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# Commutative Parasemifields Finitely Generated as Semirings

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Commutative parasemifields that are finitely generated as semirings are studied in more detail.

This short article continues immediately [2] and [3] and the reader is fully referred to the cited papers as concerns all necessary and/or helpful prerequisities.

## 1. Introduction

By a parasemifield we mean a non-trivial algebraic structure with two commutative and associative binary operations, addition and multiplication, where the multiplication forms an (abelian) group and distributes over the addition. Familiar examples of such a structure are the parasemifields of positive rational or real numbers. Both these parasemifields are congruence-simple and they are not finitely generated as semirings. In fact, according to [1, 14.3], every congruence-simple finitely generated commutative semiring is either finite or additively idempotent. A corresponding result for ideal-simple finitely generated commutative semirings seems to be an open problem. According to [2, 5.1], it is sufficient to solve the problem only for parasemifields.

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Since every parasemifield is infinite, it would mean that a parasemifield is additively idempotent, provided that it is a finitely generated semiring.

## 2. Parasemifields and subsemigroups of $\mathbb{N}_0^m$

In the paper, let $S$ be a commutative parasemifield that is not additively idempotent (i.e., $1_S \neq 1_S + 1_S = 2_S$).

First, observe that the prime subparasemifield $T$ of $S$ (i.e., the subparasemifield generated by the unit $1_S$) is a copy of the parasemifield $\mathbb{Q}^+$ of positive rationals. It is quite easy to show that $\mathbb{Q}^+$ is a congruence-simple semiring (i.e., $\mathrm{id}_{\mathbb{Q}^+}$ and $\mathbb{Q}^+ \times \mathbb{Q}^+$ are the only semiring congruences of $\mathbb{Q}^+$) and that $\mathbb{Q}^+$ is not a finitely generated semiring. Consequently, if $\rho$ is a congruence of $S$, then either $\rho \restriction T = \mathrm{id}_T$ or $T$ is contained in a block of $\rho$ and the factor-semiring $S/\rho$ is additively idempotent.

For every $u \in S$, the set $I_u = (S + u) \cup \{u\}$ is the principal ideal of the additive semigroup $S(+)$ generated by the element $u$. We denote by $Q$ the set of the elements $u \in S$ such that $I_u \cap T \neq \emptyset$. Furthemore, we put $R = (S + T) \cup T$ and $P = Q \cap R$; notice that $R$ is the ideal of $S(+)$ generated by $T$.

**2.1 Proposition.** ([3])

(i) *Both $Q$ and $R$ are subsemirings of $S$.*
(ii) $R = Q^{-1} = \{u^{-1} | u \in Q\}$.
(iii) $S = QR = \{uv | u \in Q, v \in R\}$.
(iv) $T \subseteq P = Q + T = Q \cap R$.
(v) *$P$ is an additively archimedean and cancellative parasemifield.*
(vi) *Neither $Q$ nor $P$ is a finitely generated semiring,*
(vii) *If $u_1, \dots, u_n \in S, n \geq 1$ are such that $u_1 + \dots + u_n \in Q$, then $u_1, \dots, u_n \in Q$.*
(viii) *If $u \in S$ and $n \geq 1$ are such that $u^n \in Q$ ($R, P$, resp.), then $u \in Q$ ($R, P$, resp.).*
(ix) *If $u, v, w \in Q$ are such that $u + v = u + w$, then $v + t = w + t$ for every $t \in T$.*

*Proof.* See [3, 4.3], [3, 4.8], [3, 3.11], [3, 4.10], [3, 4.18], [3, 4.4], [3, 4.6], and [3, 4.15]. □

In the remaining part of the paper, assume that $S$ is finitely generated as a semiring. Let $\{z_1, \dots, z_m\}, m \geq 1$, be a finite set of generators of $S$.

**2.2 Lemma.**

(i) $Q \neq S \neq R$.
(ii) $Q \neq P \neq R$.

*Proof.* Combine 2.1(vi) and [3, 4.9]. □

Put $A = \{(k_1, \dots, k_m) \in \mathbb{N}_0^m | z_1^{k_1} \cdots z_m^{k_m} \in Q\}$, $A' = \{(k_1, \dots, k_m) \in \mathbb{N}_0^m | z_1^{k_1} \cdots z_m^{k_m} \in R\}$, and $B = \{(k_1, \dots, k_m) \in \mathbb{N}_0^m | z_1^{k_1} \cdots z_m^{k_m} \in P\}$ ($\mathbb{N}_0$ denotes the semiring of non-negative integers).

**2.3 Proposition.**

(i) $0 \in A, 0 \neq A \neq \mathbb{N}_0^m$ *and* $A$ *is a pure subsemigroup of* $\mathbb{N}_0^m(+)$ *(i.e.,* $nA = A \cap n\mathbb{N}_0^m$ *for every* $n \geq 1$*).*

(ii) *$A$ is not a finitely generated semigroup.*

*Proof.* (i) Clearly, $z_1^0 \cdots z_m^0 = 1_S \in T \subseteq Q$, and so $0 \in A$. Since $Q(\cdot)$ is a subsemigroup of the multiplicative group $S(\cdot)$, we see that $A$ is a subsemigroup of the additive semigroup $\mathbb{N}_0^m(+)(= \mathbb{N}_0(+)^m)$. From 2.1(viii) follows that $A$ is a pure subsemigroup.

(ii) See [3, 4.19(iii)].

□

**2.4 Lemma.** *Let* $k \geq 1$ *and* $a_i = (k_{i,1}, \ldots, k_{i,m}) \in \mathbb{N}_0^m, 1 \leq i \leq k$.

(i) *If* $\sum_i z_1^{k_{i,1}} \cdots z_m^{k_{i,m}} \in Q$*, then* $a_i \in A$ *for every* $i, 1 \leq i \leq k$.

(ii) *If* $q_i \in \mathbb{Q}_0^+, 1 \leq i \leq k$ *(the semifield of non-negative rationals) are such that* $a = \sum_i q_i a_i \in \mathbb{N}_0^m(+)$ *and if* $a_i \in A$ *for every* $i$*, then* $a \in A$.

*Proof.* (i) The assertion follows easily from 2.1(vii).

(ii) We have $q_i = r_i/s_i$ for suitable $r_i \in \mathbb{N}_0$ and $s_i \in \mathbb{N}$. If $s = s_1 \cdots s_k$ then $sq_i \in \mathbb{N}_0, b_i = sq_i a_i \in A$ and $sa = \sum b_i \in A \cap s\mathbb{N}_0^m = sA$. Thus $a \in A$.

□

**2.5 Proposition.** $0 \in A', A' \neq \mathbb{N}_0^m$ *and* $A'$ *is a pure subsemigroup of* $\mathbb{N}_0^m(+)$.

*Proof.* Similar to that of 2.3(i). □

**2.6 Proposition.**

(i) $0 \in B$ *and* $B$ *is a pure subsemigroup of* $\mathbb{N}_0^m(+)$.

(ii) $B = A \cap A'$.

(iii) $B \neq A$.

*Proof.* (i) Similar to 2.3(i).

(ii) We have $P = Q \cap R$.

(iii) If $B = A$ then $A \subseteq A'$, and hence $R = S$, a contradiction with 2.2(i).

□

**2.7 Lemma.** *Let* $b \in A$*. Then* $b \in B$ *if and only if* $a - b \in A$ *for every* $a \in A$ *such that* $a - b \in \mathbb{N}_0^m$.

*Proof.* Let $b = (k_1, \ldots, k_m)$ and $u = z_1^{k_1} \cdots z_m^{k_m} \in Q$. If $b \in B$ then $u \in P$, $u^{-1} \in$ $\in Q$ and $z_1^{l_1-k_1} \cdots z_m^{l_m-k_m} = u^{-1}v \in Q$, where $a = (l_1, \ldots, l_m) \in A$ and $a - b \in \mathbb{N}_0^m$. Consequently, $a - b \in A$.

Now we are going to show the converse implication. We have $u^{-1} = \sum_{i=1}^k z_1^{k_{i,1}} \cdots$ $\cdots z_m^{k_{i,m}}$ for some $k \geq 1$ and $a_i = (k_{i,1}, \ldots, k_{i,m}) \in \mathbb{N}_0^m, 1 \leq i \leq k$. Then $1_S = uu^{-1} =$ $= \sum_i z_1^{k_1+k_{i,1}} \cdots z_m^{k_m+k_{i,m}} \in Q$, and it follows from 2.4(i) that $b + a_i \in A$ for every $i$. On the other hand, $a_i = (b + a_i) - a_i \in \mathbb{N}_0^m$ and we get that $a_i \in A$. Thus $z_1^{k_{i,1}} \cdots z_m^{k_{i,m}} \in Q$ for every $i$ and, finally, $u^{-1} \in Q$. Thus $u \in P$ and $b \in B$. □

**2.8 Lemma.** *Let $a \in A'$ and $b \in B$ be such that $a - b \in \mathbb{N}_0^m$. Then $a - b \in A'$.*

*Proof.* See the first part of the proof of 2.7. □

**2.9 Lemma.** *Let $a \in A$ and $a_1, \dots, a_k \in \mathbb{N}_0^m, k \geq 1$, be such that $a + a_i \in A$ for every $i, 1 \leq i \leq k$. Assume that there exist positive integers $n_1, \dots, n_k$ such that $(n_i - 1)a + n_i a_i \in A$ for every $i$. Then:*

(i) *$(n - 1)a + na_i \in A$ for all $i$ and positive integers $n \geq \max(n_i)$.*

(ii) *$(n - 1)a + \sum r_i a_i \in A$ for all $n \geq \max(n_i)$ and $r_1, \dots, r_k \in \mathbb{N}_0, \sum r_i = n$.*

*Proof.* (i) We have $n = n_i + l_i$ for some $l_i \in \mathbb{N}_0$ and $(n - 1)a + na_i = (n_i - 1)a + n_i a_i + l_i(a + a_i) \in A$.

(ii) We have $(n-1)a + \sum r_i a_i = \sum (r_i/n)((n-1)a + na_i) \in \mathbb{N}_0^m$. It remains to combine (i) and 2.4(ii).

□

**2.10 Lemma.** *Let $a = (k_1, \dots, k_m) \in A$ and $u = z_1^{k_1} \cdots z_m^{k_m} \in Q$. Let $a_i = (k_{i,1}, \dots, k_{i,m}) \in \mathbb{N}_0^m, 1 \leq i \leq k$, be such that $u^{-1} = \sum v_i$, where $v_i = z_1^{k_{i,1}} \cdots z_m^{k_{i,m}} \in S$. Then:*

(i) *$a + a_i \in A$ for every $i$.*

(ii) *$a \in B$ and $u \in P$, provided that there exist positive integers $n_i$ such that $(n_i - 1)a + n_i a_i \in A$ for every $i$.*

*Proof.* (i) Easy (see the second part of the proof of 2.7).

(ii) Put $n = \sum n_i$. Then $u^{-n} = (\sum v_i)^n = \sum_r t_r \prod_{i=1}^k v_i^{r_i}$, $r = (r_1, \dots, r_k) \in \mathbb{N}_0^k, \sum r_i = n, t_r \in \mathbb{N}, u^{n-1} = z_1^{(n-1)k_1} \cdots z_m^{(n-1)k_m}$ and $u^{-1} = u^{n-1}u^{-n}$. On the other hand, $u^{n-1} \prod_{i=1}^k v_i^{r_i} = z_1^{s_1} \cdots z_m^{s_m}$, where $s_j = (n - 1)k_j + \sum_{i=1}^k r_i k_{i,j}$ for every $j = 1, \dots, m$. Since $(n - 1)a + \sum_i r_i a_i \in A$ by 2.9(ii), we have $u^{n-1} \prod_{i=1}^k v_i^{r_i} \in Q$ and consequently, $u^{-1} = \sum_r t_r (u^{n-1} \prod_{i=1}^k v_i^{r_i}) \in Q$. Thus $u \in P$ and $a \in B$.

□

**2.11 Remark.** Consider the situation from 2.10. If $a \in A \setminus B$ (i.e., $u \notin P$), then there exists $i_0, 1 \leq i_0 \leq k$, such that $(n - 1)a + na_{i_0} \notin A$ for every positive integer $n$. In particular, $a_{i_0} \neq 0$. Now, if $a_{i_0} = qa$ for some $q \in \mathbb{Q}^+$, then $q = r/s, r, s \in \mathbb{N}$, and we get $(s - 1)a + sa_{i_0} = (s - 1)a + ra = (s + r - 1)a \in A$, a contradiction. Thus $a_{i_0} \notin \mathbb{Q}_0^+ a$.

**2.12 Lemma.** *Let $a, a_1, \dots, a_k \in A, k \geq 1, b \in \mathbb{N}_0^m, r, s \in \mathbb{Q}^+$ and $q_1, \dots, q_k \in \mathbb{Q}_0^+$ be such that $rb - sa = \sum_{i=1}^k q_i a_i$. Then $(n - 1)a + nb \in A$ for a positive integer $n$ (and hence $a + b \in A$).*

*Proof.* There are positive integers $n, l, t$ such that $r = n/t$ and $s = l/t$. Now, $nb - la = \sum_i q_i a_i \in \mathbb{N}_0^m$ and $nb - la \in A$ by 2.4(ii). But $(n-1)a + nb = (nb - la) + + (n + l - 1)a \in A$. □

**2.13 Lemma.** *The following conditions are equivalent for all $a \in A$ and $b \in \mathbb{N}_0^m$:*

(i) *$(n-1)a + nb \in A$ for some $n \in \mathbb{N}$.*

(ii) *There are $r, s \in \mathbb{Q}^+, k \in \mathbb{N}, a_1, \dots, a_k \in A$ and $q_1, \dots, q_k \in \mathbb{Q}_0^+$ such that $r(a + b) - sa = \sum_{i=1}^k q_i a_i$.*

*Moreover, if these equivalent conditions are satisfied, then $a + b \in A$.*

*Proof.* If (i) is true, then $(n-1)a + nb = a_1 \in A$ and $n(a + b) - a = a_1$, so we can put $r = n, s = 1 = k, q_1 = 1$. Moreover, $n(a + b) = a_1 + a \in A$ and $a + b \in A$ since $A$ is a pure subsemigroup of $\mathbb{N}_0^m(+)$.

Now, assume that (ii) is satisfied. We have $r = k_1/t$ and $s = k_2/t$ for suitable $k_1, k_2, t \in \mathbb{N}$. Then $c = (k_1 - k_2)a + k_2 b = t(r(a+b) - sa) = \sum_i t q_i a_i \in \mathbb{Z}^m \cap (\mathbb{Q}_0^+)^m = \mathbb{N}_0^m$ and $c \in A$ by 2.4(ii). Consequently, $(k_1 - 1)a + k_1 b = c + (k_2 - 1)a \in A$. □

**2.14 Lemma.** *Let $a \in A$ be such that for every $b \in \mathbb{N}_0^m$ with $a + b \in A$ there exist $a_1, \dots, a_k \in A, k \geq 1, r, s \in \mathbb{Q}^+$ and $q_1, \dots, q_k \in \mathbb{Q}_0^+$ with $r(a + b) - sa = \sum_i q_i a_i$. Then $a \in B$.*

*Proof.* Combine 2.13 and 2.10. □

**2.15 Corollary.** (cf. 2.11) Let $a \in A \backslash B$ (see 2.6(iii)). Then there exists $b \in \mathbb{N}_0^m$ such that $a + b \in A$ and $r(a + b) - sa \neq \sum q_i a_i$ for all $a_1, \dots, a_k \in A, k \geq 1, r, s \in \mathbb{Q}^+$ and $q_1, \dots, q_k \in \mathbb{Q}_0^+$. In particular, $a + b \notin \mathbb{Q}a$ and $b \notin \mathbb{Q}a$.

**2.16 Remark.** Let $\sigma$ be a congruence of $S$ maximal with respect to $(1_S, 2_S) \notin \sigma$, Then $S/\sigma$ is a parasemifield that is not additively idempotent.

As in [3], define a relation $\mu_S$ on $S$ by $(a, b) \in \mu_S$ if and only if $b = a + z$ for some $z \in S \cup \{0\}$ and define a relation $\eta_S$ on $S$ by $(a, b) \in \eta_S$ if and only if there exist $m, n \in \mathbb{N}$ such that $(a, mb) \in \mu_S$ and $(b, na) \in \mu_S$. Then $\eta_S$ is the smallest congruence of $S$ such that the corresponding factor is additively idempotent (see [3, 1.5]).

Hence, $\eta_S \subseteq \sigma_1$, whenever $\sigma_1$ is a congruence of $S$ such that $\sigma \subsetneq \sigma_1$. In particular, the factor-semiring $S/\sigma$ is subdirectly irreducible.

## 3. Mapping to $\mathbb{R}$

The preceding section is immediately continued. Since $S$ is a non-trivial finitely generated semiring, $S$ possesses at least one (proper) maximal congruence $\rho$. Combining [1, 14.3], [1, 10.1], [1, 5.3], we conclude that there exists a mapping $\varphi : S \to \mathbb{R}$ (the field of real numbers) such that $\ker(\varphi) = \rho, \varphi(u + v) = \min(\varphi(u), \varphi(v))$ and

$\varphi(uv) = \varphi(u) + \varphi(v)$ for all $u, v \in S$. Then $\varphi(1_S) = 0$ and $\varphi(S)(+)$ is a non-zero finitely generated subgroup of $\mathbb{R}(+)$. In fact, if the semiring $S$ is generated by $\{z_1, \dots, z_m\}, m \geq 1$, then the semigroup $\varphi(S)(+)$ is generated by the real numbers $\varphi(z_1), \dots, \varphi(z_m)$.

Put $V = \varphi^{-1}(\varphi(S) \cap \mathbb{R}_0^+)$, $U = \varphi^{-1}(\varphi(S) \cap \mathbb{R}_0^-)$ and $W = \varphi^{-1}(0)$.

**3.1 Proposition.**

(i) *$V$ and $U$ are subsemirings of $S$.*
(ii) *$W$ is a subparasemifield of $S$.*
(iii) $U = V^{-1}$.
(iv) *$S + U = U$ and $W + V = W$.*
(v) *$V \cup U = S$ and $V \cap U = W$.*
(vi) $V \neq S \neq U$.
(vii) $V \neq W \neq U$.
(viii) *$Q \subseteq V, R \subseteq U$ and $P \subseteq W$.*

*Proof.* The first seven assertions follow easily from the properties of the mapping $\varphi$. It remains to show the last one.

First, $T \subseteq W = V \cap U$, since $T$ is the prime subparasemifield of $S$. If $v \in Q \backslash T$, then $v + w \in T$ for some $w \in S$ and we have $0 = \varphi(v + w) = \min(\varphi(v), \varphi(w))$. Consequently, $\varphi(v) \geq 0$ and $v \in V$. This means that $Q = (Q \backslash T) \cup T \subseteq V$. If $u \in R$, then $u^{-1} \in V$, and so $u \in U$ by (iii). □

**3.2 Lemma.** *Let $u_1, \dots, u_n \in S, n \geq 1$, and $u = u_1 + \cdots + u_n$.*

(i) *If $u \in V$, then $u_1, \dots, u_n \in V$.*
(ii) *If $u \in U$, then $u_i \in U$ for at least one $i$.*
(iii) *If $u \in W$, then $u_1, \dots, u_n \in V$ and $u_i \in W$ for at least one $i$.*

*Proof.* It is easy. □

**3.3 Lemma.** *If $u \in S$ and $n \geq 1$ are such that $u^n \in V$ ($U, W$, resp.), then $u \in V$ ($U, W$, resp.).*

*Proof.* It is easy. □

**3.4 Lemma.** *Both $V' = V \backslash W$ and $U' = U \backslash W$ are subsemirings of $S$.*

*Proof.* It is easy. □

**3.5 Lemma.** $\varphi(S) = \varphi(Q) - \varphi(Q)$.

*Proof.* We have $\varphi(S) = \varphi(QR) = \varphi(QQ^{-1}) = \varphi(Q) + \varphi(Q^{-1}) = \varphi(Q) - \varphi(Q)$. □

Put $\overline{A} = \{(k_1, \dots, k_m) \in \mathbb{N}_0^m | z_1^{k_1} \cdots z_m^{k_m} \in V\}$, $\widetilde{A} = \{(k_1, \dots, k_m) \in \mathbb{N}_0^m | z_1^{k_1} \cdots z_m^{k_m} \in U\}$, and $\overline{B} = \{(k_1, \dots, k_m) \in \mathbb{N}_0^m | z_1^{k_1} \cdots z_m^{k_m} \in W\}$.

**3.6 Proposition.**

(i) $0 \in \overline{A}, 0 \neq \overline{A} \neq \mathbb{N}_0^m$ *and* $\overline{A}$ *is a pure subsemigroup of* $\mathbb{N}_0^m(+)$.

(ii) $A \subseteq \overline{A} \neq \mathbb{N}_0^m$.

(iii) *If* $\overline{A}$ *is a finitely generated semigroup, then* $V$ *is a finitely generated semiring.*

*Proof.* (i) An easy consequence of the definition of $\overline{A}$.

(ii) $A \subseteq \overline{A}$, since $Q \subseteq V$, and $\overline{A} \neq \mathbb{N}_0^m$ since $V \neq S$.

(iii) Use 3.2(i). □

**3.7 Lemma.** *Let* $k \geq 1, a_1, \dots, a_k \in \overline{A}$ *and* $q_1, \dots, q_k \in \mathbb{Q}_0^+$ *be such that* $a = \sum_i q_i a_i \in \mathbb{N}_0^m$. *Then* $a \in \overline{A}$.

*Proof.* Similar to that of 2.4(ii). □

**3.8 Proposition.**

(i) $0 \in \widetilde{A}, 0 \neq \widetilde{A} \neq \mathbb{N}_0^m$ *and* $\widetilde{A}$ *is a pure subsemigroup of* $\mathbb{N}_0^m(+)$.

(ii) $\overline{A} \cup \widetilde{A} = \mathbb{N}_0^m$.

*Proof.* It is easy (use 3.1(v)). □

**3.9 Lemma.** *Let* $k \geq 1$ *and* $a_i = (k_{i,1}, \dots, k_{i,m}) \in \mathbb{N}_0^m, 1 \leq i \leq k$, *be such that* $\sum_i z_1^{k_{i,1}} \cdots z_m^{k_{i,m}} \in V$ *(*$U$*, resp.). Then* $a_i \in \overline{A}$ *for every* $i$ *(*$a_j \in \widetilde{A}$ *for at least one* $j$*, resp.).*

*Proof.* It is easy. □

**3.10 Proposition.**

(i) $0 \in \overline{B}$ *and* $\overline{B}$ *is a non-zero pure subsemigroup of* $\mathbb{N}_0^m(+)$.

(ii) $\overline{B} = \overline{A} \cap \widetilde{A}$.

(iii) $\overline{A} \neq \overline{B} \neq \widetilde{A}$.

*Proof.* (i) Clearly, $0 \in \overline{B}$ and $\overline{B}$ is a pure subsemigroup of $\mathbb{N}_0^m(+)$. Since the semiring $S$ is generated by the set $\{z_1, \dots, z_m\}$ there are $k \geq 1$ and $0 \neq a_i \in \mathbb{N}_0^m, i = 1, \dots, k, a_i = (k_{i,1}, \dots, k_{i,m})$, such that $1_S = \sum_i z_1^{k_{i,1}} \cdots z_m^{k_{i,m}}$. By 3.2(iii), $a_i \in \overline{B}$ for at least one $i$. Thus $\overline{B} \neq 0$.

(ii) We have $W = V \cap U$ by 3.1(v).

(iii) If $\overline{B} = \overline{A}$, then $\overline{A} \subseteq \widetilde{A}$, and hence $V \subseteq U$ and $U = S$, a contradiction. If $\overline{B} = \widetilde{A}$, then $\widetilde{A} \subseteq \overline{A}$, and hence $\overline{A} = \mathbb{N}_0^m$, again a contradiction. □

**3.11 Lemma.**

(i) *Let* $b \in \overline{A}$. *Then* $b \in \overline{B}$ *if and only if* $a - b \in \overline{A}$ *for every* $a \in \overline{A}$ *such that* $a - b \in \mathbb{N}_0^m$.

(ii) *Let* $b \in \overline{B}$. *Then* $a - b \in \widetilde{A}$ *for every* $a \in \widetilde{A}$ *such that* $a - b \in \mathbb{N}_0^m$.

*Proof.* Similar to that of 2.7. □

**3.12 Remark.** By 3.1(iv), we have $W + V = W$. We are going to show that $w + V \neq W$ for every $w \in W$.

Assume, on the contrary, that $w_1 + V = W$ for some $w_1 \in W$. Then $w_1^{-1} \in W$, and hence $1_S + V = 1_S + w_1^{-1}V = w_1^{-1}(w_1 + V) = w_1^{-1}W = W$. Furthemore, $w + V = w + wV = w(1_S + V) = wW = W$ for every $w \in W$. In particular, $w + 2_S + V = W$, and then $w + 1_S + V + 2_S = W + 1_S$. But $V + 2_S = V + 1_S + 1_S \subseteq W + 1_S$ and we see that $w + 1_S + W + 1_S = W + 1_S$. Now, it is clear that $W + 1_S$ is a subgroup of $S(+)$. If $z$ is the neutral element of the subgroup, then $2z = z$, and hence $2_S = 1_S$, a contradiction.

## References


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