Tomáš Kepka; Petr Němec

Ideal-simple semirings. III.

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 53 (2012), No. 2, 31--46

Persistent URL: http://dml.cz/dmlcz/143698

2012 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 53, NO. 2

IDEAL-SIMPLE SEMIRINGS III

TOMÁŠ KEPKA, PETR NĚMEC

Praha

Received April 25, 2012

Ideal-simple endomorphism semirings of semilattices are investigated.

This note is an immediate continuation of [2] and [3].

1. Semilattices

In this section, let M (= $M(+)$) be a semilattice (i.e., an idempotent commutative semigroup). Setting $a \le b$ iff $b \in S + a$, we get a compatible ordering and $a + b = \sup(a, b)$ for all $a, b \in M$. An element w is the smallest (greatest, resp.) element iff w is neutral (absorbing, resp.). We denote this fact by $w = 0 = 0_M$ ($w = 1 = 1_M$, resp.).

A non-empty subset N of M is an ideal of M if $M + N \subseteq N$. Such an ideal is called prime if $a + b \notin N$ for all $a, b \in M \setminus N$ (i.e., either $N = M$ or $M \setminus N$ is a subsemilattice of M). We denote by $\underline{P}(M)$ the set of proper prime ideals of M.

For every $a \in M$, the set $\{ x \in M \mid a \le x \}$ is an ideal of M. The set $\{ y \in M \mid a < y \}$ is either empty or an ideal.

A one-element set $\{w\}$ is an ideal iff $w = 1_M$. This ideal is prime iff 1 is irreducible.

Department of Algebra, MFF UK, Sokolovská 83, 186 75 Praha 8, Czech Republic (T. Kepka)
Department of Mathematics, Czech University of Life Sciences, Kamýcká 129, 165 21 Praha 6 – Suchdol, Czech Republic (P. Němec)

Supported by the Grant Agency of the Czech Republic, grant GAČR 201/09/0296.

2000 Mathematics Subject Classification. 16Y60

Key words and phrases. semilattice, semiring, ideal, ideal-simple

E-mail address: kepka@karlin.mff.cuni.cz, nemec@tf.czu.cz

For every $a \in M$, the set $Q_a = \{z \in M \,|\, z \nleq a\}$ is either empty or a prime ideal of M. Anyway, $Q_a = \emptyset$ iff $a = 1_M$. Moreover, $a \notin Q_a$, and hence $Q_a \in \underline{P}(M)$ for every $a \neq 1_M$. Notice that $M \setminus Q_a = \{u \in M \,|\, u \leq a\}$.

1.1 Lemma. *Let $a \in M$ and let I be an ideal of M such that $a \notin I$. Then $I \subseteq Q_a$.*

Proof. It is obvious. □

1.2 Lemma.. *Let $\in \underline{P}(M)$. Then $P = \cap Q_a$, $a \in M \setminus P$.*

Proof. Use 1.1. □

1.3 Lemma. (i) *$Q_a \subseteq Q_b$ iff $b \leq a$.*
(ii) *$Q_a = Q_b$ iff $a = b$.*

Proof. It is easy. □

1.4 Lemma. *Let $P \in \underline{P}(M)$, Then:*
(i) *If $P = Q_a$ for some $a \in M$, $a \neq 1_M$, then $u \leq a$ for every $u \in M \setminus P$.*
(ii) *If $a \in M \setminus P$ is such that $u \leq a$ for every $u \in M \setminus P$ then $P = Q_a$.*

Proof. Use 1.2 and 1.3. □

1.5 Corollary. *Let $P \in \underline{P}(M)$ be such that the set $M \setminus P$ is finite. Then $P = Q_a$, where $a = \sum x$, $x \in M \setminus P$.* □

1.6 Corollary. *If M is finite then $\underline{P}(M) = \{ Q_a \,|\, a \in M,\ a \neq 1_M \}$.* □

2. Endomorphisms of semilattices (a)

Let M be a semilattice and $\underline{E} = \mathrm{End}(M)$ the full endomorphism semiring of M.

2.1 Proposition. (i) *The semiring $\underline{E}$ is additively idempotent and the identity automorphism* id_M *is the multiplicatively neutral element of $\underline{E}$.*
(ii) *$\underline{E}$ has an additively neutral element if and only if $0_M \in M$. Then the constant endomorphism $x \mapsto 0$ is the additively neutral element and it is left multiplicatively absorbing.*
(iii) *$\underline{E}$ has an additively absorbing element if and only if $1_M \in M$. Then the constant endomorphism $x \mapsto 1$ is the additively absorbing element and it is left mulitplicatively absorbing.*
(iv) *If $|M| \geq 2$ then $\underline{E}$ has no right multiplicatively absorbing element.*
(v) *$\underline{E}$ is non-trivial iff M is so.*

Proof. It si easy. □

For every $n \geq 1$, let $\underline{R}^{(n)} = \{ f \in \underline{E} \mid |f(M)| \leq n \}$ and $\underline{R}^{(\omega)} = \cup \underline{R}^{(n)}$, $n \geq 1$. For every $1 \leq n \leq \omega$, let $\underline{E}^{(n)}$ be the subsemiring of $\underline{E}$ generated by $\underline{R}^{(n)}$.

2.2 Proposition. (i) $\underline{R}^{(1)} = \underline{E}^{(1)} \subseteq \underline{E}^{(2)} \subseteq \underline{E}^{(3)} \subseteq \cdots \subseteq \underline{E}^{(\omega)} = \underline{R}^{(\omega)}$.
(ii) *All the semirings* $\underline{E}^{(1)}, \underline{E}^{(2)}, \dots, \underline{E}^{(\omega)}$ *are ideals of the semiring* $\underline{E}$.

Proof. It is easy. □

2.3 Proposition. *The following conditions are equivalent:*

(i) *M is finite.*
(ii) *$\underline{E}$ is finite.*
(iii) $\underline{E}^{(\omega)} = \underline{E}$.
(iv) $\mathrm{id}_M \in \underline{E}^{(\omega)}$.
(v) $\underline{E}^{(m)} = \underline{E}$ *for some* $m \geq 1$.

Proof. It is easy. □

2.4 Proposition. $\underline{E}^{(n)} = \{ \sum_{i=1}^{m} f_i \mid m \geq 1, f_i \in \underline{R}^{(n)} \}$ *for every* $1 \leq n \leq \omega$.

Proof. It is easy. □

For all $a, x \in M$, let $\sigma_a(x) = a$; we have $\sigma_a \in \underline{E}^{(1)}$.

2.5 Proposition. (i) $\underline{E}^{(1)} = \underline{R}^{(1)} = \{ \sigma_a \mid a \in M \}$.
(ii) $\sigma_a + \sigma_b = \sigma_{a+b}$ *for all* $a, b \in M$.
(iii) $\sigma_a f = \sigma_a$ *and* $f\sigma_a = \sigma_{f(a)}$ *for all* $a \in M$ *and* $f \in \underline{E}$.

Proof. It is easy. □

2.6 Corollary. (i) *The semiring* $\underline{E}^{(1)}$ *is ideal-simple if and only if* $|M| \geq 2$. *Then* $\underline{E}^{(1)}$ *is left-ideal-free.*
(ii) *The semiring* $\underline{E}^{(1)}$ *is right-ideal-simple if and only if* $|M| = 2$. □

2.7 Lemma. *The following conditions are equivalent.*

(i) $|M| = 1$.
(ii) $\mathrm{id}_M \in \underline{E}^{(1)}$.
(iii) $|\underline{E}^{(1)}| = 1$.
(iv) $\underline{E}^{(1)} = \underline{E}$.
(v) $\underline{E}^{(1)} = \underline{E}^{(n)}$ *for some* $n \geq 2$.

Proof. It is obvious. □

2.8 Proposition. *The full endomorphism semiring* $\underline{E}$ *is never ideal-simple.*

Proof. If $\underline{E}$ is non-trivial then $|M| \geq 2$ and $\underline{E}^{(1)}$ is a proper non-trivial ideal of $\underline{E}$ (combine 2.2 and 2.7). □

2.9 Proposition. *Let $a, b \in M$, $a \leq b$, and let $P \in \underline{P}(M)$. Define a transformation $\varrho = \varrho_{a,b,P}$ of M by $\varrho(P) = \{b\}$ and $\varrho(M \setminus P) = \{a\}$. Then $\varrho \in \underline{R}^{(2)}$ and:*
(i) *$\varrho(M) = \{a, b\}$ and $\varrho \in \underline{E}^{(2)}$.*
(ii) *If $a = b$ then $\varrho = \sigma_a$.*
(iii) *If $0 \in M$ then $\varrho(0) = 0$ iff $a = 0$.*
(iv) *If $1 \in M$ then $\varrho(1) = 1$ iff $b = 1$.*
(v) *If $0, 1 \in M$ then $\varrho(0) = 0$ and $\varrho(1) = 1$ iff $a = 0$ and $b = 1$.*

Proof. It is easy. □

2.10 Proposition. *Let $f \in \underline{R}^{(2)}$. Then:*
(i) *There are $a, b \in M$ such that $f(M) = \{a, b\}$ and $a \leq b$.*
(ii) *$P = \{\, x \in M \mid f(x) = b \,\}$ is a prime ideal and $P \in \underline{P}(M)$ iff $a \neq b$.*
(iii) *If $a \neq b$ then $f = \varrho_{a,b,P}$.*
(iv) *If $a = b$ and $|M| \geq 2$ then $\underline{P}(M) \neq \emptyset$ and $f = \sigma_a = \varrho_{a,a,Q}$ for any $Q \in \underline{P}(M)$.*

Proof. It is easy. □

2.11 Corollary. *Let $|M| \geq 2$. Then $\underline{P}(M) \neq \emptyset$ and $\underline{R}^{(2)} = \{\, \varrho_{a,b,P} \mid a, b \in M, a \leq b, P \in$ $\in \underline{P}(M) \,\}$.* □

2.12 Propostion. *The semiring $\underline{E}^{(2)}$ is never ideal-simple.*

Proof. We can proceed similarly as in the proof of 2.8. □

2.13 Lemma. *Let $a, b \in M$, $a \leq b$, and let $P \in \underline{P}(M)$ and $f \in \underline{E}$. Then $f\varrho_{a,b,P} =$ $= \varrho_{f(a),f(b),P}$ and we put $g = \varrho_{a,b,P}f$, $K = \{\, x \in M \mid f(x) \notin P \,\}$ and $L = \{\, x \in M \mid f(x) \in$ $\in P \,\}$. Now:*
(i) *$M = K \cup L$ and $K \cap L = \emptyset$.*
(ii) *If $K = M$ (or $L = \emptyset$) then $g = \sigma_a = \varrho_{a,a,P}$.*
(iii) *If $K = \emptyset$ (or $L = M$) then $g = \sigma_b = \varrho_{b,b,P}$.*
(iv) *If $K \neq \emptyset \neq L$ then $L \in \underline{P}(M)$ and $g = \varrho_{a,b,L}$.*

Proof. It is easy. □

For every triple a, b, c of elements from M, where $a \leq b$, denote by $\varrho_{a,b,c}$ the transformation of M defined by $\varrho_{a,b,c}(x) = a$ if $x \leq c$ and $\varrho_{a,b,c}(x) = b$ otherwise.

2.14 Lemma. *Let $a, b, c \in M$, $a \leq b$. If $c \neq 1_M$ then $\varrho_{a,b,c} = \varrho_{a,b,Q_c}$. If $c = 1_M$ then $\varrho_{a,b,c} = \sigma_a$.*

Proof. It is obvious. □

Denote by $\underline{F}$ the subsemiring of $\underline{E}$ generated by all the endomorphisms $\varrho_{a,b,c}$, $a, b, c \in M$, $a \leq b$.

2.15 Proposition. (i) $\underline{E}^{(1)} \subseteq \underline{F} \subseteq \underline{E}^{(2)}$.
(ii) $\underline{F} = \underline{E}^{(1)}$ *iff* $|M| = 1$.

Proof. It is easy. □

2.16 Proposition. *The semiring $\underline{F}$ is never ideal-simple.*

Proof. Use 2.15. □

2.17 Proposition. *Let E be an ideal-simple subsemiring of $\underline{E}$, $E^{(1)} = E \cap \underline{E}^{(1)}$ and $E^{(2)} = E \cap \underline{E}^{(2)}$. Then:*
(i) *If $E^{(1)} \neq \emptyset$ then $E^{(1)}$ is an ideal of E.*
(ii) *If $|E^{(1)}| = 1$ then $E^{(1)} = \{\sigma_v\}$ for some $v \in M$ and $f(v) = v$ for every $f \in E$.*
(iii) *If $|E^{(1)}| \geq 2$ then $E = E^{(1)} \subseteq \underline{E}^{(1)}$.*
(iv) *If $|E^{(2)}| \geq 2$ then $E = E^{(2)} \subseteq \underline{E}^{(2)}$.*

Proof. It is easy. □

2.18 Lemma. *Let $a, b, c \in M$, $a \leq b$, and let $f \in \underline{E}$. Then:*
(i) $f\varrho_{a,b,c} = \varrho_{f(a),f(b),c}$.
(ii) $\varrho_{a,b,c}f = g$, *where $g = \sigma_a$ if $f(M) \leq c$, $g = \sigma_b$ if $f(x) \nleq c$ for every $x \in M$ and $g = \varrho_{a,b,L}$ if $\emptyset \neq L = \{\, x \in M \mid f(x) \nleq c \,\} \neq M$ (then $L \in \underline{P}(M)$).*

Proof. It is easy. □

2.19 Lemma. *Let $a_1, a_2, b_1, b_2, c_1, c_2 \in M$, $a_1 \leq b_1$, $a_2 \leq b_2$. Put $h = \varrho_{a_1,b_1,c_1}\varrho_{a_2,b_2,c_2}$. Then:*
(i) *If $b_2 \leq c_1$ then $h = \sigma_{a_1}$.*
(ii) *If $b_2 \nleq c_1$ and $a_2 \leq c_1$ then $h = \varrho_{a_1,b_1,c_2}$.*
(iii) *If $a_2 \nleq c_1$ then $h = \sigma_{b_1}$.*

Proof. It is easy. □

2.20 Lemma. *Let $a_1, a_2, b_1, b_2, c_1, c_2 \in M$, $a_1 \leq b_1$, $a_2 \leq b_2$. Let $f \in \underline{E}$ and $k = \varrho_{a_1,b_1,c_1} f \varrho_{a_2,b_2,c_2}$. Then:*
(i) *If $f(b_2) \leq c_1$ then $k = \sigma_{a_1}$.*
(ii) *If $f(b_2) \nleq c_1$ and $f(a_2) \leq c_1$ then $k = \varrho_{a_1,b_1,c_1}$.*
(iii) *If $f(a_2) \nleq c_1$ then $k = \sigma_{b_1}$.*

Proof. It follows from 2.19, since $k = \varrho_{a_1,b_1,c_1}\varrho_{f(a_2),f(b_2),f(c_2)}$. □

2.21 Lemma. (cf. 2.14) *Let $a, b \in M$, $a \leq b$, and let $P \in \underline{P}(M)$. The following conditions are equivalent:*

(i) $\varrho_{a,b,P} = \varrho_{a,b,c}$ *for some* $c \in M$.

(ii) $\varrho_{a,b,P} = \varrho_{a,b,c}$ *for some* $c \in M$, $c \neq 1_M$.

(iii) *There is* $c \in M$ *such that* $M \setminus P = \{ x \in M \mid x \leq c \}$.

(iv) *The set* $M \setminus P$ *has the greatest element (if* $M \setminus P$ *is finite then* $\sum M \setminus P$ *is the greatest element).*

Proof. It is easy. □

2.22 Lemma. *Let $a_1, a_2, b_1, b_2, c \in M$ be such that $a_1 \leq b_1$, $a_2 \leq b_2$, and let $P \in$ $\in \underline{P}(M)$. Put $g = \varrho_{a_1,b_1,P}\varrho_{a_2,b_2,c}$. Then:*

(i) *If* $a_2 \in P$ *then* $g = \sigma_{b_1}$.

(ii) *If* $b_2 \notin P$ *then* $g = \sigma_{a_1}$.

(iii) *If* $a_2 \notin P$ *and* $b_2 \in P$ *then* $g = \varrho_{a_1,b_1,c}$.

Proof. Use 2.13. □

2.23 Proposition. (i) *$\underline{F}$ is a left ideal of $\underline{E}$.*

(ii) $\underline{F} = \{ \sum_{i=1}^{n} \varrho_{a_i,b_i,c_i} \mid n \geq 1,\ a_i, b_i, c_i \in M, a_i \leq b_i \}$.

(iii) *$\underline{E}^{(2)}$ is generated by $\underline{F}$ as an ideal of itself.*

(iv) *If M is finite then $\underline{F} = \underline{E}^{(2)}$.*

Proof. (i) Use 2.18(i).

(ii) Use 2.19.

(iii) We have $\varrho_{a,b,a}\varrho_{a,b,P} = \varrho_{a,b,P}$ by 2.13.

(iv) Use 2.21. □

2.24 Remark. (i) Let $a_0 \in M$ and $R_0 = \{ x \in M \mid a_0 \leq x \}$. Then $a_0 \in R_0$ and R_0 is an ideal of M. Clearly, R_0 is a proper ideal iff $a_0 \neq 0_M$. Similarly, R_0 is a prime ideal iff $u + v + a_0 \neq u + v$ whenever $u, v \in M$ are such that $u + a_0 \neq u$ and $v + a_0 \neq v$ (then a_0 is irreducible). Now, if $R_0 \in \underline{P}(M)$ and $a, b \in M$ are such that $a \leq b$ then $\varrho_{a,b,R_0} = a$ if $a \nleq x$ and $\varrho_{a,b,R_0}(x) = b$ if $a_0 \leq x$.

(ii) Let $a_1 \in M$ and $R_1 = \{ x \in M \mid a_1 < x \}$. Clearly, $a_1 \notin R_1$ and if $R_1 \neq \emptyset$ then R_1 is a proper ideal. If R_1 is a prime ideal and $a, b \in M$ are such that $a \leq b$ then $\varrho_{a,b,R_1}(x) = a$ if $a_1 \nless x$ and $\varrho_{a,b,R_1}(x) = b$ if $a_1 < x$.

2.25 Remark. The following results are proved in [1] ([1, 3.2, 3.3, 3.4, 4.2]).

(i) The full endomorphism semiring $\underline{E}$ (that is not ideal-simple by 2.3) is congruence-simple if and only if $0_M, 1_M \in M$ and $0_M \neq 1_M$.

(ii) If M is finite then $\underline{E}$ is congruence-simple if and only if $|M| \geq 2$ and $0_M \in M$.

(iii) The semiring $\underline{F}$ (that is not ideal-simple by 2.16) is congruence-simple if and only if $|M| \geq 2$.

(iv) The following conditions are equivalent:

(a) $\underline{F} = \underline{E}$.
(b) The semiring $\underline{F}$ has a left (right, resp.) multiplicatively neutral element.
(c) $\mathrm{id}_M \in \underline{F}$ ($\mathrm{id}_M \in \underline{E}^{(2)}$).
(d) M is finite, $0_M \in M$ and M is distributive as a lattice.

(v) Let $|M| \geq 2$. Proceeding similarly as in the proof of [1,3.4], one can show that the semiring $E^{(2)}$ is congruence-simple. If $0_M \in M$ then all the semirings $\underline{E}^{(2)}, \underline{E}^{(3)}, \dots, \underline{E}^{(\omega)}$ are congruence-simple. If $|M| = 3$ and $0_M \notin M$ then $E^{(3)} = \underline{E}$ is not congruence-simple. The semiring $E^{(1)}$ is ideal-simple and it is congruence-simple if and only if $|M| = 2$.

3. Endomorphisms of semilattices (b)

Let M be a semilattice such that $0 = 0_M \in M$. Put $\underline{E}_0 = \{ f \in \underline{E} \,|\, f(0) = 0 \}$. Clearly, $\underline{E}_0$ is a subsemiring of the full endomorphisms semiring $\underline{E}$ and $\mathrm{id}_M \in \underline{E}_0$. If $|M| \geq 2$ then $\underline{E}^{(1)} \nsubseteq \underline{E}_0$, and hence $\underline{E}_0 \neq \underline{E}$.

3.1 Proposition. (i) *The semiring $\underline{E}_0$ is additively idempotent and the identity automorphism* id_M *is the multiplicatively neutral element of $\underline{E}_0$.*
(ii) *The constant endomorphism $\sigma_0 \in \underline{E}_0$ is both additively neutral and multiplicatively absorbing.*
(iii) $\{\sigma_0\} = \underline{E}^{(1)} \cap \underline{E}_0$ *is an ideal of $\underline{E}_0$.*

Proof. It is easy. □

For every $n \geq 1$, let $\underline{R}_0^{(n)} = \{ f \in \underline{E}_0 \,|\, |f(M)| \leq n \}$ and we put $\underline{R}_0^{(\omega)} = \cup \underline{R}_0^{(n)}$, $n \geq 1$. For every $1 \leq n \leq \omega$, let $\underline{E}_0^{(n)}$ be the subsemiring of $\underline{E}_0$ generated by $\underline{R}_0^{(n)}$.

3.2 Proposition. (i) $\underline{R}_0^{(n)} = \underline{R}^{(n)} \cap \underline{E}_0$ *for every $a \leq n \leq \omega$.*
(ii) $\underline{E}_0^{(n)} = \underline{E}_0 \cap \underline{E}^{(n)}$ *for every $1 \leq n \leq \omega$.*
(iii) $\{\sigma_0\} = \underline{R}_0^{(1)} = \_0^{(1)} \subseteq \underline{E}_0^{(2)} \subseteq \underline{E}_0^{(3)} \subseteq \cdots \subseteq \underline{E}_0^{(\omega)} = \underline{R}_0^{(\omega)}$.
(iv) *All the semirings $\underline{E}_0^{(1)}, \underline{E}_0^{(2)}, \dots, \underline{E}_0^{(\omega)}$ are ideals of the semiring $\underline{E}_0$.*

Proof. It is easy (use 2.2 and the fact that if $f, g \in \underline{E}$ are such that $f + g \in \underline{E}_0$ then $f, g \in \underline{E}_0$). □

3.3 Proposition. *The following conditions are equivalent.*
(i) *M is finite.*
(ii) *$\underline{E}_0$ is finite.*
(iii) $\underline{E}_0^{(\omega)} = \underline{E}_0$.
(iv) $\mathrm{id}_M \in \underline{E}_0^{(\omega)}$.
(v) $\underline{E}_0^{(m)} = \underline{E}_0$ *for some $m \geq 1$.*

Proof. It is easy. □

3.4 Proposition. $\underline{E}_0^{(n)} = \{ \sum_{i=1}^m f_i \mid m \geq 1, f_i \in \underline{R}_0^{(n)} \}$ *for every* $1 \leq n \leq \omega$.

Proof. It is easy. □

3.5 Lemma. $\underline{R}_0^{(2)} = \{\sigma_0\} \cup \{ \varrho_{0,a,P} \mid a \in M, P \in \underline{P}(M) \}$.

Proof. Combine 2.9 and 2.10. □

In the sequel, we put $\varrho_{a,P} = \varrho_{0,a,P}$. We have $\varrho_{0,P} = \sigma_0$ and if $|M| \geq 2$ then $\underline{R}_0^{(2)} = \{ \varrho_{a,P} \mid a \in M, P \in \underline{P}(M) \}$.

3.6 Corollary. *Let* $|M| \geq 2$. *Then* $\underline{E}_0^{(2)} = \{ \sum_{i=1}^m \varrho_{a_i,P_i} \mid m \geq 1, a_i \in M, P_i \in \underline{P}(M) \}$. □

3.7 Lemma. *Let* $a \in M$, $P \in \underline{P}(M)$ *and* $f \in \underline{E}_0$. *Then* $f\varrho_{a,P} = \varrho_{f(a),P}$ *and we put* $g = \varrho_{a,P}f$, $K = \{ x \in M \mid f(x) \notin P \}$ *and* $L = \{ x \in M \mid f(x) \in P \}$. *Then:*
(i) $0 \in K$, $M = K \cup L$ *and* $K \cap L = \emptyset$.
(ii) *If* $K = M$ *(or* $L = \emptyset$*) then* $g = \sigma_0$.
(iii) *If* $K \neq M$ *(or* $L \neq \emptyset$*) then* $L \in \underline{P}(M)$ *and* $g = \varrho_{a,L}$.

Proof. Use 2.13. □

Put $\varrho_{a,b} = \varrho_{0,a,b}$ for all $a, b \in M$. That is, $\varrho_{a,b}(x) = 0$ if $x \leq b$ and $\varrho_{a,b}(x) = a$ otherwise. We have $\varrho_{0,b} = \sigma_0$.

3.8 Lemma. *Let* $a, b \in M$. *If* $b \neq 1_M$ *then* $\varrho_{a.b} = \varrho_{0,a,Q_b} = \varrho_{a,Q_b}$. *If* $b = 1_M$ *then* $\varrho_{a,b} = \sigma_0$.

Proof. Use 2.14. □

Denote by $\underline{F}_0$ the subsemiring of $\underline{E}_0$ generated by all the endomorphisms $\varrho_{a,b}$, $a, b \in M$.

3.9 Proposition. (i) $\underline{E}_0^{(1)} \subseteq \underline{F}_0 \subseteq \underline{E}_0^{(2)}$.
(ii) $\underline{F}_0 = \underline{E}_0^{(1)}$ *iff* $|M| = 1$.

Proof. It is easy. □

3.10 Lemma. *Let* $a, b \in M$ *and* $f \in \underline{E}_0$. *Then:*
(i) $f\varrho_{a,b} = \varrho_{f(a),b}$.
(ii) $\varrho_{a,b}f = g$, *where* $g = \sigma_0$ *if* $f(M) \leq b$ *and* $g = \varrho_{a,b}$ *if* $\emptyset \neq L = \{ x \in M \mid f(x) \nleq b \}$ *(then* $L \in \underline{P}(M)$*).*

Proof. Use 2.18 (or 3.7). □

3.11 Lemma. *Let* $a_1, a_2, b_1, b_2 \in M$ *and* $h = \varrho_{a_1,b_1}\varrho_{a_2,b_2}$. *Then* $h = \sigma_0$ *if* $a_2 \leq b_1$ *and* $h = \varrho_{a_1,b_2}$ *otherwise.*

Proof. Use 2.19 (or 3.10). □

3.12 Lemma. *Let* $a_1, a_2, b_1, b_2 \in M$, $f \in \underline{E}_0$ *and* $k = \varrho_{a_1,b_1} f \varrho_{a_2,b_2}$. *Then* $k = \sigma_0$ *if* $f(a_2) \leq b_1$ *and* $k = \varrho_{a_1,b_2}$ *otherwise.*

Proof. Use 2.20 (or 3.11 and the fact that $k = \varrho_{a_1,b_1}\varrho_{f(a_2),b_2}$). □

3.13 Lemma. (cf. 3.8) *Let* $a \in M$ *and* $P \in \underline{P}(M)$. *The following conditions are equivalent:*

(i) $\varrho_{a,P} = \varrho_{a,b}$ *for some* $b \in M$.

(ii) $\varrho_{a,P} = \varrho_{a,b}$ *for some* $b \in M$, $b \neq 1_M$.

(iii) *There is* $b \in M$ *such that* $M \setminus P = \{ x \in M \mid x \leq b \}$.

(iv) *The set* $M \setminus P$ *has the greatest element (if* $M \setminus P$ *is finite then* $\sum M \setminus P$ *is the greatest element).*

Proof. Use 2.21. □

3.14 Proposition. (i) $\underline{F}_0$ *is a left ideal of* $\underline{E}_0$.
(ii) $\underline{F}_0 = \{ \sum_{i=1}^{n} \varrho_{a_i,b_i} \mid n \geq 1,\ a_i, b_i \in M \}$.
(iii) $\underline{E}_0^{(2)}$ *is generated by* $\underline{F}_0$ *as an ideal of itself.*
(iv) *If* M *is finite then* $F_0 = \underline{E}_0^{(2)}$.

Proof. (i) Use 3.10(i).
(ii) Use 3.11.
(iii) By 3.7, $\varrho_{a,0}\varrho_{a,P} = \varrho_{a,P}$.
(iv) See 3.13. □

3.15 Lemma. *Let* E *be a subsemiring of* $\underline{E}_0$ *such that* $\underline{F}_0 \subseteq E$. *If* I *is a non-trivial ideal of* E *then* $\underline{F}_0 \subseteq I$.

Proof. Since I is non-trivial, there is $f \in I$, $f \neq \sigma_0$. Then $f(u) = v \neq 0$ for some $u, v \in M$. Of course, $u \neq 0$ as well. Now, $g_{a,b} = \varrho_{a,0}\varrho_{v,b} = \varrho_{a,0} f \varrho_{u,b} \in I$ for all $a.b \in M$. But $g_{a,b} = \varrho_{a,b}$ by 3.11. □

3.16 Corollary. *Let* $|M| \geq 2$ *and* E *be a subsemiring of* $\underline{E}_0$ *such that* $\underline{F}_0 \subseteq E$ *and* E *is generated by* $\underline{F}_0$ *as an ideal of itself. Then* E *is ideal-simple and* $E \subseteq \underline{E}_0^{(2)}$. □

3.17 Proposition. *Let* $|M| \geq 2$. *Then the semirings* $\underline{F}_0$ *and* $\underline{E}_0^{(2)}$ *are ideal-simple.*

Proof. Use 3.16 and 3.14(iii). □

3.18 Lemma. *Let* E *be a subsemiring of* $\underline{E}_0$ *such that* $\underline{E}_0^{(2)} \subseteq E$. *If* I *is a non-trivial ideal of* E *then* $\underline{E}_0^{(2)} \subseteq I$.

Proof. We have $\varrho_{a,0} f \varrho_{u,P} = \varrho_{a,0}\varrho_{v,P} - \varrho_{a,P}$, $f(u) = v \neq 0$ (see the proof of 3.15). □

3.19 Corollary. *Let $|M| \geq 2$ and E be a subsemiring of $\underline{E}_0$ such that $\underline{E}_0^{(2)} \subseteq E$. Then $\underline{E}_0^{(2)}$ is the smallest non-trivial ideal of E and E is ideal-simple if and only if $E = \underline{E}_0^{(2)}$.* □

3.20 Lemma. *The semiring $\underline{E}_0$ ($\underline{E}_0^{(2)}$, $\underline{F}_0$, resp.) has an additively absorbing element iff $1_M \in M$ (e.g., M finite).*

Proof. It is easy. □

3.21 Remark. The following results are proved in [4]:
(i) The semiring $\underline{E}_0$ is congruence-simple if and only if $1_M \in M$ and $0_M \neq 1_M$.
(ii) The semiring $\underline{F}_0$ ($\underline{E}_0^{(2)}$, resp.) is congruence-simple if and only if $|M| \geq 2$.
(iii) The following conditions are equivalent:
- (a) $\underline{F}_0 = \underline{E}_0$ ($\underline{E}_0^{(2)} = \underline{E}_0$, resp.).
- (b) The semiring $\underline{F}_0$ ($\underline{E}_0^{(2)}$, resp.) has a left (right, resp.) multiplicatively neutral element.
- (c) $\mathrm{id}_M \in \underline{F}_0$ ($\mathrm{id}_M \in \underline{E}_0^{(2)}$, resp.).
- (d) M is finite and distributive as a lattice.

3.22 Proposition. *The semiring $\underline{E}_0$ is ideal-simple if and only if M is non-trivial finite and distributive as a lattice.*

Proof. First, assume that $\underline{E}_0$ is ideal-simple. Then $|M| \geq 2$ and $\underline{E}_0 = \underline{E}_0^{(2)}$ by 3.2(iv). Now, M is finite by 3.3 and $\underline{F}_0 = \underline{E}_0^{(2)} = \underline{E}_0$ by 3.14(iv) and 3.19. Consequently, M is a distributive lattice by 3.21 (iii).

Conversely, assume that M is a finite distributive lattice. Then $\underline{E}_0 = \underline{F}_0$ and 3.17 applies. □

3.23 Remark. Assume that M is finite and not distributive as a lattice. Then $|M| = m \geq 5$ and $\underline{E}_0 = \underline{E}_0^{(m)}$. By 3.22, $\underline{E}_0^{(m)}$ is not ideal-simple.

4. Endomorphisms of semilattices (c)

Let M be a semilattice such that $1 = 1_M \in M$. Put $\underline{E}_1 = \{ f \in \underline{E} \mid f(1) = 1 \}$. Clearly, $\underline{E}_1$ is a subsemiring of the full endomorphism semiring $\underline{E}$ and $\mathrm{id}_M \in \underline{E}_1$. If $|M| \geq 2$ then $\underline{E}^{(1)} \nsubseteq \underline{E}_1$, and hence $\underline{E}_1 \neq \underline{E}$.

4.1 Proposition. (i) *The semiring $\underline{E}_1$ is additively idempotent and the identuty automorphism* id_M *is the multiplicatively neutral element of $\underline{E}_1$.*
(ii) *The constant endomorphism $\sigma_1 \in \underline{E}_1$ is bi-absorbing.*
(iii) $\{\sigma_1\} = \underline{E}^{(1)} \cap \underline{E}_1$ *is an ideal of $\underline{E}_1$.*

Proof. It is easy. □

For every $n \geq 1$, let $\underline{R}_1^{(n)} = \{ f \in \underline{E}_1 \,|\, |f(M)| \leq n \}$ and we put $\underline{R}_1^{(\omega)} = \cup \underline{R}_1^{(n)}$, $n \geq 1$. For every $1 \leq n \leq \omega$, let $\underline{E}_1^{(n)}$ be the subsemiring of $\underline{E}_1$ generated by $\underline{R}_1^{(n)}$.

4.2 Proposition. (i) $\underline{R}_1^{(n)} = \underline{R}^{(n)} \cap \underline{E}_1$ *for every* $1 \leq n \leq \omega$.
(ii) $\underline{E}_1^{(n)} = E_1 \cap \underline{E}^{(n)}$ *for every* $1 \leq n \leq \omega$.
(iii) $\{\sigma_1\} = \underline{R}_1^{(1)} = \underline{E}_1^{(1)} \subseteq \underline{E}_1^(2) \subseteq \underline{E}_1^{(3)} \subseteq \cdots \subseteq \underline{E}_1^{(\omega)} = \underline{R}_1^{(\omega)}$.
(iv) *All the subsemirings* $\underline{E}_1^{(1)}, \underline{E}_1^{(2)}, \dots, \underline{E}_1^{(\omega)}$ *are ideals of the semiring* $\underline{E}_1$.

Proof. Everything is easy (use 2.2), nevertheless (ii) deserves a short proof (perhaps).

Clearly, $\underline{E}_1^{(n)} \subseteq \underline{E}_1 \cap \underline{E}^{(n)}$. On the other hand, if $f \in \underline{E}_1 \cap \underline{E}^{(n)}$ then $f = \sum f_i$, $f_i \in \underline{R}^{(n)}$. Now, define $\overline{f_i}$ by $\overline{f_i}(x) = f(x)$ for $x \neq 1$ and $\overline{f_i}(1) = 1$. One sees easily that $\overline{f_i} \in \underline{R}_1^{(n)}$ and $f = \sum \overline{f_i}$. Thus $f \in \underline{E}_1^{(n)}$. □

4.3 Proposition. *The following conditions are equivalent:*

(i) *M is finite.*
(ii) *$\underline{E}_1$ is finite.*
(iii) $\underline{E}_1^{(\omega)} = \underline{E}_1$.
(iv) $\mathrm{id}_M \in \underline{E}_1^{(\omega)}$.
(v) $\underline{E}_1^{(m)} = \underline{E}_1$ *for some* $m \geq 1$.

Proof. It is easy. □

4.4 Propostion. $\underline{E}_1^{(n)} = \{ \sum_{i=1}^m f_i \,|\, m \geq 1, f_i \in \underline{R}_1^{(n)} \}$ *for every* $1 \leq n \leq \omega$.

Proof. It is easy. □

4.5 Lemma. $\underline{R}_1^{(2)} = \{\sigma_1\} \cup \{ \varrho_{a,1,P} \,|\, a \in M, P \in \underline{P}(M) \}$.

Proof. Combine 2.9 and 2.10. □

In the sequel, we put $\tau_{a,P} = \varrho_{a,1,P}$. We have $\tau_{1,P} = \sigma_1$ and if $|M| \geq 2$ then $\underline{R}_1^{(2)} = \{ \tau_{a,P} \,|\, a \in M,\ P \in \underline{P}(M) \}$.

4.6 Corollary. *Let* $|M| \geq 2$. *Then* $\underline{E}_1^{(2)} = \{ \sum_{i=1}^m \tau_{a_i,P_i} \,|\, m \geq 1, a_i \in M, P_i \in \underline{P}(M) \}$. □

4.7 Lemma. *Let* $a \in M$, $P \in \underline{P}(M)$ *and* $f \in \underline{E}_1$. *Then* $f\tau_{a,P} = \tau_{f(a),P}$ *and we put* $g = \tau_{a,P} f$, $K = \{ x \in M \,|\, f(x) \notin P \}$ *and* $L = \{ x \in M \,|\, f(x) \in P \}$. *Then:*
(i) $1 \in L$, $M = K \cup L$ *and* $K \cap L = \emptyset$.
(ii) *If* $L = M$ *(or* $K = \emptyset$*) then* $g = \sigma_1$.
(iii) *If* $L \neq M$ *(or* $K \neq \emptyset$*) the* $L \in \underline{P}(M)$ *and* $g = \tau_{a,L}$.

Proof. Use 2.13. □

Put $\tau_{a,b} = \varrho_{a,1,b}$ for all $a, b \in M$, $b \neq 1$. That is, $\tau_{a,b}(x) = a$ if $x \leq b$ and $\tau_{a,b} = 1$ otherwise.

4.8 Lemma. *Let $a, b \in M$, $b \neq 1$. Then $\tau_{a,b} = \varrho_{a,1,Q_b} = \tau_{a,Q_b}$.*

Proof. Use 2.14. □

Denote by $\underline{F}_1$ the subsemiring of $\underline{E}_1$ generated by all the endomorphisms $\tau_{a,b}$, $a, b \in M$, $b \neq 1$ ($\underline{F}_1 = \{\sigma_1\}$ if $|M| = 1$).

4.9 Proposition. (i) $\underline{E}_1^{(1)} \subseteq \underline{F}_1 \subseteq \underline{E}_1^{(2)}$.
(ii) $\underline{F}_1 = \underline{E}_1^{(1)}$ *iff* $|M| = 1$.

Proof. It is obvious. □

4.10 Lemma. *Let $a, b \in M$, $b \neq 1$, and $f \in \underline{E}_1$. Then:*
(i) $f\tau_{a,b} = \tau_{f(a),b}$.
(ii) $\tau_{a,b}f = g$*, where* $g = \sigma_1$ *if* $f(x) \nleq b$ *for every* $x \in M$ *and* $g = \tau_{a,L}$ *if* $L = \{\, x \in \in M \mid f(x) \nleq b \,\} \neq M$.

Proof. Use 2.18 (or 4.7) □

4.11 Lemma. *Let $a_1, a_2, b_1, b_2 \in M$, $b_1 \neq 1 \neq b_2$, and $h = \tau_{a_1.b_1}\tau_{a_2,b_2}$. Then $h = \tau_{a_1,b_2}$ if $a_2 \leq b_1$ and $h = \sigma_1$ otherwise.*

Proof. Use 2.19 (or 4.10). □

4.12 Lemma. *Let $a_1, a_2, b_1, b_2 \in M$, $b_1 \neq 1 \neq b_2$, $f \in \underline{E}_1$ and $k = \tau_{a_1,b_1}f\tau_{a_2,b_2}$. Then $k = \tau_{a_1,b_2}$ if $f(a_2) \leq b_1$ and $k = \sigma_1$ otherwise.*

Proof. Use 2.20 (or 4.11 and the fact that $k = \tau_{a_1,b_1}\tau_{f(a_2),b_2}$). □

4.13 Lemma. (cf. 4.8) *Let $a \in M$ and $P \in \underline{P}(M)$. The following conditions are equivalent:*

(i) $\tau_{a,P} = \tau_{a,b}$ *for some* $b \in M$, $b \neq 1$.
(ii) *There is $b \in M$ such that $M \setminus P = \{\, x \in M \mid x \leq b \,\}$.*
(iii) *The set $M \setminus P$ has the greatest elemeng (if $M \setminus P$ is finite then $\sum M \setminus P$ is the greatest element).*

Proof. Use 2.21. □

4.14 Proposition. (i) $\underline{F}_1$ *is a left ideal of* $\underline{E}_1$.
(ii) $\underline{F}_1 = \{\, \sum_{i=1}^{n} \tau_{a_i,b_i} \mid n \geq 1,\ a_i, b_i \in M, b_i \neq 1 \,\}$.
(iii) $\underline{E}_1^{(2)}$ *is generated by* $\underline{F}_1$ *as an ideal of itself.*
(iii) *If M is finite then* $\underline{F}_1 = \underline{E}_1^{(2)}$.

Proof. (i) Use 4.10(i).
(ii) Use 4.11.
(iii) By 4.7, $\tau_{a,b}\tau_{a,P} = \tau_{a,P}$ for all $a, b \in M$, $b \neq 1$, $P \in \underline{P}(M)$.
(iv) See 4.13. □

4.15 Lemma. *Let E be a subsemiring of $\underline{E}_1$ such that $\underline{F}_1 \subseteq E$. If I is a non-trivial ideal of E then $\underline{F}_1 \subseteq I$.*

Proof. Since I is non-trivial, there is $f \in I$, $f \neq \sigma_1$. Then $f(u) = v \neq 1$ for some $u, v \in M$. Of course, $u \neq 1$ as well. Now, $g_{a,b} = \tau_{a,v}\tau_{v,b} = \tau_{a,v} f \tau_{u,v} \in I$ for all $a, b \in M$, $b \neq 1$. But $g_{a,b} = \tau_{a,b}$ by 4.11. □

4.16 Corollary. *Let $|M| \geq 2$ and E be a subsemiring of $\underline{E}_1$ such that $\underline{F}_1 \subseteq E$ and E is generated by $\underline{F}_1$ as an ideal of itself. Then E is ideal-simple and $E \subseteq \underline{E}_1^{(2)}$.* □

4.17 Proposition. *Let $|M| \geq 2$. Then the semirings $\underline{F}_1$ and $\underline{E}_1^{(2)}$ are ideal-simple.*

Proof. Use 4.16 and 4.14(iii). □

4.18 Lemma. *Let E be a subsemiring of $\underline{E}_1$ such that $\underline{E}_1^{(2)} \subseteq E$. If I is a non-trivial ideal of E then $\underline{E}_1^{(2)} \subseteq I$.*

Proof. We have $\tau_{a,v} f \tau_{u,P} = \tau_{a,v}\tau_{v,P} = \tau_{a,P}$, $f(u) = v \neq 1$ (see the proof of 4.15). □

4.19 Corollary. *Let $|M| \geq 2$ and E be a subsemiring of $\underline{E}_1$ such that $\underline{E}_1^{(2)} \subseteq E$. Then $\underline{E}_1^{(2)}$ is the smallest non-trivial ideal of E and E is ideal-simple if and only if $E = \underline{E}_1^{(2)}$.* □

4.20 Lemma. (i) *The semiring $\underline{E}_1$ ($\underline{E}_1^{(2)}$, resp.) has an additively neutral element iff $0_M \in M$ and the element $1 = 1_M$ is irreducible (if $|M| \geq 2$ then $\tau_{0,\{1\}}$ is the additively neutral element.*
(ii) *The semiring $\underline{F}_1$ has an additively neutral element iff either $|M| = 1$ or $|M| \geq 2$ and the set $M \setminus \{1\}$ has the greatest element (if w is that element then $\tau_{0,w}$ is the additively neutral element of $\underline{F}_1$).*

Proof. Assume that $|M| \geq 2$. Now, let $f \in \underline{F}_1$ be such that $f + \tau_{a,b} = \tau_{a,b}$ for all $a, b \in M$, $b \neq 1$. Then $0_M \in M$ and $f(x) = 0$ for every $x \in M$, $x \neq 1$.

Next, assume that $0_M \in M$ and define a transformationα of M by $\alpha(1) = 1$ and $\alpha(x) = 0$ for every $x \neq 1$. Then $\alpha \in \underline{E}_1$ iff 1 is irreducible. Then $\alpha = \tau_{0,\{1\}}$. The rest is clear. □

4.21 Example. Put $M = \omega + 1 = \{0, 1, \ldots, \omega\}$. Then $0 = 0_M$, $\omega = 1_M$ and the semiring $\underline{E}_1^{(2)}$ has an additively neutral element by 4.10. On the other hand, $\underline{F}_1$ has no additively neutral element.

4.22 Lemma. *If* $\mathrm{id}_M \in \underline{E}_1^{(2)}$ *then*$|M| \leq 2$.

Proof. Assume $|M| \geq 2$. Then $\mathrm{id}_M = \sum \tau_{a_i, P_i}$ and $M \setminus \{1\} = M \setminus \cup P_i$, $\cup P_i = \{1\}$ and $P_i = \{1\}$. Furthermore, $a_i \in M \setminus \{1\}$ and $M \setminus \{1\} = \{\sum a_i\}$. Thus $|M| = 2$. □

4.23 Corollary. $\underline{F}_1 = \underline{E}_1$ *(*$\underline{E}_1^{(2)} = \underline{E}_1$*, resp.) if and only if* $|M| \leq 2$. □

4.24 Corollary. *The semiring* $\underline{E}_1$ *is ideal-simple if and only if* $|M| = 2$ *(then* $|\underline{E}_1| =$ $= 2$*).* □

5. Endomorphisms of semilattices (d)

Let M be a semilattice such that $0 = 0_M \in M$, $1 = 1_M \in M$ and $0 \neq 1$. Put $\underline{E}_{01} = \{ f \in \underline{E} \mid f(0) = 0 \text{ and } f(1) = 1 \}$. Clearly, E_{01} is a subsemiring of the full endomorphism semiring $\underline{E}$ and $\mathrm{id}_M \in \underline{E}_{01}$. If $|M| = 2$ then $\underline{E}_{01} = \{\mathrm{id}_M\}$. Furthermore, $\underline{E}_{01} \neq \underline{E}, \underline{E}_0, \underline{E}_1$.

5.1 Proposition. *The semiring* $\underline{E}_{01}$ *is additively idempotent and the identity automorphism* id_M *is the multiplicatively neutral element of* $\underline{E}_{01}$.

Proof. It is easy. □

For every $n \geq 1$, let $R_{01}^{(n)} = \{ f \in \underline{E}_{01} \mid |f(M)| \leq n \}$ and we put $R_{01}^{(\omega)} = \cup \underline{R}_{01}^{(n)}$, $n \geq 1$. For every $2 \leq n \leq \omega$, let $\underline{E}_{01}^{(n)}$ be the subsemiring of $\underline{E}_{01}$ generated by $\underline{R}_{01}^{(n)}$ (we have $\underline{R}_{01}^{(1)} = \emptyset$).

5.2 Proposition. (i) $\underline{R}_{01}^{(n)} = \underline{R}^{(n)} \cap \underline{E}_{01}$ *for every* $1 \leq n \leq \omega$.

(ii) $\underline{E}_{01}^{(n)} = \underline{E}_{01} \cap \underline{E}^{(n)}$ *for every* $2 \leq n \leq \omega$.

(iii) $\emptyset = \underline{R}_{01}^{(1)} \subseteq \underline{E}_{01}^{(2)} \subseteq \underline{E}_{01}^{(3)} \subseteq \cdots \subseteq \underline{E}_{01}^{(\omega)} = \underline{R}_{01}^{(\omega)}$.

(iv) *All the subsemirings* $\underline{E}_{01}^{(2)}, \underline{E}_{01}^{(3)}, \ldots, \underline{E}_{01}^{(\omega)}$ *are ideals of the semiring* $\underline{E}_{01}$.

Proof. Use 4.2. □

5.3 Proposition. *The following conditions are equivalent:*

(i) *M is finite.*

(ii) $\underline{E}_{01}$ *is finite.*

(iii) $\underline{E}_{01}^{(\omega)} = \underline{E}_{01}$.

(iv) $\mathrm{id}_M \in \underline{E}_{01}^{(\omega)}$,

(v) $\underline{E}_{01}^{(m)} = \underline{E}_{01}$ *for some* $m \geq 2$.

Proof. It is easy. □

5.4 Proposition. $\underline{E}_{01}^{(n)} = \{ \sum_{i=1}^{m} f_i \,|\, m \geq 1, f_i \in \underline{R}_{01}^{(n)} \}$ *for every* $2 \leq n \leq \omega$.

Proof. It is easy. □

5.5 Lemma. $\underline{R}_{01}^{(2)} = \{ \varrho_{0,1,P} \,|\, P \in \underline{P}(M) \}$.

Proof. Use 4.5. □

In the sequel we put $\lambda_P = \varrho_{0,1,P}$ for every $P \in \underline{P}(M)$; $\lambda_P(P) = \{1\}$ and $\lambda_P(M \setminus P) = \{0\}$. We have $\lambda_P = \varrho_{1,P} = \tau_{0,P}$.

5.6 Proposition. (i) $\underline{E}_{01}^{(2)} = \{ \lambda_P \,|\, P \in \underline{P}(M) \}$.
(ii) $\lambda_{P_1} + \lambda_{P_2} = \lambda_{P_1 \cup P_2}$ *and* $\lambda_{P_1}\lambda_{P_2} = \lambda_{P_2}$ *for all* $P_1, P_2 \in \underline{P}(M)$.

Proof. It is easy. □

5.7 Corollary. *The following conditions are equivalent:*

(i) $\underline{E}_{01}^{(2)}$ *is ideal-simple.*
(ii) $\underline{E}_{01}^{(2)}$ *is right ideal-free.*
(iii) $|\underline{E}_{01}^{(2)}| \geq 2$.
(iv) $|M| \geq 3$. □

5.8 Lemma. *Let* $P \in \underline{P}(M)$ *and* $f \in \underline{E}_{01}$. *Then* $f\lambda_P = \lambda_P$ *and we put* $g = \lambda_P f$ *and* $L = \{ s \in M \,|\, f(s) \in P \}$. *Then* $1 \in L$, $0 \notin L$, $L \in \underline{P}(M)$ *and* $g = \lambda_L$.

Proof. It is easy. □

Put $\lambda_a = \varrho_{0,1,a}$ for every $a \in M$, $a \neq 1$. That is, $\lambda_a(x) = 0$ if $x \leq a$ and $\lambda_a(x) = 1$ otherwise. We have $\lambda_a = \varrho_{1,a} = \tau_{0,a}$.

5.9 Lemma. *Let* $a \in M$, $a \neq 1$. *Then* $\lambda_a = \varrho_{0,1,Q_a} = \lambda_{Q_a}$.

Proof. It is easy. □

Let $\underline{F}_{01}$ be the subsemiring of $\underline{E}_{01}$ generated by all the endomorphisms λ_a, $a \in M$, $a \neq 1$.

5.10 Lemma. (i) $\underline{F}_{01} = \{ \sum_{i=1}^{n} \lambda_{a_i} \,|\, n \geq 1, a_i \in M, a_i \neq 1 \}$.
(ii) $\lambda_a\lambda_b = \lambda_b$ *for all* $a, b \in M$, $a \neq 1 \neq b$.
(iii) $\lambda_a + \lambda_b = \lambda_{Q_a \cup Q_b}$ *for all* $a, b \in M$, $a \neq 1 \neq b$.
(iv) $\sum_{i=1}^{n} \lambda_{a_i} = \lambda_{\cup Q_{a_i}}$ *for all* $a_i \in M$, $a_i \neq 1$.

Proof. It is easy. □

5.11 Corollary. *The following conditions are equivalent:*

(i) $\underline{F}_{01}$ *is ideal-simple.*
(ii) $\underline{F}_{01}$ *is right-ideal-free.*
(iii) $|\underline{F}_{01}| \geq 2$.
(iv) $|M| \geq 3$. □

5.12 Proposition. *The semiring $\underline{E}_{01}$ is never ideal-simple.*

Proof. If $\underline{E}_{01}$ is ideal-simple then M is finite and $|M| \geq 3$. Since $\underline{E}_{01}^{(2)}$ is a non-trivial ideal of $\underline{E}_{01}$, we have $\underline{E}_{01} = \underline{E}_{01}^{(20}$ and $\mathrm{id}_M \in \underline{E}_{01}^{(2)}$. But then $|\mathrm{id}_M(M)| = 2$ by 5.6(i), and hence $|M| = 2$, a contradiction. □

References

[1] J. Ježek, T. Kepka and M. Maróti: *The endomorphism semiring of a semilattice,* Semigroup Forum **78** (2009), 21–26.
[2] T. Kepka and P. Němec: *Ideal-simple semirings I.*
[3] T. Kepka and P. Němec: *Ideal-simple semirings II.*
[4] J. Zumbrägel: *Congruence-simple semirings with zero,* J. Algebra Appl. **7** (2008), 363–377.