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On separating sets of words. III.

On Separating Sets of Words III

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Transitive closures of special replacement relations in free monoids are studied.

1. Introduction

This article is an immediate continuation of [1] and [2]. References like I.3.3 (II.3.3, resp.) lead to the corresponding section and result of [1] ([2], resp.) and all definitions and preliminaries are taken from the same sources.

2. The transitive closure of the replacement relation

Let Z be a set of words and $\psi : Z \to A^*$ a mapping. Put $(\rho_{Z,\psi} =) \rho = \{(uzv, u\psi(z)v) \mid z \in Z, u, v \in A^*\}$, $(\lambda_{Z,\psi} =) \lambda = \rho \cup \mathrm{id}_{A^*}$, denote by $(\tau_{Z,\psi} =) \tau$ the smallest transitive relation defined on A^* and containing ρ (i. e., the transitive closure of ρ) and put $(\xi_{Z,\psi} =) \xi = \tau \cup \mathrm{id}_{A^*}$.

A sequence $w_0, w_1, \ldots, w_m$ of words from A^*, $m \geq 1$, will be called a ρ-sequence (λ-sequence, resp.) if $(w_i, w_{i+1}) \in \rho$ ($(w_i, w_{i+1}) \in \lambda$, resp.) for every i, $0 \leq i < m$. The positive integer m is the length of the sequence and the sequence is said to lead from w_0 to w_m.

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Proposition 2.1
    (i) $(u, v) \in \tau$ if and only if there exists at least one ρ-sequence leading from u to v.
    (ii) $(u, v) \in \xi$ if and only if there exists at least one λ-sequence leading from u to v (and hence ξ is the transitive closure of λ).

Proof. Obvious from the definition of the relations τ and ξ.   □

Proposition 2.2
    (i) τ is stable and transitive.
    (ii) ξ is stable, reflexive and transitive (and hence ξ is a stable quasiordering of the monoid A^*).

Proof. Easy (use 2.1).   □

Put $(\nu_{Z,\psi} =) \nu = \ker(\tau)$ (i. e., $(u, v) \in \nu$ iff $(u, v) \in \tau$ and $(v, u) \in \tau$) and $(\mu_{Z,\psi} =)$ $\mu = \nu \cup \mathrm{id}_{A^*}$.

Proposition 2.3
    (i) ν is stable, symmetric and transitive.
    (ii) If $(u, v) \in \nu$, then $(u, u) \in \nu$, $(v, v) \in \nu$, $(u, u) \in \tau$ and $(v, v) \in \tau$.
    (iii) μ is a congruence of the monoid A^*.
    (iv) $\mu = \ker(\xi)$.

Proof. Easy.   □

Proposition 2.4 *The following conditions are equivalent:*
    (i) ν is reflexive.
    (ii) $\nu = \mu$.
    (iii) ν is a congruence of the monoid A^*.
    (iv) τ is reflexive.
    (v) For every $u \in A^*$ there is at least one ρ-sequence leading from u to u.

Proof. Easy.   □

3. On when the closure is antisymmetric

Proposition 3.1 *The following conditions are equivalent:*
    (i) τ is a stable near-ordering on A^*.
    (ii) τ is antisymmetric.
    (iii) $w_0 \neq w_m$ whenever $w_0, w_1, \ldots, w_m$ is a ρ-sequence of length $m \geq 2$ such that $w_i \neq w_0$ for at least one i, $1 \leq i < m$.
    (iv) $\nu \subseteq \mathrm{id}_{A^*}$.
    (v) ξ is a stable (reflexive) ordering on A^*.
    (vi) ξ is antisymmetric.
    (vii) $\mu = \mathrm{id}_{A^*}$.

Proof. Easy (use 2.2, 2.3 and 2.4).   □

Remark 3.2 The equivalent conditions of 3.1 are satisfied if $u = v$ whenever $(u, v) \in \tau$ and $(v, u) \in \rho$.

Indeed, assume that the latter condition is true. Let $w_0, w_1, \ldots, w_m$ is a ρ-sequence of length $m \geq 2$ such that $w_i \neq w_0$ for at least one i, $1 \leq i < m$. Let j be the largest number with $1 \leq j \leq m$ and $w_j \neq w_0$. If $j < m$, then $w_{j+1} = w_0$, $(w_0, w_j) \in \tau$, $(w_j, w_0) \in \rho$, a contradiction. Thus $j = m$ and $w_m \neq w_0$.

Proposition 3.3 *The following conditions are equivalent:*
- (i) *τ is a stable sharp ordering on A^*.*
- (ii) *τ is irreflexive.*
- (iii) *τ is irreflexive and antisymmetric.*
- (iv) *$w_0 \neq w_m$ whenever $w_0, w_1, \ldots, w_m$ is a ρ-sequence.*
- (v) *$\nu = \emptyset$.*

Proof. Easy (use 2.2, 2.3 and 2.4). ☐

Proposition 3.4 *Assume that $|\psi(z)| < |z|$ ($|z| < |\psi(z)|$, resp.) for every $z \in Z$. Then:*
- (i) *$|v| < |u|$ ($|v| < |u|$, resp.) for every $(u, v) \in \tau$.*
- (ii) *τ is a stable sharp ordering.*
- (iii) *ξ is a stable ordering.*

Proof. Easy (use 3.1 and 3.3). ☐

Lemma 3.5 *Let $Z \subseteq A^+$ be a strongly separating set and let $w_0, \ldots, w_m$ be a ρ-sequence. Then:*
- (i) *$\mathrm{tr}(w_0) \leq \mathrm{tr}(w_m) + m$.*
- (ii) *If, for every $z \in Z$, either $|\psi(z)| \leq 2$ or $\psi(z)$ is reduced, then $\mathrm{tr}(w_m) \leq \mathrm{tr}(w_0) + m$.*
- (iii) *If $|\psi(z)| \leq 1$ for every $z \in Z$, then $\mathrm{tr}(w_m) \leq \mathrm{tr}(w_0)$.*

Proof. The result follows by induction from I.7.6. ☐

Proposition 3.6 *Assume that $|\psi(z)| \leq 1$ for every $z \in Z$. If $w \in A^*$ is a meagre word and $(w, v) \in \xi$ then v is meagre.*

Proof. The result follows immediately from 3.5 (iii). ☐

4. Reduced and pseudoreduced words

Proposition 4.1 *The following conditions are equivalent for a word w:*
- (i) *w is reduced.*
- (ii) *$(w, x) \notin \rho$ for every $x \in A^*$.*
- (iii) *$(w, x) \notin \tau$ for every $x \in A^*$.*

Proof. Obvious. ☐

A word w will be called *strongly $((Z, \psi)$-$)$ pseudoreduced* (or *almost $((Z, \psi)$-$)$ reduced*) if $x = w$ for all $(w, x) \in \rho$.

Proposition 4.2 *The following conditions are equivalent for a word w:*
   (i) *w is strongly pseudoreduced.*
   (ii) $x = w$ *for all* $(w, x) \in \lambda$.
   (iii) $x = w$ *for all* $(w, x) \in \tau$.
   (iv) $x = w$ *for all* $(w, x) \in \xi$.
   (v) $\psi(z) = z$ *for every* $z \in Z$ *that is a factor of w.*

Proof. Easy. □

Corollary 4.3 *If* $\psi(z) \neq z$ *for every* $z \in Z$, *then every strongly pseudoreduced word is reduced.*

A word w will be called *(weakly)* $((Z, \psi)\text{-})$ *pseudoreduced* if $(w, x) \in \rho$ implies $(x, w) \in \rho$ (i. e., $(w, x) \in \ker(\rho)$).

Proposition 4.4 *Assume that* $\ker(\rho) \subseteq \mathrm{id}_{A^*}$ *(e. g.,* $\nu \subseteq \mathrm{id}_{A^*}$ *– see 3.1). Then a word w is pseudoreduced iff it is strongly pseudoreduced.*

Proof. Clearly, every strongly pseudoreduced word is pseudoreduced. On the other hand, if w is pseudoreduced and $(w, x) \in \rho$, then $(x, w) \in \rho$, $(w, x) \in \ker(\rho)$ and $w = x$. □

A word w will be called $((Z, \psi)\text{-})$ *quasireduced* if $(w, x) \in \tau$ implies $(x, w) \in \tau$ (then $(w, x) \in \nu$).

Proposition 4.5 *A word w is quasireduced iff* $(w, x) \in \xi$ *implies* $(x, w) \in \xi$

Proof. Obvious. □

Proposition 4.6
   (i) *Every strongly pseudoreduced word is quasireduced.*
   (ii) *If* $\nu \subseteq \mathrm{id}_{A^*}$ *(see 3.1), then every quasireduced word is strongly pseudoreduced.*

Proof. Obvious. □

Proposition 4.7 *Assume that* $\nu \subseteq \mathrm{id}_{A^*}$ *(e. g., if* ψ *is strictly length decreasing or strictly length increasing – see 3.4). Then the following conditions are equivalent for a word w:*
   (i) *w is pseudoreduced.*
   (ii) *w is strongly pseudoreduced.*
   (iii) *w is quasireduced.*
Moreover, if $\psi(z) \neq z$ *for every* $z \in Z$, *then these conditions are equivalent to:*
   (iv) *w is reduced.*

Proof. Combine 4.2, 4.3, 4.4 and 4.6. □

Proposition 4.8 *Assume that the mapping* ψ *is strictly length decreasing. Then for every word* $w \in A^*$ *there exists at least one reduced word* $r \in A^*$ *such that* $(w, r) \in \xi$.

Proof. Easy (by induction on $|w|$). □

5. Meagre words

A word w will be called *meagre* if $\mathrm{tr}(w) \leq 1$.

Proposition 5.1 *(II.5.4) Let Z be a strongly separating set of words such that $Z \neq \{\varepsilon\}$ and, for every $z \in Z$, either $|\psi(z)| \leq 1$ or $\psi(z)$ is reduced. Assume further that there exists no pair $(z_1, z_2) \in Z \times Z$ such that either $\psi(z_1) = z_2$, $\psi(z_2) = z_1$ or $z_1 = ur$, $z_2 = sv$, $\psi(z_1) = us$, $\psi(z_2) = rv$, $u, v, r, s \in A^+$. Then every pseudoreduced meagre word is reduced.*

A word w will be called *pseudomeagre* if $(w, x) \in \rho$ for at most one $x \in A^*$. Clearly, every meagre word is pseudomeagre.

Proposition 5.2 *(II.6.7) Let Z be a strongly separating set of words such that $Z \neq \{\varepsilon\}$. Assume further that the following two conditions are satisfied:*
  (c1) $\varepsilon \neq \psi(z) \neq z$ and $\psi(z) \neq zxz$ for all $z \in Z$ and $x \in A^*$.
  (c2) *If $z_1, z_2 \in Z$ and $x, y \in A^*$ are such that $\psi(z_1) = yxz_1$ and $\psi(z_2) = z_2xy$, then*
       $\psi(z_1) = \psi(z_2)$.
Then every pseudomeagre word is meagre.

Proposition 5.3 *(II.6.8) Let Z be a strongly separating set of words such that $Z \neq \{\varepsilon\}$.*
  (i) *If $\psi(z) \neq \varepsilon$ and z is neither a prefix nor a suffix of $\psi(z)$ for every $z \in Z$, then every pseudomeagre word is meagre.*
  (ii) *If $|\psi(z)| \leq |z|$ for every $z \in Z$, then every pseudomeagre word is meagre if and only if $\varepsilon \neq \psi(z) \neq z$ for every $z \in Z$.*

Proposition 5.4 *(II.7.3) Let Z be a strongly separating set of words as in 5.1. Assume further that there exists no triple $(z_1, z_2, z_3) \in Z \times Z \times Z$ such that $z_1 = uv$, $z_3 = gh$ and $\psi(z_2) = vpg$ for some $u, v, g, h \in A^+$ and $p \in A^*$. If $(w, v) \in \xi$ and w is meagre, then v is meagre.*

Corollary 5.5 *Let Z be a strongly separating set of words such that $\varepsilon \notin Z$ and $\psi(Z) \subseteq A$. Then:*
  (i) *A word v is meagre, provided that $(w, v) \in \xi$ for a meagre word w.*
  (ii) *If $\psi(z) \neq z$ for every $z \in Z$, then every pseudomeagre word is meagre.*
  (iii) *If there exists no pair $(z_1, z_2) \in Z \times Z$ such that $\psi(z_1) = z_2$ and $\psi(z_2) = z_1$, then every pseudoreduced pseudomeagre word is reduced.*

6. Confluency

Proposition 6.1 *Assume that for all $u, v, w \in A^*$ such that $(u, v) \in \rho$, $(u, w) \in \rho$, $(v, w) \notin \rho$, $(w, v) \notin \rho$ and $v \neq w$ there exists at least one $x \in A^*$ with $(v, x) \in \rho$ and $(w, x) \in \rho$ (then $v \neq x \neq w$ and $v \neq u \neq w$). Then the relation ξ is confluent (i. e., for all $(p, q) \in \xi$, $(p, r) \in \xi$ there exists at least one $s \in A^*$ with $(q, s) \in \xi$ and $(r, s) \in \xi$).*

Proof. It follows easily from our assumption that the relation λ is confluent. We have to show that the transitive closure ξ of λ is confluent as well.

Let $u_0, u_1, \ldots, u_m$ and $v_0, v_1, \ldots, v_n$ be a λ-sequences such that $u_0 = v_0$.

Assume first that $m = 1$. Proceeding by induction, we find words $r_1, \ldots, r_n$ in A^* in the following way: Since λ is confluent, we have $(u_1, r_1) \in \lambda$ and $(v_1, r_1) \in \lambda$ for some $r_1 \in A^*$. Now, if $1 \le j < n$ and $r_1, \ldots, r_j$ are found such that $u_1, r_1, r_2, \ldots, r_j$ is a λ-sequence and $(v_1, r_1) \in \lambda$, $(v_2, r_2) \in \lambda$, $\ldots$, $(v_j, r_j) \in \lambda$, then $(r_j, r_{j+1}) \in \lambda$ and $(v_{j+1}, r_{j+1}) \in \lambda$ for some $r_{j+1} \in A^*$. Consequently, by induction, $(v_n, r_n) \in \lambda$ and $u_1, r_1, \ldots, r_n$ is a λ-sequence. Thus $(u_m, r_n) = (u_1, r_n) \in \xi$ and $(v_n, r_n) \in \xi$.

In the general case, we proceed by induction on $m + n$. Due to the preceding part of the proof, we can assume that $m \ge 2$. Then $(u_{m-1}, r) \in \xi$ and $(v_n, r) \in \xi$ for some $r \in A^*$. Furthermore, $(u_{m-1}, u_m) \in \lambda$, and hence $(u_m, s) \in \xi$ and $(r, s) \in \xi$ for at least one $s \in A^*$. Consequently, $(u_m, s) \in \xi$ and $(v_n, s) \in \xi$. $\qquad\square$

Remark 6.2 Assume that ξ is confluent (see 6.1). If $(u, v) \in \tau$ and $(u, w) \in \tau$, then $(v, r) \in \xi$ and $(w, r) \in \xi$ for some $r \in A^*$. If $v \ne r \ne w$, then $(v, r) \in \tau$ and $(w, r) \in \tau$. If $v = r \ne w$, then $(w, v) \in \tau$. If $v \ne r = w$, then $(v, w) \in \tau$. The final case is $v = r = w$ (cf. 6.1).

Remark 6.3 Let the assumption of 6.1 be satisfied and let $w \in A^*$ be such that $(x, w) \in \tau$ whenever $(w, x) \in \rho$. We show that w is quasireduced. Indeed, if $w = w_0, w_1, \ldots, w_m = x$ is a ρ-sequence, we show by induction on m that $(x, w) \in \tau$. To this purpose, we can assume that $x \ne w$. The case $m = 1$ is clear. Let $m \ge 2$. We have $(w_{m-1}, w_m) \in \tau$ by induction and $(w_{m-1}, x) \in \rho$. Proceeding similarly as in the proof of 6.1, we find a word $r \in A^*$ such that $(w, r) \in \lambda$ and $(x, r) \in \xi$. Then $(r, w) \in \xi$, and hence $(x, w) \in \xi$. Since $x \ne w$, we get $(x, w) \in \tau$.

Proposition 6.4 *Let $Z \subseteq A^+$ be a strongly separating set. Then:*
  (i) *The relation ξ is confluent.*
  (ii) *If $(u, v) \in \tau$ and $(u, w) \in \tau$, then either $(v, r) \in \tau$ and $(w, r) \in \tau$ for some $r \in A^*$ or $(v, w) \in \tau$ or $(w, v) \in \tau$ or $v = w$.*

Proof. Combine I.7.11, 6.1 and 6.2. $\qquad\square$

Proposition 6.5 *Let $Z \subseteq A^+$ be a strongly separating set and let ψ be strictly length-decreasing. Then for every $w \in A^*$ there exists a uniquely determined reduced word r such that $(w, r) \in \xi$.*

Proof. Combine 4.8 and 6.4. $\qquad\square$

Lemma 6.6 *Let $Z \subseteq A^+$ be a strongly separating set and let ψ be strictly length-decreasing. If $(u_1 u_2 \cdots u_m, r) \in \xi$, $(u_i, v_i) \in \xi$, $1 \le i \le m$, and r is reduced, then $(v_1 v_2 \cdots v_m, r) \in \xi$.*

Proof. We have $(u_1 u_2 \cdots u_m, v_1 v_2 \cdots v_m) \in \xi$ and the rest follows from 6.4. $\qquad\square$

7. Regularity

We will say that the replacement relation ρ (or the pair (Z, ψ)) is *regular* if $m = n$ whenever $w_0, w_1, \ldots, w_m$ and $v_0, v_1, \ldots, v_n$ are ρ-sequences with $w_0 = v_0$ and $w_m = v_n$. In such a case, we put $(\text{dist}_{(Z,\psi)}(w_0, w_m) =)\ \text{dist}(w_0, w_m) = m$.

Lemma 7.1 *Assume that ρ is regular. If $(u, v) \in \tau$ and $(v, w) \in \tau$, then $\text{dist}(u, w) = = \text{dist}(u, v) + \text{dist}(v, w)$.*

Proof. Easy. □

Remark 7.2 Assume that ρ is regular. Then τ is irreflexive, and hence τ is a stable sharp ordering on A^* by 3.3. Now, setting $\text{dist}(w, w) = 0$, we have $\text{dist}(u, v)$ for all $(u, v) \in \xi$. Clearly, $\text{dist}(u, w) = \text{dist}(u, v) + \text{dist}(v, w)$ for all $(u, v) \in \xi$ and $(v, w) \in \xi$.

Lemma 7.3 *Assume that for all $u, v, w \in A^*$ such that $(u, v) \in \rho$, $(u, w) \in \rho$ and $v \neq w$ there is at least one $r \in A^*$ with $(v, r) \in \rho$ and $(w, r) \in \rho$. If $u_0, u_1, \ldots, u_m$ and $v_0, v_1, \ldots, v_n$ are ρ-sequences with $u_0 = v_0$, $u_m = v_n$ and u_m is reduced, then $m = n$*

Proof. We will proceed by induction on $m+n$. We have $m+n \geq 2$ and, if $m+n = 2$, then $m = n = 1$. Henceforth, assume that $1 \leq n \leq m$ and $2 \leq m$.

If $u_1 = v_1$, then $n \geq 2$, since v_n is reduced. Now, $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ are ρ-sequences of length $m - 1$ and $n - 1$, resp. Then $m - 1 = n - 1$ by induction, and so $m = n$.

It remains to consider the case $u_1 \neq v_1$. According to our assumption, there is $r_1 \in A^*$ with $(u_1, r_1) \in \rho$ and $(v_1, r_1) \in \rho$. Since v_n is reduced, we have $n \geq 2$ and, proceeding similarly, we find an index $1 \leq k < n$ and words $r_1, \ldots, r_k$ such that $(r_i, r_{i+1}) \in \rho$ for every $1 \leq i < k$, $(v_j, r_j) \in \rho$ for every $1 \leq j \leq k$ and $r_k = v_{k+1}$ (use again the fact that v_n is reduced). Clearly, $u_1, r_1, r_2 \ldots, r_{k-1}, v_{k+1}, \ldots, v_n$ and $u_1, \ldots, u_m$ are ρ-sequences of length $n - 1$ and $m - 1$, resp. Thus $n - 1 = m - 1$ by induction and we get $m = n$. □

Lemma 7.4 *Let the assumptions of 7.3 be satisfied. If $u_0, u_1, \ldots, u_m$ and $v_0, v_1, \ldots, v_n$ are ρ-sequences such that $u_0 = v_0$, $u_m = v_n$ and $(u_m, r) \in \xi$ for at least one reduced word $r \in A^*$, then $m = n$.*

Proof. If $u_m = r$, then u_m is reduced and the rest follows from 7.3. If $u_m \neq r$, then $(u_m, r) \in \tau$ and there is a ρ-sequence $w_0, w_1, \ldots, w_k$ such that $u_m = w_0$ and $r = w_k$. Now, $u_0, u_1, \ldots, u_m, w_1, \ldots, w_k$ and $v_0, v_1, \ldots, v_n, w_1, \ldots, w_k$ are ρ-sequences and we have $m + k = n + k$. Then $m = n$. □

Corollary 7.5 *Let the assumptions of 7.3 be satisfied and let for every $u \in A^*$ there exists at least one reduced word $r \in A^*$ with $(u, r) \in \xi$. Then the relation ρ is regular.*

Proposition 7.6 *Assume that $Z \subseteq A^+$ is strongly separating set and that for every $u \in A^*$ there exists at least one reduced word $r \in A^*$ with $(u, r) \in \xi$. Then the relation ρ is regular.*

Proof. Combine I.7.11 and 7.5. □

Theorem 7.7 *Assume that $Z \subseteq A^+$ is strongly separating set and that the mapping ψ is strictly length decreasing. Then the relation ρ is regular.*

Proof. Combine 4.8 and 7.6. □

R e f e r e n c e s

[1] FLAŠKA, V., KEPKA, T. AND KORTELAINEN, J.: *On separating sets of words I*, Acta Univ. Carolin. Math. Phys., **49/1** (2008), 33–51.
[2] FLAŠKA, V., KEPKA, T. AND KORTELAINEN, J.: *On separating sets of words II*, Acta Univ. Carolin. Math. Phys., **50/1** (2009), 15–28.