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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 53 (2012), No. 1, 77--94

Persistent URL: http://dml.cz/dmlcz/143692

A POSTERIORI ERROR ESTIMATES OF THE DISCONTINUOUS GALERKIN METHOD FOR THE HEAT CONDUCTION EQUATION

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Received June 21, 2011

Revised December 2, 2011

We deal with a numerical solution of the nonstationary heat equation with mixed Dirichlet/Neumann boundary conditions. The space semi-discretization is carried out with the aid of the interior penalty Galerkin methods and the backward Euler method is employed for the time discretization. Supposing the shape regularity and local quasi-uniformity, we derive a posteriori upper error bound. This approach is based on the Helmholtz decomposition and the Oswald interpolation operator.

1. Introduction

Our aim is to develop a sufficiently accurate and efficient numerical method for simulations of unsteady flows. A promising technique is a combination of the discontinuous Galerkin finite element method (DGFEM) for the space discretization and the backward difference formula for the time discretization, see [4]. In order to both apply an adaptive algorithm and assess the discretization error, a posteriori error estimates have to be developed.

Within this paper, we focus on simplified model problem, represented by the heat equation, which is discretized by the low order DGFEM and the backward Euler method. Our aim is to derive a posteriori error estimate of the discretization error. For a review, see, e.g., [12]. The approach based on the flux reconstruction from

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This work is a part of the research projects MSM 0021620839 (V. Dolejší). The research of I. Šebestová was supported by the grant SVV-2011-263316.

2000 Mathematics Subject Classification. 65M15

Key words and phrases. Discontinuous Galerkin finite element method, Helmholtz decomposition, Oswald interpolation operator, Euler backward scheme, a residual-based a posteriori upper error bound

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the Raviart-Thomas-Nedelec (RTN) finite element space was presented in [5]. This technique gives a guaranteed (that is, containing no undetermined constants) and fully computable error estimates however it requires a reconstruction of the flux from the RTN space.

Therefore, we were inspired by [10], where Crouzeix-Raviart finite element method is employed for spatial discretization of the heat equations. The derived a posteriori error estimates are based on the Helmholtz decomposition of the gradient of the error.

In this paper, we apply the approach from [10] to the discontinuous Galerkin discretization. Hence, it represents an extension of [11], where homogeneous Dirichlet boundary conditions are considered and a less efficient error indicator was derived. We derive upper bound of the discretization error which is simply computable but it suffers from a presence of undetermined constants.

2. Problem definition

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded multiply connected polyhedral Lipschitz domain with a boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $T > 0$ and $Q_T = \Omega \times (0, T)$. Let us consider the problem:

$$
\begin{array}{rcll}
\partial u/\partial t - \Delta u & = & f & \text{in} \quad Q_T, \\
u & = & u_D & \text{on} \quad \partial\Omega_D \times (0,T), \\
\nabla u \cdot n & = & g_N & \text{on} \quad \partial\Omega_N \times (0,T) \\
u(x,0) & = & u^0(x) & \text{in} \quad \Omega.
\end{array} \tag{1}
$$

We use a standard notation for the Lebesgue, Sobolev and Bochner spaces, see, e.g. [9]. Specially, for a function v in an appropriate space, we will use the following notation: $\|v\|_{k,\omega} = \|v\|_{H^k(\omega)}$, $\|v\|_{\omega} = \|v\|_{L^2(\omega)}$, $\|v\|_{\partial\omega} = \|v\|_{L^2(\partial\omega)}$, $\|v\|_{1/2,\partial\omega} = \|v\|_{H^{1/2}(\partial\omega)}$, $\|v\|_{-1/2,\partial\omega} = \|v\|_{H^{-1/2}(\partial\omega)}$, $|v|_{k,\omega} = |v|_{H^k(\omega)}$, where $\omega \subseteq \Omega$. Recall that $\|v\|_{H^{1/2}(\partial\omega)} := \inf_{\substack{\varphi \in H^1(\omega) \\ \varphi = v \text{ on } \partial\omega}} \|\varphi\|_{1,\omega}$ and $\|v\|_{H^{-1/2}(\partial\omega)} := \sup_{\substack{\varphi \in H^{1/2}(\partial\omega) \\ \varphi \neq 0}} \frac{((v,\varphi))}{\|\varphi\|_{1/2,\partial\omega}}$, where $((\cdot,\cdot))$ denotes duality pairing between spaces $H^{1/2}(\partial\omega)$ and $H^{-1/2}(\partial\omega)$. Moreover, $H^1_D(\Omega) \equiv \{v \in H^1(\Omega);\ v = 0 \quad \text{on} \quad \partial\Omega_D\}$, $H^1_{z,D}(\Omega) \equiv \{v \in H^1(\Omega);\ v = z \quad \text{on} \quad \partial\Omega_D\}$ for a function $z : \partial\Omega_D \to \mathbb{R}$.

3. Discretization

3.1 Time semidiscretization

Let $0 = t_0 < t_1 < \ldots < t_{\bar{N}} = T$ be a partition of the time interval $[0, T]$ and let $\tau_n = t_n - t_{n-1}$, $\tau = \max\{\tau_n : 1 \le n \le \bar{N}\}$. We use the backward Euler scheme to get the *semi-discrete problem*: Find a sequence $\{u^n\}_{1\le n\le \bar{N}}$, $u^n - u^*(t_n) \in H^1_D(\Omega)$ such that

$$(2)\quad \int_\Omega \frac{u^n - u^{n-1}}{\tau_n} v \,\mathrm{d}x + \int_\Omega \nabla u^n \cdot \nabla v \,\mathrm{d}x = \int_\Omega f^n v \,\mathrm{d}x + \int_{\partial\Omega_N} g_N^n v \,\mathrm{d}S \quad \forall v \in H^1_D(\Omega),$$

where $u^*(t_n) \in H^1(\Omega)$ has the trace $u_D^n := u_D(\cdot, t_n)$ on $\partial\Omega_D$, $f^n := f(\cdot, t_n)$ and $g_N^n :=$ $:= g_N(\cdot, t_n)$. For a simplicity, we assume that functions u_D^n, f^n and g_N^n are piecewise linear for each time t_n. The solution of (2) is called the *semi-discrete solution*.

In the rest of the paper till the end of Section 4, the index n, denoting the time level, attains values $n = 1, \dots, \bar{N}$ and we do not mention this fact explicitly.

3.2 Space discretization

We will carry out the space discretization with the aid of the first order DGFEM. On each time level t_n, $n = 1, \dots, \bar{N}$, we consider a family $\{\mathscr{T}_{h,n}\}_{h>0}$ of partitions of Ω into a finite number of closed triangles in 2D and tetrahedra in 3D with mutually disjoint interiors, possibly containing hanging nodes. These partitions are called triangulations hereafter. We assume that the following conditions are satisfied:

$$(3)\quad \text{shape regularity:}\quad \exists C_s > 0 : \ \frac{h_K}{\rho_K} \le C_s \ \forall\, K \in \mathscr{T}_{h,n},$$

$$(4)\quad \text{local quasi-uniformity:}\quad \exists C_H > 0 : \ h_K \le C_H h_{K'} \ \forall K, K' \in \mathscr{T}_{h,n} \text{ sharing a face,}$$

where $h_K = \mathrm{diam}(K)$ for $K \in \mathscr{T}_{h,n}$, ρ_K denotes the diameter of the largest d-dimensional ball inscribed into K, and ∂K denotes the boundary of element K. Moreover, we assume that there exists a triangulation $\widetilde{\mathscr{T}_{h,n}}$ satisfying (3) and (4) which is a refinement of both $\mathscr{T}_{h,n-1}$ and $\mathscr{T}_{h,n}$, $1 \le n \le \bar{N}$ and such that

$$\exists C_{HT} > 0 : \ \forall 1 \le n \le \bar{N}\ \forall K \in \widetilde{\mathscr{T}_{h,n}}\ \forall K' \in \mathscr{T}_{h,n}, K \subset K' : \ \frac{h_{K'}}{h_K} < C_{HT}.$$

This condition reflects simultaneous presence of finite element functions defined on different triangulations and restricts the coarsening rate.

By $\widetilde{\mathscr{F}}^I_{h,n}$, $\widetilde{\mathscr{F}}^D_{h,n}$ and $\widetilde{\mathscr{F}}^N_{h,n}$ we denote the set of all interior faces (edges for $d = 2$), faces (edges for $d = 2$) on $\partial\Omega_D$ and faces (edges for $d = 2$) on $\partial\Omega_N$, respectively. For a simplicity, we put $\widetilde{\mathscr{F}}^{ID}_{h,n} := \widetilde{\mathscr{F}}^I_{h,n} \cup \widetilde{\mathscr{F}}^D_{h,n}$, $\widetilde{\mathscr{F}}^{DN}_{h,n} := \widetilde{\mathscr{F}}^D_{h,n} \cup \widetilde{\mathscr{F}}^N_{h,n}$, $\widetilde{\mathscr{F}}^{IN}_{h,n} := \widetilde{\mathscr{F}}^I_{h,n} \cup \widetilde{\mathscr{F}}^N_{h,n}$, $\widetilde{\mathscr{F}}_{h,n} := \widetilde{\mathscr{F}}^I_{h,n} \cup \widetilde{\mathscr{F}}^D_{h,n} \cup \widetilde{\mathscr{F}}^N_{h,n}$. For each $\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}$ there exist two elements K^L_Γ and K^R_Γ such that $\Gamma \subset \overline{K^L_\Gamma} \cap \overline{K^R_\Gamma}$. We define a unit normal vector $\boldsymbol{n}_\Gamma$ to each $\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}$ so that it points out of K^L_Γ and we set $h_\Gamma := \max(h_{K^L_\Gamma}, h_{K^R_\Gamma})$, $\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}$. Finally, we assume that $\boldsymbol{n}_\Gamma$, $\Gamma \in \widetilde{\mathscr{F}}^{DN}_{h,n}$, has the same orientation as the outward normal to $\partial\Omega$ and we put $h_\Gamma := \max h_{K^L_\Gamma}$, $\Gamma \subset \partial K^L_\Gamma$, $\Gamma \in \widetilde{\mathscr{F}}^{DN}_{h,n}$.

Over the triangulation $\widetilde{\mathscr{T}_{h,n}}$ we define the so-called broken Sobolev space

$$H^s(\Omega, \widetilde{\mathscr{T}_{h,n}}) = \{v; v|_K \in H^s(K)\ \forall K \in \widetilde{\mathscr{T}_{h,n}}\},\ \ s \ge 1$$

equipped with the norm $\|v\|^2_{H^s(\Omega,\widetilde{\mathscr{T}_{h,n}})} = \sum_{K\in\widetilde{\mathscr{T}_{h,n}}} \|v\|^2_{H^s(K)}$. For $v \in H^1(\Omega, \widetilde{\mathscr{T}_{h,n}})$ we define the broken gradient $\nabla_h v$ of v by $(\nabla_h v)|_K := \nabla(v|_K)$ for $\forall K \in \widetilde{\mathscr{T}_{h,n}}$ and use the following notation: v^L_Γ stands for the trace of $v|_{K^L_\Gamma}$ on Γ, v^R_Γ is the trace of $v|_{K^R_\Gamma}$ on Γ, $\langle v\rangle_\Gamma :=$ $:= \frac{1}{2}(v^L_\Gamma + v^R_\Gamma)$, $[v]_\Gamma := v^L_\Gamma - v^R_\Gamma$, $\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}$. Further, for $\Gamma \in \widetilde{\mathscr{F}}^D_{h,n}$, we define v^L_Γ as the trace of $v|_{K^L_\Gamma}$ on Γ, and $\langle v\rangle_\Gamma := [v]_\Gamma := v^L_\Gamma$. If $\boldsymbol{n}_\Gamma$, $[\cdot]_\Gamma$, and $\langle\cdot\rangle_\Gamma$ appear in an integral of the form $\int_\Gamma \ldots dS$, we will omit the subscript Γ and write, respectively, $\boldsymbol{n}$, $[\cdot]$, and $\langle\cdot\rangle$ instead. Finally, we define the space of discontinuous piecewise linear functions

$$S^n_h = \{v;\, v \in L^2(\Omega), v|_K \in P^1(K)\, \forall\, K \in \widetilde{\mathscr{T}_{h,n}}\},$$

where $P^1(K)$ is the space of linear functions on K.

For $u^n_h, v^n_h \in H^2(\Omega, \widetilde{\mathscr{T}_{h,n}})$, we define the forms

$$\begin{aligned}
a^n_h(u^n_h, v^n_h) &:= \sum_{K\in\widetilde{\mathscr{T}_{h,n}}} \int_K \nabla u^n_h \cdot \nabla v^n_h \,\mathrm{d}x - \sum_{\Gamma\in\widetilde{\mathscr{F}}^{ID}_{h,n}} \int_\Gamma \langle \nabla u^n_h \cdot \boldsymbol{n}\rangle [v^n_h]\,\mathrm{d}S \\
&\quad + \theta \sum_{\Gamma\in\widetilde{\mathscr{F}}^{ID}_{h,n}} \int_\Gamma \langle \nabla v^n_h \cdot \boldsymbol{n}\rangle [u^n_h]\,\mathrm{d}S + \sum_{\Gamma\in\widetilde{\mathscr{F}}^{ID}_{h,n}} \int_\Gamma \sigma [u^n_h][v^n_h]\,\mathrm{d}S, \\
\ell^n_h(v^n_h) &:= \int_\Omega f^n v^n_h \,dx + \sum_{\Gamma\in\widetilde{\mathscr{F}}^{N}_{h,n}} \int_\Gamma g^n_N v^n_h \,dS + \theta \sum_{\Gamma\in\widetilde{\mathscr{F}}^{D}_{h,n}} \int_\Gamma \nabla v^n_h \cdot \boldsymbol{n} u^n_D \,\mathrm{d}S \\
&\quad + \sum_{\Gamma\in\widetilde{\mathscr{F}}^{D}_{h,n}} \int_\Gamma \sigma u^n_D v^n_h \,\mathrm{d}S,
\end{aligned} \tag{5}$$

where $u^n_D = u_D(\cdot, t_n)$ and the parameter $\theta = -1$, $\theta = 1$, and $\theta = 0$ corresponds to the symmetric, nonsymmetric, and incomplete variants of the DGFEM, respectively.

Now, we can state the discrete problem: For a given approximation $u^0_h \in S^0_h$ of an initial condition u^0, find a sequence $\{u^n_h\}_{1\le n\le \bar{N}}$, $u^n_h \in S^n_h$ such that

$$\int_\Omega \frac{u^n_h - u^{n-1}_h}{\tau_n} v^n_h \,\mathrm{d}x + a^n_h(u^n_h, v^n_h) = \ell^n_h(v^n_h) \quad \forall v^n_h \in S^n_h. \tag{6}$$

We call the solution of (6) the approximate solution. The reader is referred to [1] for the derivation of discontinuous Galerkin formulation.

4. A posteriori analysis

In the following, we derive a residual-based a posteriori upper error bound on the discretization error based on the Helmholtz decomposition of the gradient of the error. This approach was developed in [10], where the heat equation was solved with the aid of the combination of the Crouzeix-Raviart nonconforming finite elements in space and the backward Euler scheme in time.

In the analysis, we employ the standard results of the finite element theory, namely the *multiplicative trace inequality*

$$\|v\|^2_{\partial K} \le C_M(|v|_{1,K}\|v\|_K + h_K^{-1}\|v\|^2_K) \qquad \forall v \in H^1(K),\ K \in \widetilde{\mathcal{T}}_{h,n}, \tag{7}$$

the *inverse inequality*

$$|v|_{1,K} \le C_I h_K^{-1}\|v\|_K \qquad \forall v \in P^1(K),\ K \in \widetilde{\mathcal{T}}_{h,n}, \tag{8}$$

the *trace inequality*

$$\|\boldsymbol{n}\cdot \operatorname{curl} v\|_{-1/2,\partial K} \le C_T\|\operatorname{curl} v\|_K \qquad \forall v \in (H^1(K))^k,\ K \in \widetilde{\mathcal{T}}_{h,n}, \tag{9}$$

and the *approximation property* of the L^2-projection operator Π_h on S_h^n

$$|v - \Pi_h v|_{i,K} \le C_A h_K^{2-i}|v|_{2-i,K} \qquad \forall v \in H^{2-i}(K),\ K \in \widetilde{\mathcal{T}}_{h,n},\ i = 0, 1, \tag{10}$$

where C_M, C_I, C_T, and C_A are constants independent of K, h and n and $k = 1$ for $d = 2$ and $k = 3$ for $d = 3$. Let us recall that curl operator is defined by

$$\operatorname{curl} v := \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right),\ v : \Omega \to \mathbb{R}^2,\ d = 2,$$
$$\operatorname{curl} v := \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right), v = (v_1, v_2, v_3) : \Omega \to \mathbb{R}^3,\ d = 3.$$

We introduce the space $H(\operatorname{curl}, \Omega) := \{v \in (L^2(\Omega))^k; \operatorname{curl} v \in (L^2(\Omega))^d\}$, where $k = 1$ for $d = 2$ and $k = 3$ for $d = 3$. Moreover, $\operatorname{div}\operatorname{curl}\chi = 0$ for $\chi \in (H^1(K))^k$, meaning the operator div in the weak sense. Finally, $\operatorname{curl}\chi \cdot \boldsymbol{n}$ is meant in the following sense (for the proof see [11]):

Lemma 1 *Let $\Omega \in \mathbb{R}^d$ (d=2 or 3) be a bounded domain with Lipschitz-continuous boundary. Then there exists a unique continuous linear operator*

$$T_n : H(curl, \Omega) \to H^{-1/2}(\partial\Omega), \tag{11}$$

such that

$$\forall v \in (C^\infty(\overline{\Omega}))^k \quad T_n v = n \cdot curl\, v|_{\partial\Omega}, \tag{12}$$

where $k = 1$ for $d = 2$ and $k = 3$ for $d = 3$.

Let $\{u^n\}_{1\le n\le \bar{N}}$ be the semi-discrete solution given by (2) and $\{u_h^n\}_{1\le n\le \bar{N}}$ be the approximate solution given by (6). We set

$$\{e^n\}_{1\le n\le \bar{N}} = \{u^n - u_h^n\}_{1\le n\le \bar{N}}. \tag{13}$$

In order to derive a posteriori estimates, we introduce the interpolation operator that maps $H^1(\Omega, \widetilde{\mathcal{T}}_{h,n})$ into $S_h^n \cap H_D^1(\Omega)$ and the Helmholtz decomposition.

4.1 Oswald interpolation operator

Let $\mathscr{N}_{h,n}$ be the set of all Lagrangian vertices of the elements of $\widetilde{\mathscr{T}_{h,n}}$ such that functions from $S_h^n \cap H_D^1(\Omega)$ are uniquely determined by their values at nodes from $\mathscr{N}_{h,n}$. It means that all hanging nodes are excluded from $\mathscr{N}_{h,n}$. According to, e.g., [8], we define the Oswald interpolation operator $\mathscr{I}_{Os}^D : S_h^n \to S_h^n \cap H_D^1(\Omega)$ by

$$\begin{aligned} \mathscr{I}_{Os}^D(v_h)(\nu) &= \frac{1}{\mathrm{card}(\omega_\nu)} \sum_{K \in \omega_\nu} v_h|_K(\nu), \quad \nu \in \mathscr{N}_{h,n} \setminus \mathscr{N}_{h,n}^D \\ &= 0, \quad \nu \in \mathscr{N}_{h,n}^D \end{aligned}$$

where $\omega_\nu = \{K \in \widetilde{\mathscr{T}_{h,n}};\ \nu \in K\}$, $\mathscr{N}_{h,n}^D = \{\nu \in \mathscr{N}_{h,n}; \nu \in \partial\Omega_D\}$. Moreover, we define the interpolation operator $I_{h,n}^D : H^1(\Omega, \widetilde{\mathscr{T}_{h,n}}) \to S_h^n \cap H_D^1(\Omega)$ by

$$I_{h,n}^D(v) = \mathscr{I}_{Os}^D(\Pi_h(v)) \quad \forall\, v \in H^1(\Omega, \widetilde{\mathscr{T}_{h,n}}), \tag{14}$$

where Π_h is given by (10). The proof of the following theorem can be found in [7].

Theorem 1 *Let $\widetilde{\mathscr{T}_{h,n}}$ be conforming or nonconforming triangulation satisfying* (3) *and* (4). *Then*

$$\sum_{K \in \widetilde{\mathscr{T}_{h,n}}} \|v_h - \mathscr{I}_{Os}^D(v_h)\|_{i,K}^2 \le C_O^2 \sum_{\Gamma \in \mathscr{F}_{h,n}^{ID}} h_\Gamma^{1-2i} \|[v_h]\|_\Gamma^2 \quad \forall v_h \in S_h^n,\ i = 0, 1 \tag{15}$$

where the constant C_O is independent of h and v_h.

4.2 Helmholtz decomposition

We deal with nonconforming space S_h^n, which brings some difficulties in comparison with conforming methods. For that reason the Helmholtz decomposition of the gradient of the error is carried out as follows (see, e.g., [3]):

Theorem 2 *Let e^n be given by* (13)*, then there exists the decomposition*

$$\nabla_h e^n = \nabla \phi^n + \mathrm{curl}\,\chi^n, \tag{16}$$

where $\phi^n \in H_D^1(\Omega)$ is the solution of the problem

$$\int_\Omega \nabla \phi^n \cdot \nabla v \,\mathrm{d}x = \int_\Omega \nabla_h e^n \cdot \nabla v \,\mathrm{d}x \quad \forall v \in H_D^1(\Omega), \tag{17}$$

$\chi^n \in (H^1(\Omega))^k$ ($k = 1$ for $d = 2$ and $k = 3$ for $d = 3$) such that $\boldsymbol{n} \cdot \mathrm{curl}\,\chi^n = 0$ on $\partial\Omega_N$. Moreover, the following holds: $\|\nabla_h e^n\|_\Omega^2 = \|\nabla \phi^n\|_\Omega^2 + \|\mathrm{curl}\,\chi^n\|_\Omega^2$.

The orthogonality of the splitting is crucial because it suffices to estimate each part of the error independently. A proof of the above theorem can be found in [3]. Now, we state several relations for the error e^n given by (13) (see also [11]).

Lemma 2 *Let $v_h \in S_h^n \cap H_D^1(\Omega)$, $\phi \in H_D^1(\Omega)$ and $\chi \in (H^1(\Omega))^k$ ($k = 1$ for $d = 2$ and $k = 3$ for $d = 3$) such that $\boldsymbol{n} \cdot \operatorname{curl} \chi = 0$ on $\partial\Omega_N$. The error e^n satisfies*

$$(18) \qquad \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla v_h \, \mathrm{d}x = \int_\Omega \frac{e^{n-1} - e^n}{\tau_n} v_h \, \mathrm{d}x + \theta \sum_{\Gamma \in \widetilde{\mathscr{F}}_{h,n}^I} \int_\Gamma \langle \nabla v_h \cdot \boldsymbol{n} \rangle [u_h^n] \, \mathrm{d}S,$$

$$\sum_{K \in \widetilde{\mathscr{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla \phi \, \mathrm{d}x = \int_\Omega (f^n - \frac{u^n - u^{n-1}}{\tau_n}) \phi \, \mathrm{d}x - \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} \int_{\partial K} \nabla u_h^n \cdot \boldsymbol{n} \phi \, \mathrm{d}S,$$

$$(19) \qquad + \int_{\partial\Omega_N} g_N^n \phi \, \mathrm{d}S$$

$$(20) \qquad \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} \int_K \nabla (e^n - \phi) \cdot \operatorname{curl} \chi \, \mathrm{d}x = \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} \int_{\partial K \setminus \partial\Omega_N} (e^n - \phi) \operatorname{curl} \chi \cdot \boldsymbol{n} \, dS.$$

Proof: By subtracting (6) from (2) (with $v := v_h$) and the fact that $[v_h]_\Gamma = 0 \; \forall \Gamma \in \in \widetilde{\mathscr{F}}_{h,n}^{ID}$ since $v_h \in H_D^1(\Omega)$, we get assertion (18). Further,

$$\sum_{K \in \widetilde{\mathscr{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla \phi \, dx = \int_\Omega \nabla u^n \cdot \nabla \phi \, \mathrm{d}x - \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} \int_{\partial K} \nabla u_h^n \cdot \boldsymbol{n} \phi \, \mathrm{d}S =$$

$$= \int_\Omega (f^n - \frac{u^n - u^{n-1}}{\tau_n}) \phi \, \mathrm{d}x + \int_{\partial\Omega_N} g_N^n \phi \, \mathrm{d}S - \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} \int_{\partial K} \nabla u_h^n \cdot \boldsymbol{n} \phi \, \mathrm{d}S,$$

where the first equality follows from Green's theorem, the fact $\Delta u_h^n = 0$, and (13) and the second equality follows from the definition of the semi-discrete problem (2). Finally, (20) follows by integrating by parts and the fact that div curl$\chi = 0$. □

4.3 Auxiliary results

For time level $n \geq 1$ we define the *local error indicators*

$$(21) \qquad \eta_{K,1}^n = h_K \left\| f^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_K + \sum_{\Gamma \in \mathscr{F}_K^N} h_\Gamma^{1/2} \left\| g_N^n - \boldsymbol{n} \cdot \nabla u_h^n \right\|_\Gamma$$

$$+ \sum_{\Gamma \in \mathscr{F}_K^I} h_\Gamma^{1/2} \|[\boldsymbol{n} \cdot \nabla u_h^n]\|_\Gamma + \sum_{\Gamma \in \mathscr{F}_K^I} h_\Gamma^{-1/2} \|[u_h^n]\|_\Gamma + \sum_{\Gamma \in \mathscr{F}_K^D} h_\Gamma^{-1/2} \|u_D^n - u_h^n\|_\Gamma,$$

$$\eta_{K,2}^n = \sum_{\Gamma \in \mathscr{F}_K^I} h_\Gamma^{1/2} \|[u_h^n]\|_\Gamma + \sum_{\Gamma \in \mathscr{F}_K^D} h_\Gamma^{1/2} \|u_D^n - u_h^n\|_\Gamma,$$

where $\mathscr{F}_K^I$, $\mathscr{F}_K^N$, $\mathscr{F}_K^D$ denote the set of all interior faces (edges for $d = 2$) of element K , faces (edges for $d = 2$) on $\partial\Omega_N \cap \partial K$, and faces (edges for $d = 2$) on $\partial\Omega_D \cap \cap \partial K$, respectively. The indicators reflect the residuum of the equation, the jump in the boundary conditions, the interelement jumps of the approximate solution and the jump of its normal component of the gradient.

We introduce, in addition, for $z \in H^s(\Omega, \widetilde{\mathscr{T}}_{h,n})$ and $g : \partial\Omega_D \to \mathbb{R}$ the following notation:

$$(22) \qquad J(z)^g_{\pm\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} := \left(\sum_{\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}} h_\Gamma^{\pm 1} \|[z]\|_\Gamma^2 + \sum_{\Gamma \in \widetilde{\mathscr{F}}^D_{h,n}} h_\Gamma^{\pm 1} \|z - g\|_\Gamma^2 \right)^{1/2}.$$

Further, we define:

$$(23) \qquad \begin{aligned} (\eta^n_{\mathrm{R}})^2 &:= \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} h_K^2 \left\| f^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_K^2, \\ (\eta^n_{\mathrm{J}})^2 &:= \sum_{\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}} h_\Gamma^{-1} \|[u_h^n]\|_\Gamma^2, \\ (\eta^n_{\mathrm{Jd}})^2 &:= \sum_{\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}} h_\Gamma \|[\boldsymbol{n} \cdot \nabla u_h^n]\|_\Gamma^2, \\ (\eta^n_{\mathrm{Jdb}})^2 &:= \sum_{\Gamma \in \widetilde{\mathscr{F}}^N_{h,n}} h_\Gamma \left\| g_N^n - \boldsymbol{n} \cdot \nabla u_h^n \right\|_\Gamma^2. \end{aligned}$$

Before we proceed to the proof of the a posteriori estimate, we prove auxiliary assertions. Constant c occurring in the estimates is a generic positive constant which can differ from formula to formula and is independent of h and τ.

Lemma 3 *Let* $w \in H^1(\Omega, \widetilde{\mathscr{T}}_{h,n})$ *and* Π_h *given by* (10), *then, for* $i = 0, 1$

$$(24) \qquad \sum_{\Gamma \in \widetilde{\mathscr{F}}^{ID}_{h,n}} h_\Gamma^{1-2i} \|[\Pi_h w]\|_\Gamma^2 \le c \left(\sum_{K \in \widetilde{\mathscr{T}}_{h,n}} h_K^{-2i+2} |w|_{1,K}^2 + \left(J(w)^0_{\frac{1}{2}-i,\widetilde{\mathscr{F}}_{h,n}} \right)^2 \right).$$

Proof: The following sequence of inequalities holds:

$$\begin{aligned} \sum_{\Gamma \in \widetilde{\mathscr{F}}^{ID}_{h,n}} h_\Gamma^{1-2i} \|[\Pi_h w]\|_\Gamma^2 &\le c \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} h_K^{1-2i} \|\Pi_h w - w\|_{\partial K}^2 + c \sum_{\Gamma \in \widetilde{\mathscr{F}}^{ID}_{h,n}} h_\Gamma^{1-2i} \|[w]\|_\Gamma^2 \\ &\le c \sum_{K \in \widetilde{\mathscr{T}}_{h,n}} h_K^{-2i+2} |w|_{1,K}^2 + c \sum_{\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}} h_\Gamma^{1-2i} \|[w]\|_\Gamma^2 + c \sum_{\Gamma \in \widetilde{\mathscr{F}}^D_{h,n}} h_\Gamma^{1-2i} \|w\|_\Gamma^2, \end{aligned}$$

where the first inequality follows from the triangle inequality and the local quasi-uniformity and the second one from (7) and (10). Hence, due to definition (22), we have the assertion. □

Lemma 4 *Let* $w \in H^1(\Omega, \widetilde{\mathscr{T}}_{h,n})$ *and* $I^D_{h,n}$ *given by* (14), *then*

$$(25) \qquad \sum_{\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}} h_\Gamma \|\langle \nabla I^D_{h,n}(w) \cdot \boldsymbol{n} \rangle\|_\Gamma^2 \le c \left(|w|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + \left(J(w)^0_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} \right)^2 \right).$$

Proof: The following sequence of inequalities holds:

$$\sum_{\Gamma\in\widetilde{\mathscr{F}}^{I}_{h,n}} h_\Gamma \|\langle \nabla I^D_{h,n}(w)\cdot \boldsymbol{n}\rangle\|^2_\Gamma \le c \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} h_K \|\nabla I^D_{h,n}(w)\|^2_{\partial K}$$
$$\le c \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} h_K (\|\nabla I^D_{h,n}(w)\|_K |\nabla I^D_{h,n}(w)|_{1,K} + h_K^{-1}\|\nabla I^D_{h,n}(w)\|^2_K)$$
$$\le c \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\nabla I^D_{h,n}(w)\|^2_K \le c \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\nabla (I^D_{h,n}(w) - \Pi_h(w))\|^2_K + \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\nabla \Pi_h(w)\|^2_K$$
$$\le \sum_{\Gamma\in\widetilde{\mathscr{F}}^{ID}_{h,n}} h_\Gamma^{-1}\|[\Pi_h(w)]\|^2_\Gamma + c|w|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} \le c|w|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + c\left(J(w)^0_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2,$$

where the first inequality follows from the local quasi-uniformity, the second one from (7), the third one from (8) and the fourth one from the triangle inequality, the fifth one from (15) and the boundedness of Π_h, and the last one from (24) with $i := 1$. □

Corollary 1 *Let $v \in H^1_{g_D,D}(\Omega)$ and $z \in S^n_h$ be arbitrary, where g_D is the restriction to $\partial\Omega_D$ of a function in $S^n_h \cap H^1(\Omega)$. Then*

$$\sum_{\Gamma\in\widetilde{\mathscr{F}}^{ID}_{h,n}} h_\Gamma^{1-2i}\|[\Pi_h(v-z)]\|^2_\Gamma \le c\left(\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} h_K^{2-2i}|v-z|^2_{1,K} + \left(J(z)^{g_D}_{\frac{1}{2}-i,\widetilde{\mathscr{F}}_{h,n}}\right)^2\right) \tag{26}$$

and

$$\sum_{\Gamma\in\widetilde{\mathscr{F}}^{I}_{h,n}} h_\Gamma \|\langle \nabla I^D_{h,n}(v-z)\cdot \boldsymbol{n}\rangle\|^2_\Gamma \le c\left(|v-z|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + \left(J(z)^{g_D}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2\right). \tag{27}$$

Moreover, let e^n and ϕ^n be from (16), *then*

$$\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\nabla I^D_{h,n}(e^n-\phi^n)\|^2_K \le c\left(|e^n-\phi^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + \left(J(u^n_h)^{u^n_D}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2\right). \tag{28}$$

Proof: The estimates (26) and (27) follow directly from (24) and (25) where we put $w := v - z$ and use that fact that $J(v-z)^0_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} = J(z)^{g_D}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}$ since $v = g_D$ on $\partial\Omega_D$. The estimate (28) follows from the two last lines of the proof of Lemma 4 with $w := e^n - \phi^n$ and the fact that $J(e^n-\phi^n)^0_{\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} = J(u^n_h)^{u^n_D}_{\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}$ since $e^n - \phi^n = u^n_D - u^n_h$ on $\partial\Omega_D$. □

Lemma 5 *Let $w^n \in H^1(\Omega, \widetilde{\mathscr{T}_{h,n}})$ and $\phi \in H^1_D(\Omega)$ be arbitrary, then*

$$\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} \|w^n - I^D_{h,n}(w^n)\|^2_K \le c\left(\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} h_K^2 |w^n|^2_{1,K} + \left(J(w^n)^0_{\frac{1}{2},\widetilde{\mathscr{F}_{h,n}}}\right)^2\right), \tag{29}$$

$$\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} h_K^{-2}\|\phi - I^D_{h,n}(\phi)\|^2_K \le c|\phi|^2_{1,\Omega}. \tag{30}$$

Proof: Obviously,

$$\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} \|w^n - I^D_{h,n}(w^n)\|^2_K \le 2\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} \|w^n - \Pi_h w^n\|^2_K + 2\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} \|\Pi_h w^n - I^D_{h,n}(w^n)\|^2_K.$$

We estimate the first term by (10) and the second one with the aid of the combination of (15) for $i := 1$ and (24) for $i := 0$. This proves (29). The relation (30) can be estimated similarly as (29) together with $J(\phi)^0_{\frac{1}{2},\widetilde{\mathscr{F}_{h,n}}} = 0$. □

Lemma 6 *Let $\phi \in H^1_D(\Omega)$ be arbitrary. Then the following holds*

$$\sum_{\Gamma\in\widetilde{\mathscr{F}^{IN}_{h,n}}} h_\Gamma^{-1}\|\phi - I^D_{h,n}\phi\|^2_\Gamma \le c|\phi|^2_{1,\Omega}. \tag{31}$$

Proof: The following sequence of inequalities holds:

$$\begin{aligned}
&\sum_{\Gamma\in\widetilde{\mathscr{F}^{IN}_{h,n}}} h_\Gamma^{-1}\|\phi - I^D_{h,n}\phi\|^2_\Gamma \le c\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} h_K^{-1}\|\phi - I^D_{h,n}\phi\|^2_{\partial K}\\
&\le c\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} h_K^{-1}(\|\phi - I^D_{h,n}\phi\|_K|\phi - I^D_{h,n}\phi|_{1,K} + h_K^{-1}\|\phi - I^D_{h,n}\phi\|^2_K)\\
&\le c(\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} h_K^{-2}\|\phi - I^D_{h,n}\phi\|^2_K)^{1/2}(\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} |\phi - I^D_{h,n}\phi|^2_{1,K})^{1/2}\\
&+ c\sum_{K\in\widetilde{\mathscr{T}_{h,n}}} h_K^{-2}\|\phi - I^D_{h,n}\phi\|^2_K \le c|\phi|^2_{1,\Omega},
\end{aligned}$$

where the first inequality follows from the local quasi-uniformity, the second one from (7), the third one from the Cauchy inequality, the fourth one from (30) and the two last lines of the proof of Lemma 4 with $w := \phi$. □

Lemma 7 *Let $z \in S^n_h$. Then the following holds*

$$\inf_{v\in H^1_{u^n_D,D}(\Omega)} \sum_{K\in\widetilde{\mathscr{T}_{h,n}}} \|v - z\|^2_{1/2,\partial K\cap\widetilde{\mathscr{F}^{ID}_{h,n}}} \le c\left(J(z)^{u^n_D}_{-\frac{1}{2},\widetilde{\mathscr{F}_{h,n}}}\right)^2, \tag{32}$$

$$\inf_{v\in H^1_{u^n_D,D}(\Omega)} \sum_{K\in\widetilde{\mathscr{T}_{h,n}}} h_K^2\|v - z\|^2_{1/2,\partial K\cap\widetilde{\mathscr{F}^{ID}_{h,n}}} \le c\left(J(z)^{u^n_D}_{\frac{1}{2},\widetilde{\mathscr{F}_{h,n}}}\right)^2, \tag{33}$$

where c is independent of h and $H^1_{u^n_D,D}(\Omega) = \{v \in H^1(\Omega);\ v = u^n_D \quad \text{on} \quad \partial\Omega_D\}$.

Proof: For the proof of (32), see [2, Lemma 4]. The inequality (33) can be proved with the same technique as in the mentioned lemma. □

Lemma 8 *Let* χ^n *be the function involved in Helmholtz decomposition* (16). *Then*

$$[\operatorname{curl}\chi^n \cdot \boldsymbol{n}]_\Gamma = 0 \qquad \forall\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}, \tag{34}$$

where the trace $\operatorname{curl}\chi^n \cdot \boldsymbol{n}$ *on* Γ *is meant in the sense of Lemma 1.*

Proof: Let us assume that $e^n - \phi^n \in C^\infty(\overline{\Omega})$. Writing (17) elementwise, recalling that $\nabla_h(e^n - \phi^n) = \operatorname{curl}\chi^n$ and integrating by parts give

$$\begin{aligned} 0 = &- \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_K \operatorname{div}\nabla_h\chi^n v \,\mathrm{d}x + \sum_{\Gamma\in\widetilde{\mathscr{F}}^I_{h,n}} \int_\Gamma [\nabla_h\chi^n \cdot \boldsymbol{n}] v \,\mathrm{d}S \\ &+ \sum_{\gamma\in\widetilde{\mathscr{F}}^N_{h,n}} \nabla_h\chi^n \cdot \boldsymbol{n} v \,\mathrm{d}S \qquad \forall v \in H^1_D(\Omega). \end{aligned} \tag{35}$$

Obviously, the first term and the last one in (35) vanish since $\operatorname{div}\operatorname{curl}\chi^n = 0$ and $\nabla_h\chi^n \cdot \boldsymbol{n} = 0$ on $\partial\Omega_N$, respectively. Now, it suffices to choose a non-zero test function v on an interior face Γ and we have that $\nabla_h\chi^n \cdot \boldsymbol{n}$ is continuous on Γ. Finally, the assertion of the Lemma follows from the density of $C^\infty(\overline{\Omega})^k$ in $H(\operatorname{curl},\Omega)$, see [6, Theorem 2.4 & 2.10]. □

Lemma 9 *Let* ϕ^n *and* χ^n *be functions involved in Helmholtz decomposition* (16). *There exists a constant* $c > 0$ *such that*

$$\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} h_K^{2i}|e^n - \phi^n|^2_{1,K} = \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} h_K^{2i}\|\operatorname{curl}\chi^n\|^2_K \le c\left(J(u_h^n)^{u_D^n}_{i-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2, \quad i = 0, 1. \tag{36}$$

Proof: Let $v \in H^1_{u_D^n,D}(\Omega)$ be arbitrary and u^n be the semi-discrete solution at t_n. First, we present a property separately for $i = 0$ and $i = 1$. For $i = 0$

$$\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_{\partial K\setminus\partial\Omega_N} (u^n - \phi^n)\operatorname{curl}\chi^n \cdot \boldsymbol{n} \,\mathrm{d}S = \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_{\partial K\setminus\partial\Omega_N} v \operatorname{curl}\chi^n \cdot \boldsymbol{n} \,\mathrm{d}S, \tag{37}$$

since $\operatorname{curl}\chi^n$ has continuous normal traces on $\Gamma \in \widetilde{\mathscr{F}}^I_{h,n}$ according to Lemma 8, $\phi^n = 0$ on $\Gamma \in \widetilde{\mathscr{F}}^D_{h,n}$, and $u^n = v = u^n_D$ on $\Gamma \in \widetilde{\mathscr{F}}^D_{h,n}$.
For $i = 1$, by using (4) and arguments above, we can write

$$\begin{aligned} &\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} h_K^2 \int_{\partial K\setminus\partial\Omega_N} (u^n - \phi^n)\operatorname{curl}\chi^n \cdot \boldsymbol{n} \,\mathrm{d}S \\ &\le \sum_{\Gamma\in\widetilde{\mathscr{F}}^I_{h,n}} C_H^2 h_\Gamma^2 \int_\Gamma (u^n - \phi^n)[\operatorname{curl}\chi^n \cdot \boldsymbol{n}] \,\mathrm{d}S + \sum_{\Gamma\in\widetilde{\mathscr{F}}^D_{h,n}} C_H^2 h_\Gamma^2 \int_\Gamma u^n_D \operatorname{curl}\chi^n \cdot \boldsymbol{n} \,\mathrm{d}S \\ &\le \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} C_H^4 h_K^2 \int_{\partial K\setminus\partial\Omega_N} v \operatorname{curl}\chi^n \cdot \boldsymbol{n} \,\mathrm{d}S. \end{aligned} \tag{38}$$

Finally, using (20), (37), (38) we obtain for $i = 0$ as well as $i = 1$

$$\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} h_K^{2i}|e^n - \phi^n|_{1,K}^2 = \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} h_K^{2i}\|\operatorname{curl}\chi^n\|_K^2 \tag{39}$$

$$= \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} h_K^{2i}\int_K \operatorname{curl}\chi^n\cdot\nabla(e^n - \phi^n)\,\mathrm{d}x = \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} h_K^{2i}\int_{\partial K\setminus\partial\Omega_N}(e^n - \phi^n)\operatorname{curl}\chi\cdot\boldsymbol{n}\,\mathrm{d}S$$

$$\le \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} C_H^{4i}h_K^{2i}\int_{\partial K\setminus\partial\Omega_N}(v - u_h^n)\operatorname{curl}\chi^n\cdot\boldsymbol{n}\,\mathrm{d}S$$

Therefore, with the aid of the Cauchy inequality, the trace inequality (9), and (32)–(33), we have

$$\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} C_H^{4i}h_K^{2i}\int_{\partial K\setminus\partial\Omega_N}(v - u_h^n)\operatorname{curl}\chi^n\cdot\boldsymbol{n}\,\mathrm{d}S$$

$$\le c\left(\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} h_K^{2i}\|v - u_h^n\|_{1/2,\partial K\setminus\partial\Omega_N}^2\right)^{1/2}\left(\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} h_K^{2i}\|\operatorname{curl}\chi^n\|_K^2\right)^{1/2}$$

$$\le cJ(u_h^n)_{i-\frac{1}{2},\widetilde{\mathcal{F}}_{h,n}}^{u_D^n}\left(\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} h_K^{2i}\|\operatorname{curl}\chi^n\|_K^2\right)^{1/2}.$$

The last inequality follows from (32)–(33) and the fact that v was arbitrary. Hence, we can take a sequence of $\{v_j\}\in H^1_{u_D^n,D}(\Omega)$ converging to the infimum and with the aid of the limit passage we obtain the desired inequality. Now, the assertion immediately follows. □

Corollary 2 *The combination of* (29) *and* (36) *together with* $J(e^n - \phi^n)^0_{\frac{1}{2},\widetilde{\mathcal{F}}_{h,n}} = J(u_h^n)^{u_D^n}_{\frac{1}{2},\widetilde{\mathcal{F}}_{h,n}}$ *gives*

$$\left(\sum_{K\in\widetilde{\mathcal{T}}_{h,n}}\|(e^n - \phi^n) - I_{h,n}^D(e^n - \phi^n)\|_K^2\right)^{1/2} \le cJ(u_h^n)^{u_D^n}_{\frac{1}{2},\widetilde{\mathcal{F}}_{h,n}}. \tag{40}$$

5. Upper error bound

Now, we state the main result, an upper bound on the error.

Theorem 3 *Let* $\{u^n\}_{1\le n\le\bar{N}}$ *and* $\{u_h^n\}_{1\le n\le\bar{N}}$ *be the semi-discrete solution given by* (2) *and be the approximate solution given by* (6)*, respectively. Let* $1\le N\le\bar{N}$*. Then the* *error* $e^n = u^n - u_h^n,\ n = 1, \ldots, N$ *satisfies*

$$\sum_{K\in\widetilde{\mathcal{T}}_{h,N}} \|e^N\|_K^2 + \sum_{n=1}^{N} \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|\nabla e^n\|_K^2$$

$$\le \sum_{K\in\widetilde{\mathcal{T}}_{h,1}} \|e^0\|_K^2 + \sum_{n=1}^{N} C\left(\tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} (\eta_{K,1}^n)^2 + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} (\eta_{K,2}^n)^2\right),$$

where a constant C is independent of the mesh parameter and the time step.

Proof: According to (16), we can write

$$\tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|\nabla e^n\|_K^2 = \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla \phi^n \,\mathrm{d}x \qquad (=: \psi_1)$$

(41)

$$+ \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \operatorname{curl} \chi^n \,\mathrm{d}x. \qquad (=: \psi_2)$$

Setting $\phi := \phi^n$ in (19) multiplied by τ_n and adding a τ_n-multiple of the difference of the right-hand and the left-hand sides of (18) with $v_h := I_{h,n}^D \phi^n$ yield

$$\psi_1 = \tau_n \int_\Omega (f^n - \frac{u^n - u^{n-1}}{\tau_n}) \phi^n \,\mathrm{d}x - \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_{\partial K} \nabla u_h^n \cdot \boldsymbol{n} \phi^n \,\mathrm{d}S \tag{42}$$

$$+ \tau_n \int_{\partial\Omega_N} g_N^n \phi^n \,\mathrm{d}S - \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla I_{h,n}^D \phi^n \,\mathrm{d}x$$

$$+ \tau_n \int_\Omega \frac{e^{n-1} - e^n}{\tau_n} I_{h,n}^D \phi^n \,\mathrm{d}x + \tau_n \theta \sum_{\Gamma\in\widetilde{\mathcal{F}}_{h,n}^I} \int_\Gamma \langle \nabla I_{h,n}^D \phi^n \cdot \boldsymbol{n} \rangle [u_h^n] \,\mathrm{d}S.$$

By expressing term $-\tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla I_{h,n}^D \phi^n \,\mathrm{d}x$ according to identity (19), adding and subtracting term $\tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K (f^n - \frac{u_h^n - u_h^{n-1}}{\tau_n}) \phi^n \,\mathrm{d}x$, and reordering the terms we have

$$\psi_1 = \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K (f^n - \frac{u_h^n - u_h^{n-1}}{\tau_n})(\phi^n - I_{h,n}^D \phi^n) \,\mathrm{d}x \tag{43}$$

$$- \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K (e^n - e^{n-1}) \phi^n \,\mathrm{d}x - \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_{\partial K} \nabla u_h^n \cdot \boldsymbol{n}(\phi^n - I_{h,n}^D \phi^n) \,\mathrm{d}S$$

$$+ \tau_n \int_{\partial\Omega_N} g_N^n (\phi^n - I_{h,n}^D \phi^n) \,\mathrm{d}S + \tau_n \theta \sum_{\Gamma\in\widetilde{\mathcal{F}}_{h,n}^I} \int_\Gamma \langle \nabla I_{h,n}^D \phi^n \cdot \boldsymbol{n} \rangle [u_h^n] \,\mathrm{d}S.$$

Putting (43) in (41), expressing ψ_2 with the aid of (20) and adding terms $\pm \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_K (e^n - e^{n-1})(e^n - I^D_{h,n}(e^n - \phi^n))\,\mathrm{d}x$, we obtain

$$
\begin{aligned}
&\tau_n \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\nabla e^n\|_K^2 \\
&= \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_K (e^n - e^{n-1})((e^n - \phi^n) - I^D_{h,n}(e^n - \phi^n))\,\mathrm{d}x \quad (=: \xi_1) \\
&+ \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_K (e^n - e^{n-1}) I^D_{h,n}(e^n - \phi^n)\,dx \quad (=: \xi_2) \\
&+ \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_K e^n e^{n-1}\,\mathrm{d}x - \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|e^n\|_K^2\,\mathrm{d}x \\
&+ \tau_n \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_K (f^n - \frac{u_h^n - u_h^{n-1}}{\tau_n})(\phi^n - I^D_{h,n}\phi^n)\,\mathrm{d}x \quad (=: \xi_3) \\
&- \tau_n \sum_{\Gamma\in\widetilde{\mathscr{F}}^I_{h,n}} \int_\Gamma [\nabla u_h^n \cdot \boldsymbol{n}](\phi^n - I^D_{h,n}\phi^n)\,\mathrm{d}S \quad (=: \xi_4) \\
&+ \tau_n\theta \sum_{\Gamma\in\widetilde{\mathscr{F}}^I_{h,n}} \int_\Gamma \langle \nabla I^D_{h,n}\phi^n \cdot \boldsymbol{n}\rangle [u_h^n]\,\mathrm{d}S \quad (=: \xi_5) \\
&+ \tau_n \sum_{\Gamma\in\widetilde{\mathscr{F}}^N_{h,n}} \int_\Gamma (g_N^n - \nabla u_h^n \cdot \boldsymbol{n})(\phi^n - I^D_{h,n}\phi^n)\,\mathrm{d}S \quad (=: \xi_6) \\
&+ \tau_n \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_{\partial K\setminus\partial\Omega_N} e^n \operatorname{curl}\chi^n \cdot \boldsymbol{n}\,dS \quad (=: \xi_7).
\end{aligned} \tag{44}
$$

Now, we have to estimate all terms in (44). The Cauchy inequality and (40) gives

$$
\begin{aligned}
|\xi_1| &\le \left(\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|e^n - e^{n-1}\|_K^2\right)^{1/2} \left(\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|(e^n - \phi^n) - I^D_{h,n}(e^n - \phi^n)\|_K^2\right)^{1/2} \\
&\le c\|e^n - e^{n-1}\|_\Omega \; J(u_h^n)^{u_D^n}_{\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}.
\end{aligned} \tag{45}
$$

With the aid of (18) we have

$$
\begin{aligned}
\xi_2 \;=\; & \tau_n\theta \sum_{\Gamma\in\widetilde{\mathscr{F}}^I_{h,n}} \int_\Gamma \langle \nabla I^D_{h,n}(e^n - \phi^n) \cdot \boldsymbol{n}\rangle [u_h^n]\,\mathrm{d}S \quad (=: \xi_{2a}) \\
& -\tau_n \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_K \nabla e^n \nabla I^D_{h,n}(e^n - \phi^n)\,\mathrm{d}x \quad (=: \xi_{2b})
\end{aligned}
$$

Application of the Cauchy inequality, (23), and (27) with settings $v := u^n - \phi^n$, $z := u_h^n$, $g_D := u_D^n$ yield

$$
\begin{aligned}
|\xi_{2a}|^2 &\le \tau_n^2 \sum_{\Gamma\in\widetilde{\mathscr{F}}_{h,n}^I} h_\Gamma \|\langle \nabla I_{h,n}^D (e^n - \phi^n)\cdot \boldsymbol{n}\rangle\|_\Gamma^2 \times \sum_{\Gamma\in\widetilde{\mathscr{F}}_{h,n}^I} h_\Gamma^{-1} \|[u_h^n]\|_\Gamma^2 \\
&\le \tau_n^2 \eta_{\mathrm{J}}^n c \left(|e^n - \phi^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + \left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} \right)^2 \right).
\end{aligned} \tag{46}
$$

Moreover, the Cauchy inequality and (28) gives

$$
\begin{aligned}
|\xi_{2b}|^2 &\le \tau_n^2 |e^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\nabla I_{h,n}^D (e^n - \phi^n)\|_K^2 \\
&\le c\tau_n^2 |e^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} \left(|e^n - \phi^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + \left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} \right)^2 \right).
\end{aligned} \tag{47}
$$

Further, the Cauchy inequality and (30) gives

$$
|\xi_3| \le \tau_n \eta_{\mathrm{R}}^n \left(\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} h_K^{-2} \|\phi^n - I_{h,n}^D \phi^n\|_K^2 \right)^{1/2} \le c\tau_n \eta_{\mathrm{R}}^n |\phi^n|_{1,\Omega}. \tag{48}
$$

Moreover, the Cauchy inequality and (31) yields

$$
|\xi_4| \le \tau_n \eta_{\mathrm{Jd}}^n \left(\sum_{\widetilde{\mathscr{F}}_{h,n}^I} h_\Gamma^{-1} \|\phi^n - I_{h,n}^D \phi^n\|_\Gamma^2 \right)^{1/2} \le c\tau_n \eta_{\mathrm{Jd}}^n |\phi^n|_{1,\Omega}. \tag{49}
$$

Similarly, the Cauchy inequality, the estimate $|\theta| \le 1$, and (27) with settings $v :=$ $:= \phi^n$, $z := 0$, $g_D := 0$ give

$$
|\xi_5| \le \tau_n \left(\sum_{\Gamma\in\widetilde{\mathscr{F}}_{h,n}^I} h_\Gamma \|\langle \nabla I_{h,n}^D \phi^n \cdot \boldsymbol{n}\rangle\|_\Gamma^2 \right)^{1/2} \left(\sum_{\Gamma\in\widetilde{\mathscr{F}}_{h,n}^I} h_\Gamma^{-1} \|[u_h^n]\|_\Gamma^2 \right)^{1/2} \le c\tau_n \eta_{\mathrm{J}}^n |\phi^n|_{1,\Omega}. \tag{50}
$$

Again, the Cauchy inequality and (31) imply

$$
|\xi_6| \le \tau_n \eta_{\mathrm{Jdb}}^n \left(\sum_{\widetilde{\mathscr{F}}_{h,n}^N} h_\Gamma^{-1} \|\phi^n - I_{h,n}^D \phi^n\|_\Gamma^2 \right)^{1/2} \le c\tau_n \eta_{\mathrm{Jdb}}^n |\phi^n|_{1,\Omega}. \tag{51}
$$

Due to the Lemma 8, u^n can be substituted for any function $v \in H^1_{u_D^n,D}(\Omega)$ in ξ_7 as follows:

$$
|\xi_7| = \tau_n \left| \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_{\partial K\setminus\partial\Omega_N} (v - u_h^n) \operatorname{curl} \chi^n \cdot \boldsymbol{n} \, \mathrm{d}S \right|,
$$

which together with the Cauchy inequality and (32) yield

$$|\xi_7|^2 \leq c\tau_n^2 \left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2 \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\operatorname{curl}\chi^n \cdot \boldsymbol{n}\|^2_{-1/2,\partial K\setminus\partial\Omega_N}. \tag{52}$$

Finally, using a trace inequality (9) in (52) leads to

$$|\xi_7|^2 \leq c\tau_n^2 \left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2 \|\operatorname{curl}\chi^n\|^2_\Omega. \tag{53}$$

Now, relation (44) with the particular estimates of ξ_l, $l = 1,\dots,7$ given in (45)–(53) gives

$$\begin{aligned}
&\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|e^n\|_K^2 + \tau_n \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\nabla e^n\|_K^2 \\
&\leq c\tau_n |\phi^n|_{1,\Omega}(\eta^n_{\mathrm{R}} + \eta^n_{\mathrm{Jdb}} + \eta^n_{\mathrm{Jd}} + \eta^n_{\mathrm{J}}) \\
&+ c\|e^n - e^{n-1}\|_\Omega \, J(u_h^n)^{u_D^n}_{\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} \\
&+ c(\tau_n |e^n|_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + \eta^n_{\mathrm{J}})\left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} + |e^n - \phi^n|_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})}\right) \\
&+ c\tau_n J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}} \|\operatorname{curl}\chi^n\|_\Omega + \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \int_K e^n e^{n-1}\,\mathrm{d}x.
\end{aligned} \tag{54}$$

Multiplying (54) by 2, application of Young's inequality, and the relation

$$\|e^n - e^{n-1}\|^2_\Omega = \|e^n\|^2_\Omega - 2\int_\Omega e^n e^{n-1}\,\mathrm{d}x + \|e^{n-1}\|^2_\Omega,$$

give

$$\begin{aligned}
&2\sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|e^n\|_K^2 + 2\tau_n \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|\nabla e^n\|_K^2 \\
&\leq \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|e^n\|_K^2 + \sum_{K\in\widetilde{\mathscr{T}}_{h,n}} \|e^{n-1}\|_K^2 \\
&+ c\tau_n\left((\eta^n_{\mathrm{R}})^2 + (\eta^n_{\mathrm{Jdb}})^2 + (\eta^n_{\mathrm{Jd}})^2 + (\eta^n_{\mathrm{J}})^2 + \left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2\right) + \frac{\tau_n}{4}|\phi^n|^2_{1,\Omega} \\
&+ c\left(\left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2 + \left(J(u_h^n)^{u_D^n}_{\frac{1}{2},\widetilde{\mathscr{F}}_{h,n}}\right)^2 + (\eta^n_{\mathrm{J}})^2\right) \\
&+ \frac{\tau_n}{2}|e^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + c\tau_n\left(|e^n - \phi^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + \|\operatorname{curl}\chi^n\|^2_\Omega\right).
\end{aligned} \tag{55}$$

Moving some terms from the right-hand side of (55), using (36), and

$$|\phi^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} \leq 2|\phi^n - e^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})} + 2|e^n|^2_{H^1(\Omega,\widetilde{\mathscr{T}}_{h,n})},$$

we derive

$$\sum_{K\in\widetilde{\mathscr{T}_{h,n}}}\|e^n\|_K^2+\tau_n\sum_{K\in\widetilde{\mathscr{T}_{h,n}}}\|\nabla e^n\|_K^2$$
$$\le\sum_{K\in\widetilde{\mathscr{T}_{h,n}}}\|e^{n-1}\|_K^2+c\tau_n\left((\eta_{\mathrm{R}}^n)^2+(\eta_{\mathrm{Jdb}}^n)^2+(\eta_{\mathrm{Jd}}^n)^2+(\eta_{\mathrm{J}}^n)^2+\left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}_{h,n}}}\right)^2\right)$$
$$+c\left(\left(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathscr{F}_{h,n}}}\right)^2+\left(J(u_h^n)^{u_D^n}_{\frac{1}{2},\widetilde{\mathscr{F}_{h,n}}}\right)^2+(\eta_{\mathrm{J}}^n)^2\right)$$

which together with the definitions (21), (22), and (23) finally yield

$$\sum_{K\in\widetilde{\mathscr{T}_{h,n}}}\|e^n\|_K^2+\tau_n\sum_{K\in\widetilde{\mathscr{T}_{h,n}}}\|\nabla e^n\|_K^2$$
$$\le\sum_{K\in\widetilde{\mathscr{T}_{h,n}}}\|e^{n-1}\|_K^2+c\left(\tau_n\sum_{K\in\widetilde{\mathscr{T}_{h,n}}}(\eta_{K,1}^n)^2+\sum_{K\in\widetilde{\mathscr{T}_{h,n}}}(\eta_{K,2}^n)^2\right).$$

Summing the inequality on $n=1,\dots,\bar{N}$, we come to the assertion of the theorem. □

6. Conclusion

We derived the error upper bound for the heat conduction equation discretized by the low order DGFEM in space and the backward Euler scheme in time. Analogously to [10], the Helmholtz decomposition was used to overcome difficulties arising due to the nonconformity of the DGFEM.

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