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Grupoids and the Associative Law VIIB: SH-Grupoids and Simply Generated Congruences

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Szász-Hájek groupoids (shortly SH-groupoids) are groupoids containing just one non-associative (ordered) triple of elements. These groupoids were studied by G. Szász in [10] and [11], P. Hájek in [2] and [3], and later in [6], [7], [8], and [9]. Minimal SH-groupoids of type (a, a, a) and their semigroup distances were described in [6]. The present paper is a continuation of [13] and [14]. Congruences generated by at most three-element set are used and semigroup distances of the corresponding minimal SH-groupoids of type (a, b, a) are investigated.

1. Preliminaries

A groupoid $G(\cdot)$ is called *SH-groupoid* if the set $\{(a, b, c) \in G^{(3)} \mid a \cdot bc \neq ab \cdot c\}$ of non-associative triples contains just one element.

Let $H(\cdot)$ be a subgroupoid of an SH-groupoid $G(\cdot)$ having the non-associative triple (a, b, c). Then either $\{a, b, c\} \in H$ and $H(\cdot)$ is an SH-groupoid having the non-associative triple (a, b, c), or $H(\cdot)$ is a semigroup in the opposite case.

An SH-groupoid $G(\cdot)$ is called *SH-groupoid of type* (a, b, c) if there are elements $a, b, c \in G$ such that (a, b, c) is the only non-associative (ordered) triple of the groupoid $G(\cdot)$.

An SH-groupoid $G(\cdot)$ of type (a, b, a) is called *minimal SH-groupoid of type* (a, b, a) if $a \neq b$ and the groupoid $G(\cdot)$ contains no SH-groupoid $H(\cdot)$ as a proper

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subgroupoid. It is obvious that each minimal SH-groupoid $G(\cdot)$ of type (a, b, a) is a groupoid generated by the two-element set $\{a, b\}$.

Let κ be a congruence on SH-groupoid $G(\cdot)$ and let (a, b, c) be the corresponding non-associative triple. Then either $(a \cdot bc, ab \cdot c) \in \kappa$ and then the corresponding groupoid $G(\cdot)/\kappa$ is a semigroup, or $(a \cdot bc, ab \cdot c) \notin \kappa$ and then the corresponding groupoid $G(\cdot)/\kappa$ is an SH-groupoid.

1.1 Szász's theorem. *Let $G(\cdot)$ be an SH-groupoid and let (a, b, c) be the only non-associative triple of $G(\cdot)$. If $x, y \in G$ are such that $x \cdot y \in \{a, b, c\}$ then $x \cdot y \in \{x, y\}$.*

Let $G(\diamond)$ and $G(*)$ be groupoids having the same underlying set G. Then dist(G($\diamond$), G($*$)) denotes $\mathrm{card}\{(x, y) \in G^2 \mid x \diamond y \neq x * y\}$.

Let $G(\cdot)$ be a groupoid. Let sdist(G($\cdot$)) be the minimum of cardinal numbers dist(G($\cdot$), G($*$)), where $G(*)$ runs through the set of all semigroups having the underlying set G. The number sdist(G($\cdot$)) is called *semigroup distance* of the groupoid $G(\cdot)$.

2. Minimal free SH-groupoids of the type (a, b, a)

Let $F(\cdot)$ denote the absolutely free groupoid generated by a two-element set $\{a, b\}$. If $G(\cdot)$ is an arbitrary minimal SH-groupoid of type (a, b, a) then there is a congruence κ on $F(\cdot)$ such that $G(\cdot)$ is an isomorphic image of the groupoid $F/\kappa(\cdot)$. Of course, the corresponding congruence κ satisfies the conditions $(a \cdot ba, ab \cdot a) \notin \kappa$ and $(x \cdot yz, xy \cdot z) \in \kappa$ for every $x, y, z \in F$, $(x, y, z) \neq (a, b, a)$.

Further, let κ denote the least congruence on the absolutely-free groupoid $F(\cdot)$ satisfying the conditions $(a \cdot ba, ab \cdot a) \notin \kappa$ and, $(x \cdot yz, xy \cdot z) \in \kappa$ for every $x, y, z \in F$, $(x, y, z) \neq (a, b, a)$. Denote, for simplicity, by $U(\cdot)$ the corresponding groupoid $F/\kappa(\cdot)$, where $a, b \in U$ will denote the images of a, b in the natural projection of F onto U.

2.1. Lemma. *If $v, w \in F$ are arbitrary elements such that $(a, v) \in \kappa$ and $(b, w) \in \kappa$ then $a = v$ and $b = w$. Consequently, $a \neq xy \neq b$ in $U(\cdot)$ for all $x, y \in U$.*

Proof. Suppose that there is $v \in F$ such that $a \neq v$ and $(a, v) \in \kappa$. Then $(a \cdot ba, v \cdot bv) \in \kappa$, $(ab \cdot a, vb \cdot v) \in \kappa$, and hence $(a \cdot ba, ab \cdot a) \in \kappa$, a contradiction.

The second case $(b, w) \in \kappa$ is similar to that one and the rest is clear.

2.2. Lemma. *Let $v, x, y, z \in U$. Then $v \cdot (xy \cdot z) = (v \cdot xy) \cdot z$ and $v \cdot (x \cdot yz) = vx \cdot yz = (vx \cdot y) \cdot z$.*

Proof. $U(\cdot)$ is an SH-groupoid of type (a, b, a) and $a \neq xy \neq b$ for every two $x, y \in U$. The rest is clear.

2.3. Lemma. *For each positive integer $n > 3$ and every $x_1, x_2 \dots, x_n \in \{a, b\}$, $x_1 \cdot (x_2 \dots x_n) = x_1 x_2 \cdot (x_3 \dots x_n) = \dots = (x_1 x_2 \dots x_{n-l}) \cdot x_n$ holds in U.*

Proof. Obvious.

2.4 Construction. Consider a two-element set a, b and let S be the free semigroup of words generated by the set $\{a, b\}$. Further, let $g \notin S$, $U = S \cup \{g\}$, $S_n = \{w \in S \mid \lambda(w) = n\}$ be the set of all words in S of length n and put $\lambda(g) = 3$. On each set S_n, we have a lexicographic ordering given by $a < b$. Now, we put $U_n = S_n$ for each $n \neq 3$ and $U_3 = S_3 \cup \{g\}$. Finally, define a binary operation $*$ on U by $ab * a = g$, $u * v = uv$ whenever $u, v \in S$, $ab \neq u$, $a \neq v$, and $g * u = abau$, $u * g = uaba$ for every $u \in S$. However, from now on, for the sake of simplicity, we shall denote this operation on U by $\cdot$ instead of $*$.

2.5 Lemma. *The groupoid $U(\cdot)$ is (up to isomorphism) the only minimal-free SH-groupoid of type (a, b, a).*

Proof. It follows immediately from the construction 2.5

2.6 Lemma. $\mathrm{sdist}(\mathrm{U}(\cdot)) = 1$.

Proof. Define on U a new binary operation $\star$ such that $c \star a = f \neq g = c \cdot a$ and $x \star y = x \cdot y$ whenever $(x, y) \neq (c, a)$. Let $x, y \in U$ are such that $x \neq a \neq z$. Then:
(i) $(a \star b) \star a = (a \cdot b) \star a = c \star a = f = a \cdot d = a \star d = a \star (b \cdot a) = a \star (b \star a)$;
(ii) $x \star (c \star a) = x \star f = x \cdot f = x \cdot ad = x \cdot (a \cdot ba) = xa \cdot ba = (xa \cdot b) \cdot a = (x \cdot ab) \cdot a = xc \cdot a = xc \star a = (x \cdot c) \star a = (x \star c) \star a$;
(iii) $(c \star a) \star z = f \star z = f \cdot z = (a \cdot d) \cdot z = a \cdot (d \cdot z) = a \cdot (ba \cdot z) = a \cdot (b \cdot az) = ab \cdot az = c \cdot az = c \star az = c \star (a \cdot z) = c \star (a \star z)$.

If $x, y, z \in U$ are such that $(x, y, z) \neq (a, b, a)$ and $(x, c, a) \neq (x, y, z) \neq (c, a, z)$. Then $(x \star y) \star z = (x \cdot y) \star z = xy \cdot z = x \cdot yz = x \star yz = x \star (y \star z)$.
It means that $U(\star)$ is a semigroup having $\mathrm{dist}(\mathrm{U}(\cdot), \mathrm{U}(\star)) = 1$ and the rest is clear.

2.7 Theorem. *Let $G(\cdot)$ be an arbitrary minimal SH-groupoid of the type (a, b, a) such that the equation $x \cdot y = a \cdot ba$ or the equation $x \cdot y = ab \cdot a$ has only one solution in $G(\cdot)$. Then* $\mathrm{sdist}(\mathrm{G}(\cdot)) = 1$.

Proof. Suppose, for example, that the equation $x \cdot y = a \cdot ba$ has only one solution in $G(\cdot)$. Define on $G(\cdot)$ a binary operation $\star$ such that $c \star a = a \cdot ba$ and $x \star y = x \cdot y$ whenever $(x, y) \neq (c, a)$. Then $a \star (b \star a) = a \star ba = a \star d = a \cdot ba = c \star a = ab \star a = (a \star b) \star a$. It is possible to check that the remaining triples are associative in $G(\star)$. Thus, $G(\star)$ is a semigroup and the rest is clear.

2.8 Lemma. *If each of the equations $x \cdot y = a \cdot ba$ and $x \cdot y = ab \cdot a$ has only one solution in $G(\cdot)$ then each of the equations $x \cdot y = ab$ and $x \cdot y = ba$ has also only one solution in $G(\cdot)$.*

Proof. Obvious.

2.9 Lemma. *If there exists minimal SH-groupoid $G(\cdot)$ such that* $\mathrm{dist}(\mathrm{G}(\cdot)) \neq 1$ *then each of the equations $xy = a \cdot ba$ and $xy = ab \cdot a$ has to have at least two different solutions.*

Proof. Obvious.

3. Minimal SH-groupoids and congruences generated by one-element set

3.1 Construction. Let $U(\cdot)$ be the minimal free SH-groupoid constructed in 2.5. Consider the least congruence κ on $U(\cdot)$ such that $(ab, ba) \in \kappa$. Denote by $V(\cdot)$ the corresponding groupoid $U(\cdot)/\kappa$.

The groupoid $V(\cdot)$ contains an infinite semigroup $A(\cdot)$ generated by the one-element set $\{a\}$ as a proper subgroupoid. Similarly, $V(\cdot)$ contains an infinite semigroup $B(\cdot)$ generated by the one-element set $\{b\}$ as a proper subgroupoid.

Of course, the underlying sets A and B of these semigroups $A(\cdot)$ and $B(\cdot)$ are disjoint. Put $W = U - (A \cup B)$.

We have $ab = c = ba$. Therefore, the groupoid $V(\cdot)$ contains only one element $w \in W$ having the length 2 and this is just the element c. Put $W_2 = \{c\}$. Further, it holds

(i) $f = ad = a \cdot ba = a \cdot ab = a^2 \cdot b$;

(ii) $g = ca = ab \cdot a = ba \cdot a = b \cdot a^2$;

(iii) $h = a \cdot b^2 = ab \cdot b = ba \cdot b = b \cdot ab = b \cdot ba = b^2 \cdot a$.

It is clear that the elements $f, g, h \in W$ are pair-wise diferent and put $W_3 = \{f, g, h\}$.

Each word $w \in W$ of the length $n > 3$ can be written in $V(\cdot)$ in the form $a^k \cdot b^{n-k}$ for certain positive integer $0 < k < n$. Denote as $w_{n,k}$ the word $a^k \cdot b^{n-k}$ for each $1 < k < n$ and let W_n denote all words $w_{n,k}$ of the length n. Then we have $V = A \cup B \cup W_2 \cup \ldots W_n \cup W_{n+1} \cup \ldots$.

3.2 Lemma. *The groupoid $V(\cdot)$ is an SH-groupoid of type (a, b, a) generated by the two-element set $\{a, b\}$ and having* $\mathrm{sdist}(V(\cdot)) = 2$.

Proof. The congruence κ is generated by the one-element set $\{(ab, ba)\}$. Further, $(a \cdot ba, ab \cdot a) \notin \kappa$. Thus, the groupoid $V(\cdot)$ is a minimal SH-groupoid of the type (a, b, a). It contains an infinite semigroup $A(\cdot)$ as a proper subgroupoid. Therefore, the underlying set V of the SH-groupoid $V(\cdot)$ is an infinite countable set. Furthermore, $\lambda(x \cdot y) = \lambda(x) + \lambda(y)$ for every $x, y \in V$.

Define on V a binary operation $\star$ such that $b \star a^2 = c \star a = f \neq g = c \cdot a = b \cdot a^2$ and $x \star y = x \cdot y$ in the remaining cases. Then $a \star (b \star a) = a \star ba = a \cdot d = f = c \star a = (a \cdot b) \star a = (a \star b) \star a$. It is easy to see that the remaining triples $(x, y, z) \in V^{(3)}$ are associative in $V(\star)$. Therefore $V(\star)$ is a semigroup, and thus $\mathrm{sdistV}(\cdot) \leq 2$.

Finally, we have to prove that $\mathrm{sdist}(G(\cdot)) \neq 1$. If the opposite takes place then there is a semigroup $V(\nabla)$ having $\mathrm{dist}(V(\cdot), V(\nabla)) = 1$. Therefore, just one of the following four conditions takes place: (i) $a\nabla b \neq a \cdot b = c$, (ii) $b\nabla a \neq b \cdot a = c$, (iii) $a\nabla b = a \cdot b = c = b \cdot a = b\nabla a$ and $a\nabla c \neq a \cdot c = f$ or (iv) $a\nabla b = a \cdot b = c = b \cdot a = b\nabla a$ and $c\nabla a \neq c \cdot a = g$. The rest of the proof needs just a tedious checking.

3.3 Lemma. *There is at least one minimal SH-groupoid $G(\cdot)$ of type (a, b, a) having* $\mathrm{sdist}(G(\cdot)) = 2$ *and such that $a \neq x \cdot y \neq b$ for every $x, y \in G$.*

Proof. It follows immediately from 3.2.

3.4 Proposition. *Let ϱ be an arbitrary congruence on the SH-groupoid $V(\cdot)$. Then* sdist(V/($\varrho(\cdot)$)) ≤ 2.

Proof. We have $ab = c = ba$. First, suppose that $(a^n, c) \in \varrho$ for some positive integer $n > 1$. Then $f = ac = a \cdot a_n = a^{n+1} = a^n \cdot a = ca = g$, a contradiction. Further, suppose that $(b^k, c) \in \varrho$ for some positive integer $k > 1$. Then $f = ac = = a \cdot b^k = ab \cdot b^k - 1 = b^k \cdot b^k - 1 = b^{2k-1} = b^{k-1} \cdot b^k = b^{k-1} \cdot c = b^{k-1} \cdot ba = b^k \cdot a = = ca = g$, a contradiction. Thus a, a^2, b, b^2, c are pair-wise different elements of V/ϱ.

Suppose that there is $u \in V$ having $\lambda(u) > 2$. If $u = x \cdot vw$, where $x \in \{a, b\}$ and $v, w \in V$, we obtain $f = ac = a \cdot (x \cdot vw = ax \cdot vw = w_{n,k} = (x \cdot vw) \cdot a = ca = g$, a contradiction. In the remaining case, $u = g = ca$ and $f = ac = a \cdot g = a \cdot ca = = ac \cdot a = a^3 b = ca \cdot a = ga = ca = g$, a contradiction again.

It was proved that the equation $xy = c$ has in $V/\varrho(\cdot)$ just two different solutions (a, b) and (b, a). Now, define on V/ϱ a new binary operation $\star$ such that $b \star a^2 = = c \star a = f \neq g = c \cdot a = b \cdot a^2$ and $x \star y = x \cdot y$ whenever $x, y \in V/\varrho$ and $(b, a^2) \neq \neq (x, y) \neq (c, a)$. It is tedious but straightforward to check that $V/\varrho(\star)$ is a semigroup. Therefore, sdist(V/$\varrho(\cdot)$) ≤ 2.

3.5 Construction. Consider the eight-element set $H = \{a, b, c, d, e, f, g, h\}$ and define on H a binary operation $\cdot$ by the following table:

H	a	b	c	d	e	f	g	h
a	c	d	h	f	h	h	h	h
b	d	e	g	h	h	h	h	h
c	h	f	h	h	h	h	h	h
d	g	h	h	h	h	h	h	h
e	h	h	h	h	h	h	h	h
f	h	h	h	h	h	h	h	h
g	h	h	h	h	h	h	h	h
h	h	h	h	h	h	h	h	h

3.6 Lemma. *$H(\cdot)$ is a minimal SH-groupoid of the type (a, b, a).*

Proof. It follows immediately from the construction 3.5 that the groupoid $H(\cdot)$ is generated by the two-element set $\{a, b\}$. Further, $a \neq x \cdot y \neq b$ for every $x, y \in H$. Finally, it is easy to check that the groupoid $H(\cdot)$ contains only one non-associative triple (a, b, a).

3.7 Lemma. sdist(H($\cdot$)) ≤ 2.

Proof. Define on H a new binary operation $\star$ such that $b \star c = f \neq g = b \cdot c$, $d \star a = f \neq g = d \cdot a$ and $x \star y = x \cdot y$ in the remaining cases. Then $(a \star b) \star a = = d \star a = f = a \cdot d = a \star d = a \star (b \cdot a) = a \star (b \star a)$. It is easy to check that the remaining triples of $H^{(3)}$ are associative. Thus $H(\star)$ is a semigroup and the rest is clear.

3.8 Lemma. sdist(H(·)) $\neq 1$.

Proof. Suppose, on the contrary, that there is a semigroup $H(\nabla)$ having the same underlying set H and such that dist(H(·), H(∇)) $= 1$. Then either $a \cdot b \neq a\nabla b$ or $b \cdot a \neq b\nabla a$.
(i) If $v = a\nabla b \neq a \cdot b = d$ then $v \cdot a = v\nabla a = a\nabla b\nabla a = a\nabla(b \cdot a) = a\nabla d = a \cdot d = f$. However, the equation $x \cdot a = f$ has no solution in $H(\cdot)$, a contradiction.
(ii) If $w = b\nabla a \neq b \cdot a = d$ then $a \cdot w = a\nabla w = a\nabla(b\nabla a) = (a\nabla b)\nabla a = (a \cdot b)\nabla a = = d\nabla a = d \cdot a = g$. However, the equation $a \cdot y = g$ has no solution in $H(\cdot)$, a contradiction.

3.9 Corollary. sdist(H(·)) $= 2$.

4. Milan's minimal SH-groupoid and its semigroup distance

Consider the minimal free SH-groupoid $U(\cdot)$ of type (a, b, a) constructed in 2.1 and let κ be an arbitrary congruence on $U(\cdot)$ containing the ordered pairs (ab, a^2) and (ba, a^4). Then $a^2 \cdot ba = a \cdot (a \cdot ba) = a^2 \cdot ba = a^2 \cdot ba = a^2 \cdot a^4 = a^6$ and $a^2 \cdot ba = = a \cdot (ab \cdot a) = (a \cdot a^2) \cdot a = a^4$ holds in $U/\kappa(\cdot)$.

Thus the subgroupoid $A(\cdot)$ generated in $U(\cdot)$ by the one-element set $\{a\}$ is a five-element semigroup satisfying the condition $a^6 = a^4$. Further, suppose that κ is the least congruence on $U(\cdot)$ containing the ordered pairs (ab, a^2), (ba, a^4) and $b^5 = b^3$. Of course, the corresponding groupoid $U(\cdot)/\kappa$ is either a semigroup or it is a minimal SH-groupoid of type (a, b, a).

4.1 Construction. Put, for simplicity, $a^2 = e$, $a^3 = f$, $a^4 = g$, $a^2 = h$ and $b^2 = r$, $b^3 = s$, $b^4 = t$ and denote by M the nine-element set $\{a, e, f, g, h, b, r, s, t\}$. Define on M a binary operation $\cdot$ by the following table:

M	a	e	f	g	h	b	r	s	t
a	e	f	g	h	g	e	f	g	h
e	f	g	h	g	h	f	g	h	g
f	g	h	g	h	g	g	h	g	h
g	h	g	h	g	h	h	g	h	g
h	g	h	g	h	g	g	h	g	h
b	g	h	g	h	g	r	s	t	s
r	h	g	h	g	h	s	t	s	t
s	g	h	g	h	g	t	s	t	s
t	h	g	h	g	h	s	t	s	t

4.2 Lemma. *$M(\cdot)$ is a minimal SH-groupoid of type (a, b, a).*

Proof. It follows immediately from the construction 4.1 that the groupoid $M(\cdot)$ is generated by the two-element set $\{a, b\}$. Further, $a \neq x \cdot y \neq b$ for every $x, y \in M$. Finally, it is tedious but straightforward to check that the groupoid $M(\cdot)$ contains only one non-associative triple (a, b, a).

4.3 Lemma. sdist(M(·)) ≤ 3.

Proof. Define on M a new binary operation $\star$ such that $a \star b = g \neq e = a \cdot b$, $e \star b = h \neq f = e \cdot b$, $a \star r = h \neq f = a \cdot r$ and $x \star y = x \cdot y$ in the remaining cases.
(i) It is obvious that $a \star (b \star a) = a \star (b \cdot a) = a \star g = a \cdot g = h$ and $h = g \cdot a = = g \star a = (a \star b) \star a$. Further, $a \star (a \star b) = a \star g = ag = h$ and $h = e \star b = = (a \cdot a) \star b = (a \star a) \star b$. Similarly, i$a \star (b \star b) = a \star b \cdot b = a \star r = h$ and $h = g \cdot b = g \star b = (a \star b) \star b$. Finally, if $x, y, z \in M$ are such that $(x, y) \neq \neq (a, b), (a, r), (e, b) \neq (y, z)$ and $e \neq x \cdot y \neq r$ then $x \star (y \star z) = x \cdot y \cdot z = x \cdot y \cdot z = = (x \star y) \star z$.
(ii) In this part of the proof, let $x \in M$, $x \neq a$ and consider the triple (x, a, b). It is easy to see that $x \star (a \star b) = x \star g = x \cdot g = h$ for each $x \in \{a, f, h, b, t\}$. Now, $(f \star a) \star b = (f \cdot a) \star b = g \star b = g \cdot b = h$ and $(h \star a) \star b = (h \cdot a) \star b = g \star b = = g \cdot b = h$. Further, we have $(b \star a) \star b = (b \cdot a) \star b = g \star b = g \cdot b = h$ and $(t \star a) \star b = (t \cdot a) \star b = g \star b = g \cdot b = h$.

Similarly, $e \star (a \star b) = e \star g = e \cdot g = g = f \cdot b = (e \cdot a) \star b = (e \star a) \star b$ and $g \star (a \star b) = g \star g = g \cdot g = g = h \cdot b = (g \cdot a) \star b = (g \star a) \star b$. Finally, $s \star (a \star b) = s \star g = s \cdot g = g = h \cdot b = (s \cdot a) \star b = (s \star a) \star b$ and $x \star (a \star b) = = (x \star a) \star b$.
(iii) In this part of the proof, let $z \in M$, $b \neq z$ and consider the triple (a, b, z). It is obvious that $(a \star b) \star z = g \star z = g \cdot z \in \{g, h\}$. Furthermore, $g \cdot z = g$ if and only if $z \in \{e, g, r, t\}$. But then we have $a \star (b \star e) = a \star (b \cdot e) = a \star h = a \cdot h = g$ and $a \star (b \star g) = a \star (b \cdot g) = a \star h = a \cdot h = g$. Further, we have $a \star (b \star r) = = a \star (b \cdot r) = a \star s = a \cdot s = g$ and $a \star (b \star t) = a \star (b \cdot t) = a \star s = a \cdot s = g$.

Similarly, $g \cdot z = h$ if and only if $z \in \{a, f, h, b, s\}$. If $z = f$, then we have $a \star (b \star f) = a \star (b \cdot f) = a \star g = h$. If $y = h$ then we obtain $a \star (b \star h) = = a \star (b \cdot h) = a \star g = a \cdot g = h$. Finally, if $z = s$ then there is $a \star (b \star s) = = a \star (b \cdot s) = a \star t = a \cdot s = h$. Thus $a \star (b \star z) = (a \star b) \star z$, because $a \neq x \cdot y \neq b$ for every $x, y \in M$.
(iv) In this part of the proof, let $x \in M$ and consider the triple (x, a, r). It follows from the construction that again $x \star (a \star r) = x \star h = x \cdot h \in \{g, h\}$. Furthermore, $x \cdot h = g$ if and only if $x \in \{a, f, h, b, s\}$. But if $x = a$ then $(a \star a) \star r = = e \star r = e \cdot r = g$. If $x = f$ then $(f \star a) \star r = f \cdot (a \star r) = g \star r = g$. For $x = h$ we have $(h \star a) \star r = (h \cdot a) \star r = g \star r = g$. Similarly, for $x = b$ we have $(b \star a) \star r = (b \cdot a) \star r = g \star r = g$. Finally, for $x = s$ we have $(s \star a) \star r = = (s \cdot a) \star r = g \star r = g$. In the remaining cases, $x \cdot h = h$ whenever $x \in \{e, g, r, t\}$. If $x = e$ then we have $(e \star a) \star r = (e \cdot a) \star r = f \star r = h$. Further, for $x = g$

we have $(g \star a) \star r = (g \cdot a) \star r = h \star r = h \cdot r = h$. Similarly, if $x = r$ then we obtain $(r \star a) \star r = (r \cdot a) \star r = h \star r = h$. If $x = t$ then we have again $(t \star a) \star r = (t \cdot a) \star r = h \star r = h$.

Finally, the equation $x \cdot y = r$ is solvable in $M(\cdot)$ and it has only one solution $(x, y) = (b, b)$. But it was proved in (i) that the triple (a, b, b) is associative.

(v) In this part of the proof, let $x \in M$ and consider the triple (x, e, b). Similarly as above, $x \star (e \star b) = x \star h = x \cdot h \in \{g, h\}$. Now, $x \cdot h = g$ if and only if $x \in$ $\in \{a, f, h, b, s\}$. For $x = a$, we have $(a \star e) \star b = f \star b = g$. Further, $f \star e = h =$ $= h \star e =$ and $h \star b = h \cdot b = g$. Finally, $b \star e = b \cdot e = h$ and $s \star e = s \cdot b = t$. But in both cases we have again $h \star b = g = t \star b$. Thus the triple (x, e, b) is associative.

Similarly, $x \cdot h = h$ if and only if $x \in \{e, g, r, t\}$. Now, if $x = e$ or $x = b$ then $(e \star e) \star b = g \star b = h$ and $(g \star e) \star b = g \star b = h$. Further, if $x = r$ or $x = t$ then $(r \star e) \star b = g \star b = h$ and $(t \star e) \star b = g \star b = h$. Therefore, the triple (x, b, e) is also associative.

Finally, the equation $x \cdot y = e$ is solvable in $M(\cdot)$ and it has just two solutions (a, a) and (a, b). It was proved in (i) that the corresponding triples (a, a, b) and (a, b, b) are also associative.

(vi) In the last part of the proof, let $z \in M$ and consider the triple (e, b, z). Of course, $(e \star b) \star z = (h \star z) = h \cdot z \in \{g, h\}$. Further, $h \cdot z = g$ if and only if $z \in \{a, f, h, b, s$. Now, if $z = a$ then $e \star (b \star a) = e \star g = g$. If $z = f$ then again $e \star (b \star f) = e \star g =$ $= g$. Similarly, if $z = h$ then $e \star (b \star h) = e \star g = g$. If $z = b$ then $e \star (b \star b) =$ $= e \star r = g$. Finally, if $z = s$ then $e \star (b \star s) = e \star g = g$. It means that the triple (e, b, z) is associative.

In the remaining cases $h \cdot z = h$ and thus $y \in \{e, g, r, t$. Now, if $z = e$ then $e \star (b \star e) = e \star h = h$. If $z = g$ then $e \star (b \star g) = e \star h = h$. Further, if $z = r$ then $e \star (b \star r) = e \star s = h$. Finally, if $z = t$ then tagain $e \star (b \star t) = e \star s = h$. Thus the triple (e, b, z) is also associative.

Finally, taking into account the preceding parts of the proof, we see that $M(\star)$ is a semigroup and the rest is clear.

4.4 Lemma. sdist(M(·)) ≠ 1.

Proof. Suppose, on the contrary, that there is a semigroup $M(\nabla)$ having the same underlying set M and such that dist(M(·), M(∇)) = 1. Then either $a \cdot b \neq a \nabla b$ or $b \cdot a \neq b \nabla a$.

(i) If $v = a \nabla b \neq a \cdot b$ then $v \cdot a = v \nabla a = a \nabla b \nabla a = a \nabla (b \cdot a) = a \nabla g = a \cdot g = h$. It follows from this that either $v = g$, or $v = s$. Now, if $v = a \nabla b = g$ then $f = e \cdot b = e \nabla b = a \nabla a \nabla b = a \nabla g = a \cdot g = h$, a contradiction. Further, if $v = a \nabla b = s$ then to be $f = e \cdot b = e \nabla s = a \nabla a \nabla b = a \nabla s = a \cdot s = g$, a contradiction again. Thus $a \nabla b = a \cdot b$.

(ii) If $w = b \nabla a \neq b \cdot a$ then $w \cdot b = w \nabla b = b \nabla a \nabla b = g \nabla b = g \cdot b = h$. But the equation $x \cdot b = h$ has only one solution in $M(\cdot)$ and thus $w = g = b \cdot a$, a contradiction.

4.5 Proposition. sdist(M(·)) ≠ 2.

Proof. Suppose, on the contrary, that the opposite case takes place. Then there is a semigroup $M(\circ)$ such that $\mathrm{dist}(\mathrm{M}(\cdot), \mathrm{M}(\circ)) = 2$. It is obvious that $a \cdot b \neq a \circ b$ or $b \cdot a \neq b \circ a$.

(i) First, suppose that there are $v, w \in M$ such that $v = a \circ b \neq a \cdot b = e$ and $w = b \circ a \neq b \cdot a = g$. Then $x \circ y = x \cdot y$ whenever $(a, b) \neq (x, y) \neq (b, a)$. In particular, $a \cdot v = a \circ v = a \circ a \circ b = e \circ b = e \cdot b = f$. It follows from this that $e \neq v \in \{e, r\}$. Thus we have $a \circ r = f$. But then $f = a \circ r = a \circ (b \cdot b) = a \circ b \circ b = r \circ b = r \cdot b = s$, a contradiction. We have proved that either $v = a \circ b \neq a \cdot b = e$ or $w = b \circ a \neq b \cdot a = g$.

(ii) Further, suppose that there are $u, v \in M$ such that $u = a \circ a \neq a \cdot a = e$ and $v = a \circ b \neq a \cdot b = e$. Then $x \circ y = x \cdot y$ whenever $x, y \in M$ are such that $(a, a) \neq (x, y) \neq (a, b)$. It will be proved that in this case $x \circ y = x \cdot y$ for every $x, y \in A$ where $A = \{a, e, f, g, h\}$.

We have $v \cdot a = v \circ a = (a \circ b) \circ a = a \circ (b \cdot a) = a \circ g = a \cdot g = h$. Thus either $v = g$ or $v = r$. Now, if $v = g$ then $h = g \cdot b = g \circ b = a \circ b \circ b = a \circ (b \cdot b) = a \circ r = a \cdot r = f$, a contradiction. But if $v = r$ then $s = r \cdot b = r \circ b = a \circ b \circ b = a \circ (b \cdot b) = a \circ r = a \cdot r = f$, a contradiction again.

It was proved above that from $a \circ b \neq a \cdot b$ follows that $a \circ a = e = a \cdot a$. But then we have $a \circ e = a \circ a \circ a = e \circ a$. Now, at least one of the equalities $a \circ e = f = a \cdot e$ and $e \circ a = f = e \cdot a$ holds. Therefore, $a \circ e = a \cdot e = f = e \cdot a = e \circ a$. It follows immediately from this that also $a \circ f = a \circ (a \cdot e) = a \circ a \circ e = e \circ e = e \cdot e = g$ and $g = e \circ e = e \circ a \circ a = f \circ a$.

Now, it is possible to prove that $a \circ g = e \circ f = f \circ e = f \circ e = g \circ a = h$. Further, in a similar way, $a \circ h = e \circ g = f \circ f = g \circ e = h \circ a = g$. Then, step by step, we have also $e \circ h = f \circ g = g \circ f = h \circ e = h$, $f \circ h = g \circ g = h \circ f = g$, $g \circ h = h \circ g = h$ and, finally, $h \circ h = h \cdot h = g$. It means that from $a \circ b \neq a \cdot b$ follows $x \circ y = x \cdot y$ for every $x, y \in \{a, e, f, g, h\}$.

(iii) Similarly, suppose that there are $u, w \in M$ such that $u = a \circ a \neq a \cdot a = e$ and $w = b \circ a \neq b \cdot a = g$. Then $x \circ y = x \cdot y$ whenever $(a, a) \neq (x, y) \neq (b, a)$. In particular, $f = e \cdot a = e \circ a = a \circ b \circ a = a \circ w = a \cdot w$. Therefore, $w \in \{e, r\}$. Now, if $w = e$ then we obtain $f = e \cdot a = e \circ a = b \circ a \circ a = b \circ e = b \cdot e = h$, a contradiction. But if $w = r$ then $h = b \cdot e = b \circ e = b \circ a \circ b = r \circ b = r \cdot b = s$, a contradiction again. It follows from this that $a \circ a = e = a \cdot a$.

Now, similarly as in (ii), it needs just a tedious checking to prove that also $x \circ y = x \cdot y$ for every $x, y \in \{a, e, f, g, h\}$.

(iv) Further, suppose that there are $u, w \in M$ such that $u = b \circ b \neq b \cdot b = r$ and $w = b \circ a \neq b \cdot a = g$. Then $x \circ y = x \cdot y$ whenever $(b, a) \neq (x, y) \neq (b, b)$. In particular, with respect to (i), $w \cdot a = w \circ a = a \circ b \circ a = a \circ (b \cdot a) = a \circ e = a \cdot e = f$. Thus either $w = e$ or $w = r$. Further, $f = e \cdot b = e \circ b = a \circ b \circ b = a \circ u = a \cdot u$. It follows from the construction that $r \neq u \in \{e, r\}$ and, thus, $u = e$.

But in this case either $w = e$ or $w = r$. Now, if $w = e$ then we obtain $h = b \cdot e = b \circ e = b \circ b \circ b \circ a = e \circ a = e \cdot a = f$, a contradiction. Similarly, if $w = r$ then $s = b \cdot r = b \circ r = b \circ b \circ a = e \circ a = e \cdot a = f$, a contradiction again.

Thus, from $b \circ a \neq b \cdot a$ follows that $b \circ b = r = b \cdot b$.

(v) Suppose, finally, that there are $u, w \in M$ such that $w = b \circ a \neq b \cdot a = g$, and either $u = b \circ r \neq b \cdot r = s$ or $u = r \circ b \neq r \cdot b = s$.

Then $a \cdot w = a \circ w = a \circ b \circ a = (a \cdot b) \circ a = e \circ a = f$. Thus either $w = e$ or $w = r$. However, we have also $h = b \cdot e = b \circ e = b \circ ()a \cdot b = b \circ a \circ b = w \circ b = w \cdot b$, and therefore $w \in \{g, t\}$ at the same time, but this is impossible. Thus, from $b \circ a \neq b \cdot a$ follows that $b \circ r = s = b \cdot r$.

Similarly, if $r \circ b \neq r \cdot b = s$ then $f = e \cdot a = e \circ a = a \circ b \circ a = a \circ w = a \cdot w$. It means that $w \in \{e, r\}$. But, at the same time, $w \cdot b = w \circ b = b \circ a \circ b = b \circ e = b \cdot e = h$. Thus $w = g$ and this is impossible, a contradiction again.

Furthermore, if $b \circ b = r = b \cdot b$ then $b \circ r = b \circ b \circ b = r \circ b$. Now, at least one of the equalities $b \circ r = b \cdot r$ and $r \circ b = r \cdot b$ has to be valid. Therefore, $b \circ r = s = r \circ b$. Now, it is possible to check that $b \circ s = b \circ b \circ r = r \circ r = r \circ b \circ b = s \circ b$. Further, it follows from this that then also $b \circ t = b \circ b \circ s = r \circ s = s \circ r = s \circ b \circ b = t \circ b = s$. Now, it is tedious but straightforward to check step by step that also $r \circ t = s \circ s = t \circ r = t$, $s \circ t = t \circ s = s$ and $t \circ t = t = t \cdot t$.

Taking into account all that was proved above, we see that there is no semigroup $M(\circ)$ having the same underlying set M as the SH-groupoid $M(\cdot)$ and such that $\mathrm{dist}(\mathrm{M}(\cdot), \mathrm{M}(\circ)) = 2$.

4.6 Theorem. *There are finite minimal SH-groupoids of type* (a, b, a) *having the semigroup distance equal to* 3.

Proof. It follows immediately from 4.1, 4.2, 4.3, 4.4, and 4.5.

5. Comments and open problems

5.1 It has been proved in [6] that the semigroup distance of minimal SH-groupoids of type (a, a, a) is equal to 1 or 2. Furthermore, for each minimal SH-groupoid $G(\cdot)$ of type (a, a, a) having $\mathrm{sdistG}(\cdot) = 2$ there is a congruence κ on $G(\cdot)$ such that $\mathrm{sdist}(\mathrm{G}/\kappa(\cdot)) = 1$.

5.2 It was proved above that there are finite minimal SH-groupoids of type (a, b, a) having the semigroup distance equal to 3. The construction 4.1 is based on a special congruence κ generated by the three-element set $\{(ab, a^2), (ba, a^3)(b^5, b^3)\}$. Is it possible to prove that the result is similar if the corresponding congruence is generated by the three-element set $\{(ab, a^2), (ba, a^3), (b^r, b^s)\}$ for positive integers r, s such that $r > s > 2$?

5.3 Let $U(\cdot)$ be a minimal SH-groupoid of type (a, b, a). Let κ be the congruence on $U(\cdot)$ generated by the two-element set $\{(ab, a^k), (ba, a^m)\}$ for positive integers k, m

such that $k > m + 1$. Show that the corresponding groupoid $U(\cdot)/\kappa$ is a minimal SH-groupoid of type (a, b, a) having infinite underlying set. Is the semigroup distance of this SH-groupoid also equal to 3?

5.4 Does it exist a minimal SH-groupoid of type (a, b, a) having its semigroup distance greater then 3?

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