear combination of homogeneous $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular vectors, and therefore Theorem 4.2, 1) and 2) give all $i(\text{Lie }G_2)\cap\mathfrak{p}$-singular vectors (namely, the linear combinations of v_λ and the vector given by (15)).

We note that an alternative proof of Theorem 4.2, 3) can be given as follows. From a well known example (see e.g., [3], [10], [11]) of singular vectors in conformal geometry of dimension 5 describing conformally invariant powers of the Laplace operator, we know that for $\lambda \in \{-3/2, -1/2, 1/2, \dots\}$ there exists one $\mathrm{so}(7) \cap \mathfrak{p}$-singular vector in $M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$. On the other hand points 1) and 2) of Theorem 4.2 present us with only one such candidate, so that candidate must be the $\mathrm{so}(7) \cap \mathfrak{p}$-singular vector in question.

For $\lambda \in \{-3/2,\ -1/2, 1/2, \dots\}$, the $\mathfrak{h}$-weight of the $\mathrm{so}(7) \cap \mathfrak{p}$-singular vector in $M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$ given by Theorem 4.2 equals $(\lambda - 2N)\varepsilon_1 = (\lambda - 2(\lambda + 5/2))\varepsilon_1 = (-\lambda - 5)\varepsilon_1$. Therefore the vector from Theorem 4.2 corresponds to the homomoprhism of generalized Verma modules

$$M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda) \hookrightarrow M^{\mathrm{so}(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_{-\lambda-5})\,. \tag{20}$$

In an analogous fashion we conclude that Theorem 4.2 gives a homomorphism of generalized Verma modules

$$M^{\mathrm{Lie}\ G_2}_{\mathfrak{p}'_{(1,0)}}(\mathbb{V}_{\lambda\psi_1}) \to M^{\mathrm{Lie}\ G_2}_{\mathfrak{p}'_{(1,0)}}(\mathbb{C}_{(-\lambda-5)\psi_1})\,. \tag{21}$$

We conclude this paper with the following observation from [14], a proof of which we include for completeness.

Proposition 4.3. *Suppose* $\lambda \in \{-3/2,\ -1/2, 1/2, \dots\}$. *Then both* (20) *and* (21) *are non-standard homomorphisms.*

Proof. 1. Let $\rho_{\mathfrak{l}}$ be the half-sum of the positive roots of $\mathfrak{l}$, i.e., $\rho_{\mathfrak{l}} := 3/2\varepsilon_2 + 1/2\varepsilon_3$, and let s_{η_3} denote the reflection with respect to the simple root $\eta_3 = \varepsilon_3$. Then

$$s_{\eta_3}(\lambda\varepsilon_1 + \rho_{\mathfrak{l}}) - ((-\lambda - 5)\varepsilon_1 + \rho_{\mathfrak{l}}) = (2\lambda + 5)\varepsilon_1 - \varepsilon_3\,. \tag{22}$$

As $\lambda \in \{-3/2,\ -1/2, 1/2, \dots\}$, the expression (22) is a sum of positive roots of so(7). Therefore by [2, Chapter 7] the non-generalized Verma module with highest weight $(-\lambda - 5)\varepsilon_1 + \rho_{\mathfrak{l}}$ lies in the non-generalized Verma module with highest weight $s_{\eta_3}(\lambda\varepsilon_1 + \rho_{\mathfrak{l}})$. Therefore by [13, Proposition 3.3] the homomorphism (20) is non-standard.

2. Let $\rho_{\mathfrak{l}'} = 1/2\alpha_2$. Let s_{α_2} denote the reflection with respect to the simple root α_2 (in $\mathfrak{h}'^*$). Then

$$s_{\alpha_1}(\lambda\psi_1 + \rho_{\mathfrak{l}'}) - ((-5 - \lambda)\psi_1 + \rho_{\mathfrak{l}'}) = (2\lambda + 6)\alpha_2 + (4\lambda + 16)\alpha_1$$

is clearly a positive integral combination of positive roots of Lie G_2 and the statement follows again by [2, Chapter 7] and [13, Proposition 3.3]. □

Acknowledgement. The authors gratefully acknowledge the support by the Czech Grant Agency through the grant GA CR P 201/12/G028.

References

[1] Čap, A., Slovák, J., *Parabolic geometries, I: Background and General Theory*, Mathematical Surveys and Monographs, American Mathematical Society, 2009.

[2] Dixmier, J., *Algebres Enveloppantes*, Gauthier-Villars Editeur, Paris–Bruxelles–Montreal, 1974.

[3] Eastwood, M. G., Graham, C. R., *Invariants of conformal densities*, Duke Math. J. **63** (1991), 633–671.

[4] Graham, R. C., Willse, T., *Parallel tractor extension and ambient metrics of holonomy split* G_2, http://xxx.lanl.gov/abs/1109.3504.

[5] Humphreys, J. E., Jr., *Representations of Semisimple Lie Algebras in the BGG Category* $\mathcal{O}$, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, 2008.

[6] Juhl, A., *Families of conformally covariant differential operators, Q–curvature and holography*, Progress in Math., Birkhäuser, 2009.

[7] Kobayashi, T., *Discrete decomposability of the restriction of* $A_{\mathfrak{q}}(\lambda)$ *with respect to reductive subgroups and its applications*, Invent. Math. **117** (1994), 181–205, Part II, Ann. of Math. (2) **147** (1998), 709–729; Part III, Invent. Math. **131** (1998), 229–256.

[8] Kobayashi, T., *Multiplicity–free theorems of the restriction of unitary highest weight modules with respect to reductive symmetric pairs*, Progress in Math, vol. 280, Birkhäuser, 2007, pp. 45–109.

[9] Kobayashi, T., *Restrictions of generalized Verma modules to symmetric pairs*, Transform. Groups **17** (2012), 523–546.

[10] Kobayashi, T., Ørsted, B., Somberg, P., Souček, V., *Branching laws for Verma modules and applications in parabolic geometry, I*, preprint.

[11] Kobayashi, T., Ørsted, B., Somberg, P., Souček, V., *Branching laws for Verma modules and applications in parabolic geometry, II*, preprint.

[12] Kostant, B., *Verma modules and the existence of quasi–invariant differential operators*, Lecture Notes in Math., Springer Verlag, 1974, pp. 101–129.

[13] Lepowsky, J., *A generalization of the Bernstein–Gelfand–Gelfand resolution*, J. Algebra **49** (1977), 496–511.

[14] Matumoto, H., *The homomorphisms between scalar generalized Verma modules associated to maximal parabolic subalgebras*, Duke Math. J. **131** (2006), 75–118.

[15] Milev, T., Somberg, P., *The branching problem for generalized Verma modules, with application to the pair* $(\mathrm{so}(7), \mathrm{Lie}\ G_2)$, http://xxx.lanl.gov/abs/1209.3970.

Mathematical Institute of Charles University,
Sokolovská 83,
Praha 8 - Karlín, Czech Republic
E-mail: somberg@karlin.mff.cuni.cz

Department of Mathematics, University of Massachusetts Boston,
100 William T. Morrissey Boulevard,
Boston, MA 02125, USA
E-mail: todor.milev@gmail.com