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IDEAL-SIMPLE SEMIRINGS I

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Basic properties of ideal-simple semirings are collected and stabilized.

The present pseudoexpository paper collects general observations and some other results concerning ideal-simple semirings. All the stuff is fairly basic, and therefore we will only scarcely attribute the results to particular sources.

1. Preliminaries

A semiring is an algebraic structure equipped with two binary operations, traditionally denoted as addition and multiplication, where the addition is commutative and associative, the multiplication is associative and distributes over the addition from both sides. The semiring is called commutative if the multiplication has this property.

Let S $(= S(+,\cdot))$ be a semiring. We say that S is additively idempotent (constant, cancellative, etc.) if the additive semigroup $(+)$ has the property. Similarly, S is multiplicatively idempotent (constant, cancellative, etc.) if the multiplicative semigroup $S(\cdot)$ is such. The semiring S is bi-idempotent if both the semigroups are idempotent.

The semiring S need not have any additively and/or multiplicatively neutral element. If S has such an element, the uniquely determined additively neutral element

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is denoted by 0_S (or only 0) and the uniquely determined multiplicatively neutral element is denoted by 1_S (or only 1); we will also write $0_S \in S$ or $1_S \in S$. The additively absorbing element (if it exists) is denoted by o_S or o. There is no specific way how to denote multiplicatively absorbing elements.

If A, B are subsets of S then $A + B = \{a + b \mid a \in A, b \in B\}$ and $AB = \{ab \mid a \in A, b \in B\}$. A non-empty subset I of S is a left (right, resp.) ideal of S if $(I+I) \cup SI \subseteq I$ ($(I+I) \cup IS \subseteq I$, resp.). A non-empty subset I of S is an ideal if $(I+I) \cup SI \cup IS \subseteq I$ (i.e., I is a left and right ideal). Clearly, S is an ideal of itself and a (left, right) ideal I is called proper if $I \neq S$. A (left, right) ideal I is called trivial if $|I| = 1$.

The semiring S is called

- (left-, right-) ideal-simple if $|S| \geq 2$ and S has no non-trivial proper (left, right) ideal;
- (left-, right-) ideal-free if $|S| \geq 2$ and S has no proper (left, right) ideal.

Clearly, every two-element semiring is left- and right-ideal-simple and the class of (left-, right-) ideal-simple (-free) semirings is closed under non-trivial homomorphic images.

The semiring S is called congruence-simple if it has just two congruence relations.

A non-empty subset I of S is a bi-ideal if $(S + I) \cup SI \cup IS \subseteq I$. The semiring S is bi-ideal-simple if $|S| \geq 2$ and S has no non-trivial proper bi-ideal. If I is a bi-ideal then $(I \times I) \cup \mathrm{id}_S$ is a congruence of S. Consequently, every congruence-simple semiring is bi-ideal-simple.

1.1 Lemma. (i) *A one-element subset $\{w\}$ is a (left, right) ideal if and only if w is (right, left) multiplicatively absorbing.*
(ii) *A one-element subset $\{w\}$ is a bi-ideal if and only if w is bi-absorbing (i.e., both additively and mulitplicatively absorbing).*

Proof. It is easy. □

1.2 Corollary. *The semiring S has at most one trivial ideal.* □

1.3 Corollary. *The semiring S is ideal-free if and only if S is ideal-simple and has no multiplicatively absorbing element.*

1.4 Lemma. *The set of additively idempotent elements of S is either empty or an ideal of S.*

Proof. It is easy. □

1.5 Lemma. *The set $S + S$ is a bi-ideal of S.*

Proof. It is easy. □

1.6 Lemma. *Let $a \in S$. Then the set $\{\sum_{i=1}^{n} b_i a c_i \mid n \geq 1, b_i, c_i \in S\}$ is an ideal of S.*

Proof. It is easy. □

1.7 Lemma. *Let n be a positive integer.*
(i) *The set $nS = \{ a \in S \mid a \in S \}$ is an ideal of S.*
(ii) *The set $\{ a \in S \mid na = a \}$ is either empty or an ideal of S.*
(iii) *If $w \in S$ is multiplicatively absorbing then the set $\{ a \in S \mid na = w \}$ is an ideal of S.*

Proof. It is easy. □

1.8 Lemma. *Let $w \in S$ be such that for every $a \in S$ there are $b, c \in S$ with $aw = b+w$ or $aw = w$, and $wa = c + w$ or $wa = w$. Then the set $S + w$ is an ideal of S.*

Proof. It is easy. □

1.9 Corollary. *Let $w \in S$ be multiplicatively absorbing. Then the set $S + w$ is an ideal of S.* □

1.10 Lemma. *Let $w \in S$ be multiplicatively absorbing. Then the set $\{ a \in S \mid w \in \in S + a \}$ is an ideal of S.*

Proof. It is easy. □

1.11 Lemma. *Let $w \in S$ be multiplicatively absorbing. The the set $\{ a \in S \mid w = = na$ for some $n \geq 1 \}$ is an ideal of S.*

Proof. It is easy. □

1.12 Lemma. *Let $w \in S$ be multiplicatively absorbing. Then the sets $\{ a \in S \mid Sa = = w \}$ and $\{ b \in S \mid bS = w \}$ are ideals of S.*

Proof. it is easy. □

1.13 Lemma. *Let S be finite and additively idempotent. Then $\sum S$ ($= \sum_{a \in S} a$) is an additively absorbing element.*

Proof. It is easy. □

1.14 Lemma. *Let $a \in S$ and $m \geq 2$ be such that $a^m = a$. Then $ab = b = ba$, where $b = a + a^2 + \cdots + a^{m-1}$.*

Proof. It is easy. □

2. Examples

2.1 Every two-element semiring is congruence-, left-ideal- and right-ideal-simple. There are just ten (up to isomorphism) two-element semirings:

$Z_1(+)$	0	1
0	0	0
1	0	0

$Z_1(\cdot)$	0	1
0	0	0
1	0	0

$Z_2(+)$	0	1
0	0	0
1	0	0

$Z_2(\cdot)$	0	1
0	0	0
1	0	1

$Z_3(+)$	0	1
0	0	0
1	0	1

$Z_3(\cdot)$	0	1
0	0	0
1	0	0

$Z_4(+)$	0	1
0	0	0
1	0	1

$Z_4(\cdot)$	0	1
0	1	1
1	1	1

$Z_5(+)$	0	1
0	0	0
1	0	1

$Z_5(\cdot)$	0	1
0	0	1
1	1	1

$Z_6(+)$	0	1
0	0	0
1	0	1

$Z_6(\cdot)$	0	1
0	0	0
1	0	1

$Z_7(+)$	0	1
0	0	1
1	1	0

$Z_7(\cdot)$	0	1
0	0	0
1	0	0

$Z_8(+)$	0	1
0	0	1
1	1	0

$Z_8(\cdot)$	0	1
0	0	0
1	0	1

$Z_9(+)$	0	1
0	0	0
1	0	1

$Z_9(\cdot)$	0	1
0	0	1
1	0	1

$Z_{10}(+)$	0	1
0	0	0
1	0	1

$Z_{10}(\cdot)$	0	1
0	0	0
1	1	1

The semirings $Z_1, \dots, Z_8$ are commutative and the semirings Z_9, Z_{10} are not (these two semirings are anti-isomorphic). The semirings $Z_3, \dots, Z_6, Z_9, Z_{10}$ are additively idempotent, the semirings $Z_2, Z_5, Z_6, Z_8, Z_9, Z_{10}$ are multiplicatively idempotent and the semirings Z_5, Z_6, Z_9, Z_{10} are bi-idempotent. The semirings Z_1, Z_2 are additively constant, the semirings Z_1, Z_3, Z_4, Z_7 are multiplicatively constant and the semiring Z_1 is bi-constant. The semirings Z_7, Z_8 are rings. The semirings Z_9, Z_{10} are ideal-free. The semiring Z_9 is right-ideal-free and the semiring Z_{10} is left-ideal-free.

2.2 Let $S(\cdot)$ be a semigroup containing an absorbing element $o \in S$. Setting $a + b = o$ for all $a, b \in S$, we get an additively constant semiring $S = S(+, \cdot)$. This semiring is ideal-simple if and only if the semigroup $S(\cdot)$ is ideal-simple.

2.3 Consider the following three-element semiring S:

$S(+)$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$S(\cdot)$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

Then S is bi-idempotent, left-ideal-simple and not congruence-simple. The element 0 is additively neutral and multiplicatively absorbing, the element 1 is additively absorbing and the elements 1, 2 are multiplicatively neutral.

2.4 Define two binary operations $*$ and $\circ$ on the set $\mathbb{Z}$ of integers by $a * b = \min(a, b)$ and $a \circ b = a + b$ for all $a, b \in \mathbb{Z}$. Then $\mathbb{Z}_1 = \mathbb{Z}(*, \circ)$ becomes an additively idempotent commutative semiring that is both ideal-free and congruence-simple. The number 0 is a multiplicatively neutral element of the semiring $\mathbb{Z}_1$.

Let $\alpha \notin \mathbb{Z}$, $\mathbb{Z}_2 = \mathbb{Z} \cup \{\alpha\}$ and $z * \alpha = \alpha * z$, $\alpha * \alpha = \alpha$, $z \circ \alpha = \alpha \circ z = \alpha$, $\alpha \circ \alpha = \alpha$ for every $z \in \mathbb{Z}$. Again, $\mathbb{Z}_2$ is an additively idempotent commutative semiring, α is additively neutral and multiplicatively absorbing and 0 is multiplicatively neutral. The semiring $\mathbb{Z}_2$ is ideal-simple, but not congruence-simple.

2.5 Remark. (i) The following result is proved in [1, 14.1]:

The following conditions are equivalent for a commutative semiring S:

(a) S is finite and congruence-simple.

(b) S is finite and ideal-simple.

(ii) It seems to be an open problem whether there is a (finite) additively idempotent semiring S such that S is ideal-simple, $0_S \in S$, $1_S \in S$, $o_S \in S$ and 0_S is multiplicatively absorbing.

2.6 Consider the following semilattice $A(+)$:

$A(+)$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

Let E be the semiring of all endomorphisms f of $A(+)$ such that $f(0) = 0$. Then E is a finite additively idempotent congruence-simple semiring that is not ideal-simple. The constant endomorphism $x \mapsto 0$ is additively neutral and multiplicatively absorbing. The constant endomorphism $x \mapsto 1$ is additively absorbing. The identity automorphism id_A is multiplicatively neutral.

3. Elementary observations (a)

In this section, let S be an ideal-simple semiring.

3.1 Proposition. *Just one of the following three cases takes place:*

(1) *S has no additively idempotent element (then S is infinite and has no additively neutral element, no additively absorbing element and no multiplicatively absorbing element);*

(2) *S has a multiplicatively absorbing element w and w is the only additively idempotent element;*

(3) *S is additively idempotent.*

Proof. Combine 1.4 and 1.1(i). □

3.2 Proposition. *Just one of the following two cases takes place:*

(1) *$S + S = S$;*

(2) *S is additively constant, $S + S = \{o\}$, where o is bi-absorbing and the multiplicative semigroup $S(\cdot)$ is ideal-simple (see 2.2).*

Proof. Combine 1.5 and 1.1(ii). □

3.3 Lemma. *Let n be a positive integer. Then just one of the following two cases takes place:*

(1) *$nS = S$;*

(2) *S contains a multiplicatively absorbing element w and $nS = \{w\}$.*

Proof. Combine 1.7(i) and 1.1(i). □

3.4 Lemma. *Let n be a positive integer. Then just one of the following three cases takes place:*

(1) *$na \neq a$ for every $a \in S$;*

(2) *$na = a$ for every $a \in S$;*

(3) *S contains a multiplicatively absorbing element w and $na \neq a$ for every $a \in S \setminus \{w\}$.*

Proof. Combine 1.7(ii) and 1.1(i). □

3.5 Lemma. *Let n be a positive integer. Then just one of the following four cases takes place:*

(1) *$na = a$ for every $a \in S$;*

(2) *$nS = S$ and $na \neq a$ for every $a \in S$;*

(3) *$nS = S$, S contains a multiplicatively absorbing element w and $w \neq na \neq a$ for every $a \in S \setminus \{w\}$;*

(4) *S contains a multiplicatively absorbing element w and $nS = \{w\}$.*

Proof. Combine 1.7(iii), 3.3 and 3.4. □

3.6 Proposition. *Let $w \in S$ be multiplicatively absorbing. Then just one of the following three cases takes place:*

(1) *$w = o_S$ is bi-absorbing;*

(2) *$w = 0_S$ is additively neutral and $0 \notin S + a$ for every $a \setminus \{0\}$ (i.e., $S \setminus \{0\}$ is a subsemigroup of $S(+)$);*

(3) *S is a ring.*

Proof. By 1.9, $S + w$ is an ideal of S and, of course, $w \in S + w$. If $|S + w| = 1$ then w is bi-absorbing. If $|S + w| \geq 2$ then $S + w = S$ and we see that $w = 0_S$. By 1.10, the set $A = \{\, a \in S \mid 0 \in S + a \,\}$ is an ideal of S and, of course, $0 \in A$. If $|A| = 1$ then (2) is true. If $|A| \geq 2$ then $A = S$ and S is a ring. □

3.7 Proposition. *Just one of the following four cases takes place:*

(1) *$nS = S$ for every positive integer n;*

(2) *S is a ring and $pS = \{0\}$ for a prime number p;*

(3) *$S + S = S$ and S contains a bi-absorbing element o such that $nS = \{o\}$ for every positive integer n (then S is infinite);*

(4) *S is additively constant (see 2.2 and 3.2(ii)).*

Proof. Assume that (1) is not true. By 3.3, S contains a multiplicatively absorbing element w such that $mS = \{w\}$ for a positive integer m. Clearly, $m \geq 2$. By 3.6, either $w = 0_S$ is additively neutral or $w = o_S$ is bi-absorbing.

First, let $w = 0_S$. Since $mS = \{0\}$ and $m \geq 2$, it follows that S is a ring. Let k be the smallest positive integer with $kS = \{0\}$. Again, $k \geq 2$, and if k is not prime then $k = k_1k_2$, where $k_1 \geq 2$ and $k_2 \geq 2$. We have $k_2 < k$, so that $k_2S = S$ by 3.3. Similarly, $k_1 < k$ and $\{0\} = kS = k_1k_2S = k_1S = S$, a contradiction. Thus (2) is true.

Next, let $w = o_S$. Again, let k be the smallest positive integer with $kS = \{o\}$. Clearly, $k \geq 2$ and there is a non-negative integer l such that $l < k$ and $k + l$ is even. We have $(k + l)S = o + S = o$. But $\frac{k+l}{2} < k$, $S = \frac{k+l}{2}S$ and $\{o\} = (2\frac{k+l}{2})S = 2S$. We have proved that $2S = \{o\}$. Since o is additively absorbing, we have $nS = \{o\}$ for every $n \geq 2$.

Assume that $S + S = S$. If $|S| = t$ were finite, $t \geq 2$ and if $a \in S \setminus \{o\}$ then $a = a_1 + \cdots + a_{t+1}$ for some $a_i \in S$. But $a_i = a_j$ for some $i < j$, $a_i + a_j = o$ and $a = o$, a contradiction. Thus (3) is true.

Assume, finally, that $S + S \neq S$. Then S is additively constant by 3.2. □

3.8 Proposition. *Assume that $nS = S$ for every positive integer n (see 3.7). Then just one of the following three cases takes place:*

(1) *S is additively idempotent;*

(2) *$na \neq ma$ for all $a \in S$ and positive integers n, m, $n \neq m$;*

(3) *S contains a multiplicatively absorbing element w such that $w \neq na \neq ma$ for all $a \in S \setminus \{w\}$ and positive integers n, m, $n \neq m$.*

Proof. Assume that S is not additively idempotent. If S has no additively idempotent element then (2) is true. Consequently, let $w \in S$ be such that $2w = w$. By 3.1, w is the only additively idempotent element of S and w is multiplicatively absorbing. If $na = w$ for some $a \in S \setminus \{w\}$ and a positive integer n then if follows from 1.7(ii) that $nS = \{w\}$, a contradiction. Thus (3) is true. □

3.9 Proposition. *Let S be ideal-free. Then either S is additively idempotent or S has no additively idempotent element and $nS = S$ for every positive integer n (see 3.8(ii)). In the latter case, S is infinite.*

Proof. Combine 3.7 and 3.8. □

3.10 Proposition. *Let S be finite. Then just one of the following three cases takes place:*

(1) *S is additively idempotent and $o_S \in S$;*
(2) *S is a ring and $pS = 0$ for a prime number p;*
(3) *S is additively constant.*

Proof. Combine 3.7 and 3.8. □

3.11 Lemma. *Let $w \in S$ be multiplicatively absorbing and let $SS \neq \{w\}$. Then $Sa \neq \{w\} \neq aS$ and $SaS \neq \{w\}$ for every $a \in S \setminus \{w\}$.*

Proof. The result follows easily from 1.12. □

3.12 Lemma. *Let $w \in S$ be multiplicatively absorbing and let $SS \neq \{w\}$. If $a \in S \setminus \{w\}$ then $S = \{ \sum_{i=1}^{n} r_i a s_i \,|\, n \geq 1,\ r_i, s_i \in S \}$.*

Proof. The result follows easily from 3.11. □

3.13 Proposition. *Let $0_s \in S$. Then just one of the following five cases takes place:*

(1) *S is additively idempotent and 0 is multiplicatively absorbing;*
(2) *S is additively idempotent, $0^2 = 0$ and $S = \{ \sum_{i=1}^{n} r_i 0 s_i \,|\, n \geq 1,\ r_i, s_i \in S \}$;*
(3) *S is a ring;*
(4) *0 is multiplicatively absorbing, it is the only additively idempotent element of S and $T + T = T$, where $T = S \setminus \{0\}$ (then T is a subsemigroup of $S(+)$ and S is infinite);*
(5) *$S \cong Z_3$.*

Proof. First, assume that S is additively idempotent. If 0 is not multiplicatively absorbing and $A = \{\sum r_i 0 s_i\}$ then A is an ideal of S and either $A = S$ or $A = \{w\}$, where $w \in S$ is multiplicatively absorbing and $w \neq 0$. In the former case, $ab = (a+0)(b+0) = ab + a0 + 0b = 0^2$, $ab + 0^2 = ab$ for all $a, b \in S$, $r_i 0 s_i + 0^2 = r_i 0 s_i$ and $0^2 = 0$. Thus (2) is true. In the latter case, we get $SS = \{w\}$ by 3.12. Now, $\{0, w\}$ is a two-element ideal of S, $S = \{0, w\}$ and $S \cong Z_3$.

Now, assume that S is not additively idempotent. By 3.1(2), 0 is multiplicatively absorbing and it is the only additively idempotent element of S. If S is not a ring then (4) follows from 1.10. □

3.14 Corollary. *If $0_S \in S$ and 0 is not multiplicatively absorbing then either $S \cong Z_3$ or 3.13(2) is true. (Notice that 0 is not multiplicatively absorbing if S is ideal-free.)* □

4. Elementary observations (b)

In this part, let S be a semiring such that $o = o_S \in S$ (o is additively absorbing).

4.1 Lemma. *For all $a, b, c \in S$ there is $d \in S$ with $oao = bac + d$.*

Proof. $oao = (b+o)a(c+o) = (ba+oa)(c+o) = bac + oac + bao + oao$ and $d = oac + bao + oao$. □

4.2 Lemma. *For all $a, b, c, d \in S$ and $n \geq 1$ such that $b + c = noao$ there is $e \in S$ with $db + e = noao$.*

Proof. $noao = n(do+o)ao = ndoao + noao = db + dc + b + c$ and $e = dc + b + c$. □

4.3 Lemma. *For all $a, b, c, d \in S$ and $n \geq 1$ such that $b + c = noao$ there is $e \in S$ with $bd + e = noao$.*

Proof. Symmetric to 4.2. □

4.4 Lemma. *For all $a, b, c, d, e \in S$ and $n \geq 1$ such that $b + c = noao$ there is $f \in S$ with $dbe + f = noao$.*

Proof. $noao = n(do+o)a(oe+o) = n(doa+oa)(oe+o) = ndoaoe + ndoao + noaoe + noao = dbe + dce + db + dc + be + ce + b + c$ and $f = dce + db + dc + be + ce + b + c$. □

4.5 Lemma. *Assume that S is ideal-simple and the element owo is multiplicatively absorbing for some $w \in S$. Then just one of the following three cases takes place:*

(1) *$owo = o$ and o is bi-absorbing;*

(2) *S is additively idempotent and $owo = w = 0_S$;*

(3) $S \cong Z_4$.

Proof. If the element owo is additively absorbing then $owo = o$ is bi-absorbing. Now, assume that $owo \neq o$. Then, by 3.6, $owo = 0$ is additively neutral. If $w = 0$ then S is additively idempotent by 3.1 and (2) is true.

Let, finally, $w \neq 0$. Again, S is additively idempotent by 3.1. Clearly, S is not a ring and, by 3.6, $T + T \subseteq T$, where $T = S \setminus \{0\}$. On the other hand, by 4.1, $0 = owo \in S + bwc$ for all $b, c \in S$. Consequently, $SwS = \{0\}$ and $SS = \{0\}$ follows from 1.12. Now, it is clear that $\{0, o\}$ is an ideal of S. Thus $\{0, o\} = S$ and $S \cong Z_4$. □

4.6 Lemma. *Assume that S is ideal-simple and the element owo is not multiplicatively absorbing for some $w \in S$. Then S is additively idempotent and $owo = o$.*

Proof. If o is multiplicatively absorbing then $owo = o$, a contradiction. Thus o is not multiplicatively absorbing and S is additively idempotent by 3.1. Put $A = \{a \in \in S \mid owo \in S + a\}$. Using 4.2, 4.3 and the fact that S is additively idempotent, we check easily that A is an ideal of S and $owo \in A$. Since owo is not mulitplicatively absorbing, we have $A \neq \{owo\}$. Consequently, $A = S$ and $owo \in S + \{o\} = \{o\}$. Thus $owo = o$. □

4.7 Proposition. *Let S be ideal-simple. Then just one of the following four cases takes place:*

(1) *S is additively idempotent, S has no multiplicatively absorbing element, $oSo = \{o\}$, $o^2 = o$, $S = \{\sum_{i=1}^{n} r_i o s_i \mid n \geq 1,\ r_i, s_i \in S\}$;*
(2) *S is additively idempotent, $0 = 0_S \in S$ is additively neutral and multiplicatively absorbing, $0 \neq o = o^2$, $oSo = \{0, o\}$, $oao = o$ for every $a \neq 0$ and $S = \{\sum_{i=1}^{n} r_i o s_i \mid n \geq 1,\ r_i, s_i \in S\}$;*
(3) *o is bi-absorbing (and so $SoS = \{o\} = oSo$);*
(4) *$S \cong Z_4$ (and so $SoS = \{0_S\} = oSo$).*

Proof. First, assume that S has no multiplicatively absorbing element. By 3.1, S is additively idempotent. By 4.6, $oSo = \}o\}$. In particular, $o^3 = o$, $o^4 = oo^2o = o$ and $o^2 = oo = oo^3 = o^4 = o$. Since o is not multiplicatively absorbing and S has no multiplicatively absorbing element at all, (1) follows from 1.6.

Next, assume that S has a multiplicatively absorbing element w. If $w = o$ then (3) is true. Henceforth, let $w \neq o$. Put $A = \{a \in S \mid oao = w\}$ and $B = \{b \in S \mid obo = o\}$. Clearly, $A \cap B = \emptyset$ and, by 4.6, we have $A \cup B = S$. Moreover, if $B \neq \emptyset$ then S is additively idempotent.

We have $w \in A$, so that $A \neq \emptyset$ and, by 4.5, $w = owo = 0_S \in S$. If $A = \{0\}$ then $B = S \setminus \{0\} \neq \emptyset$, S is additively idempotent abd $obo = o$ for every $b \in S \setminus \{0\}$. In particular, $o^3 = o$, and hence $o^2 \neq 0$, $o^4 = oo^2o = o$ and $o^2 = o$. Thus (2) is true. Finally, if $A \neq \{0\}$ then $S \cong Z_4$ by 4.5. □

4.8 Corollary. *If S is ideal-free then 4.7(1) is satisfied.* □

4.9 Remark. Assume that S is ideal-simple, $0 = 0_S \in S$ and $o = o_S \in S$. Then $0 \neq o$ and S is additively idempotent.
(i) If 0 is multiplicatively absorbing then either 4.7(2) is true or $S \cong Z_4$.
(ii) If o is multiplicatively absorbing then either 3.13(2) is true or $S \cong Z_3$.
(iii) Assume that S has no multiplicatively absorbing element (i.e., S is ideal-free). By 3.13, we have $0^2 = 0$ and $S = \{\sum_{i=1}^{n} r_i 0 s_i\}$. By 4.7, we have $o^2 = o$, $oSo = \{o\}$ and $S = \{\sum_{i=1}^{n} r_i o s_i\}$.
(iii1) Let 0 be left multiplicatively absorbing, i.e., $0S = \{0\}$. Then $S0 = S$ and it follows that $o0 = o$ (if $a0 = o$ then $o0 = (a + o)0 = o + o0 = o$) and 0 is right multiplicatively neutral. Furthermore, $oa = o(a + 0) = oa + o = o$, so that o is left multiplicatively absorbing, $So = S$ and o is right multiplicatively neutral. Of course, $ab = a0b = a0 = a$ for all $a, b \in S$.
(iii2) Similarly, if 0 is right multiplicatively absorbing then $ab = b$ for all $a, b \in S$.
(iii3) If o is left multiplicatively absorbing then $oS = \{o\}$, $S = So$ and o is right multiplicatively neutral. We have $0 = 0o = 0(a + o) = 0a + 0o = 0a + 0 = 0a$ and 0 is left multiplicatively absorbing. By (iii1), $ab = a$ for all $a, b \in S$.
(iii4) Similarly, if o is right multiplicatively absorbing then $ab = b$ for all $a, b \in S$.

4.10 Proposition. *Let S be ideal-simple and let $0 = 0_S \in S$ and $o = o_S \in S$. Assume that neither 0 nor o is mulitplicatively absorbing. Then:*
(i) *S is additively idempotent and has no multiplicatively absorbing element (then S is ideal-free).*
(ii) *$oSo = \{o\}$, $0S0 = \{0\}$, $o^2 = o$ and $0^2 = 0$.*
(iii) *If $0o = 0$ or $o0 = o$ then $ab = a$ for all $a, b \in S$.*
(iv) *If $o0 = 0$ or $0o = o$ then $ab = b$ for all $a, b \in S$.*
(v) *If $ab \neq a$ and $cd \neq d$ for some $a, b, c, d \in S$ then $0o \notin \{0, o\}$, $o0 \notin \{o, 0\}$, $0o \cdot o0 = 0$, $o0 \cdot 0o = o$.*

Proof. (i) By 3.1, S is additively idempotent. According to 3.6, S has no multiplicatively absorbing element.
(ii) By 4.7(1), we have $o^2 = o$ and $oSo = \{o\}$. By 3.13(2), $0^2 = 0$. Moreover, by 4.7(1), $0 = \sum r_i o s_i$. Since S is additively idempotent, we have $0 = ros$. But $ros = (r + 0)0(s + 0) = ros + 0os + ro0 + 0o0$ and $0o0 = 0$. Consequently, $0 = 0o0 = 0(a + o)0 = 0a0 + 0o0 = 0a0$ for every $a \in S$ and $0S0 = \{0\}$.
(iii) and (iv) See 4.9.
(v) Use (ii), (iii) and (iv). □

5. Elementary observations (c)

In this section, let S be a semiring such that $ab = ac$ for all $a, b, c \in S$.

5.1 Proposition. *There is an endomorphism α of the semiring S such that $\alpha^2 = \alpha$, $ab = \alpha(a)$ for all $a, b \in S$ and:*
(i) *$T = \alpha(S)$ is a bi-idempotent subsemiring of S and $uv = u$ for all $u, v \in T$.*
(ii) *T is an ideal of S.*
(iii) *If I is a left ideal of S then $T \subseteq I$.*
(iv) *If A is a subsemigroup of $S(+)$ with $T \subseteq A$ then A is an ideal of S.*
(v) *α is injective if and only if α is projective and if and only if $\alpha = \mathrm{id}_S$ (then $ab = a$ for all $a, b \in S$).*
(vi) *If R is a block of $\ker(\alpha)$ then R is a subsemiring of S and $|RR| = 1$.*

Proof. There is a transformation α of S such that $ab = \alpha(a)$ for all $a, b \in S$. We have $\alpha(a) = a(a + a) = a^2 + a^2 = \alpha(a) + \alpha(a)$, $\alpha(a) = a \cdot aa = aa \cdot a = \alpha(aa) = \alpha(a)$, $\alpha(a + b) = (a + b)a = a^2 + ba = \alpha(a) + \alpha(b)$ and $\alpha(ab) = \alpha^2(a) = \alpha(a)\alpha(b)$ for all $a, b \in S$. We have checked that α is an endomorphism of S, $\alpha^2 = \alpha$ and $2\alpha(a) = \alpha(a)$. The image $T = \alpha(S)$ is a subsemiring of S and $\alpha(u) = u$ for every $u \in T$. Consequently, $uv = u$ for all $u, v \in T$ and T is bi-idempotent. It is clear that T is an ideal of S. If I is a left ideal of S then $T = \alpha(S) = SI \subseteq I$. If A is a subsemigroup of $S(+)$ and $T \subseteq A$ then $SA = T \subseteq A$ and $AS = \alpha(A) \subseteq T \subseteq A$. Thus A is an ideal of S.

If α is injective (projective, resp.) then $\alpha^2 = \alpha$ implies $\alpha = \mathrm{id}_S$. Finally, if R is a block of $\ker(\alpha)$ then $RR = \alpha(R)$ and $|\alpha(R)| = 1$. □

5.2 Proposition. *S is ideal-simple if and only if at least one (and then just one) of the following three cases takes place:*

(1) *$|S| \geq 2$ and $ab = a$ for all $a, b \in S$ (then S is additively idempotent);*
(2) *S is a zero multiplication ring of finite prime order;*
(3) *S is isomorphic to one of Z_1, Z_2, Z_3.*

Proof. If $T = S$ (see 5.1) then (1) is true. If $|T| = 1$ then, due to 5.1(iv), $A = S$ whenever A is a subsemigroup of $S(+)$ with $T \subseteq A$ and $|A| \geq 2$. Now, it is easy to see that either (2) or (3) is true. □

5.3 Corollary. *Assume that S is ideal-simple and $|SS| = 1$ (i.e., S has constant multiplication). Then either S is a zero multiplication ring of finite prime order or S is isomorphic to one of Z_1, Z_2, Z_3.* □

References

[1] R. El Bashir, J. Hurt, A. Jančařík and T. Kepka: *Simple commutative semirings,* J. Algebra **236** (2001), 277–306.