Pengzhan Huang

Superconvergence of a stabilized approximation for the Stokes eigenvalue problem by projection method

Applications of Mathematics, Vol. 59 (2014), No. 4, 361–370

SUPERCONVERGENCE OF A STABILIZED APPROXIMATION FOR THE STOKES EIGENVALUE PROBLEM BY PROJECTION METHOD

Pengzhan Huang, Urumqi

(Received June 1, 2012)

Abstract. This paper presents a superconvergence result based on projection method for stabilized finite element approximation of the Stokes eigenvalue problem. The projection method is a postprocessing procedure that constructs a new approximation by using the least squares method. The paper complements the work of Li et al. (2012), which establishes the superconvergence result of the Stokes equations by the stabilized finite element method. Moreover, numerical tests confirm the theoretical analysis.

Keywords: Stokes eigenvalue problem; stabilized method; lowest equal-order pair; projection method; superconvergence

MSC 2010: 65N30, 65N25, 65B99

1. Introduction

Recently, there has been growing interest in the finite element approximations of the Stokes eigenvalue problem. At the time of writing this paper, numerous works devoted to this problem exist (see [12], [2], [20], [24], [3], [9], [10], and the references cited therein). This problem is as follows: Find $(u, p; \lambda)$ such that

$$(1.1)\qquad \begin{aligned} -\nu\Delta u + \nabla p &= \lambda u \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded, convex and open subset of $\mathbb{R}^2$ with a boundary $\partial\Omega$, $u = (u_1(x), u_2(x))$ represents the velocity vector, $p = p(x)$ the pressure, $\nu > 0$ the viscosity and $\lambda \in \mathbb{R}$ the eigenvalue.

The research has been supported by the NSF of Xinjiang Province (No. 2013211B01).

The idea of the stabilized finite element method based on the projection of the pressure onto the piecewise constant space was proposed in [1], [15] for the stationary Stokes equations. This stabilization does not require any edge-based data structure or any subdivision of a mesh into patches for the local jump formulation. Besides, it does not require any approximation of derivatives or any specification of mesh-dependent parameters. Hence, the resulting stabilized method can easily be formulated. This paper first recalls the stabilized finite element method for the Stokes eigenvalue problem approximated by using the lowest equal-order finite elements $P_1 - P_1$ [15]. Then a superconvergence result based on projection method for this stabilized finite element method of the Stokes eigenvalue problem is presented.

The main purpose of this article is to establish a superconvergence result for the stabilized finite element method for the Stokes eigenvalue problem by the projection method proposed and analyzed previously in [22], [21]. The projection method is a postprocessing procedure that constructs a new approximation by using the method of least squares surface fitting. Some details of this projection method can be found in the works of Chen and Wang [4], Heimsund et al. [8], Ye et al. [23], [5], [6], Liu and Yan [19], Li et al. [18], [14], [13], [16], [17] and Huang et al. [11]. Li et al. have used a local coarse mesh L^2-projection to establish the superconvergence of a stabilized finite element approximation for the Stokes equations. So this paper can be considered a sequel and a complement of the work of Li et al. in [17].

2. Preliminaries

We introduce the following Hilbert spaces:

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \colon \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

The spaces $L^2(\Omega)^m$, $m = 1, 2$, are equipped with the L^2-scalar product $(\cdot,\cdot)$ and the L^2-norm $\|\cdot\|_0$. The space X is endowed with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\|\nabla u\|_0$. Standard definitions are used for the Sobolev spaces $W^{m,p}(\Omega)$, with the norm $\|\cdot\|_{m,p}$, $m, p \geqslant 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$.

We define continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$, respectively, by

$$a(u, v) = \nu(\nabla u, \nabla v), \ \forall\, u, v \in X,$$

and

$$d(v, q) = (q, \operatorname{div} v), \ \forall\, v \in X, \ \forall\, q \in M,$$

and a generalized bilinear form $B((\cdot,\cdot);(\cdot,\cdot))$ on $(X \times M) \times (X \times M)$ by

$$B((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q), \quad \forall\,(u,p),(v,q) \in X \times M.$$

With the above notation, the variational formulation of problem (1.1) reads as follows: Find $(u,p;\lambda) \in (X \times M) \times \mathbb{R}$ with $\|u\|_0 = 1$, such that for all $(v,q) \in X \times M$,

$$B((u,p);(v,q)) = \lambda(u,v). \tag{2.1}$$

Moreover, the bilinear form $d(\cdot,\cdot)$ satisfies the inf-sup condition for all $q \in M$

$$\sup_{v \in X} \frac{|d(v,q)|}{\|\nabla v\|_0} \geqslant \beta_1 \|q\|_0,$$

where β_1 is a positive constant depending only on Ω.

3. A stabilized mixed finite element method

Let h be a real positive parameter tending to 0. The finite element subspace $X_h \times M_h$ of $X \times M$ is characterized by K_h, a partitioning of Ω into triangles K with the mesh size h, assumed to be uniformly regular in the usual sense. Then we define

$$X_h = \{u \in C^0(\overline{\Omega})^2 \cap X\colon\ u|_K \in P_1(K)^2,\ \forall\,K \in K_h\},$$
$$M_h = \{q \in C^0(\overline{\Omega}) \cap M\colon\ q|_K \in P_1(K),\ \forall\,K \in K_h\},$$

where $P_1(K)$ represents the space of linear functions on K.

Note that the lowest equal-order pair does not satisfy the discrete inf-sup condition

$$\sup_{v_h \in X_h} \frac{d(v_h,q_h)}{\|\nabla v_h\|_0} \geqslant \beta_2 \|q_h\|_0,\ \forall\,q_h \in M_h,$$

where the constant $\beta_2 > 0$ is independent of h. In order to fulfil this condition, a stabilized generalized bilinear term is used:

$$B_h((u_h,p_h);(v,q)) = B((u_h,p_h);(v,q)) - G(p_h,q),$$

where $G(p_h,q)$ can be defined (see [1]) by

$$G(p_h,q) = (p_h - \Pi_h p_h, q - \Pi_h q), \quad p_h, q \in M_h, \tag{3.1}$$

with the local pressure projection Π_h defined by

$$(p,q_h) = (\Pi_h p, q_h), \quad \forall\,p \in L^2(\Omega),\ q_h \in W_h.$$

Here $W_h \subset L^2(\Omega)$ denotes the piecewise constant space associated with the triangulation K_h. The following properties of the projection operator Π_h can be proved [15], [14]:

$$\|\Pi_h p\|_0 \leqslant c_1\|p\|_0, \quad \forall\, p \in L^2(\Omega), \tag{3.2}$$

$$\|p - \Pi_h p\|_0 \leqslant c_2 h\|p\|_1, \quad \forall\, p \in H^1(\Omega). \tag{3.3}$$

Subsequently, c (with or without a subscript) will denote a positive constant which is independent of mesh parameters and may stand for different values at its different occurrences.

Now, the corresponding discrete variational formulation of (2.1) for the Stokes eigenvalue problem is recast: Find $(u_h, p_h; \lambda_h) \in (X_h \times M_h) \times \mathbb{R}$ with $\|u_h\|_0 = 1$, such that for all $(v, q) \in X_h \times M_h$,

$$B_h((u_h, p_h); (v, q)) = \lambda_h(u_h, v). \tag{3.4}$$

Since the convergence of the finite element approximation to the eigenvalue problem depends on the regularity of the original eigenvalue problem, here and hereafter we assume that the regularity of the eigenfunction is $(u, p) \in H^2(\Omega)^2 \times H^1(\Omega)$. By well-established techniques for eigenvalue approximation [2], [24], [20] and for the stabilized mixed finite element method [1], [17], [15], one has the following result.

Theorem 3.1. *Let* $(u, p; \lambda)$ *be an eigenvalue pair of* (2.1). *Then,* $(u_h, p_h; \lambda_h)$ *in* (3.4) *satisfies the error estimates*

$$|\lambda - \lambda_h| \leqslant c_3(\nu\|\nabla(u - u_h)\|_0 + \|p - p_h\|_0)^2, \tag{3.5}$$

$$\nu\|u - u_h\|_0 + h(\nu\|\nabla(u - u_h)\|_0 + \|p - p_h\|_0) \leqslant c_4 h^2(\nu\|u\|_2 + \|p\|_1). \tag{3.6}$$

4. Superconvergence analysis

The L^2-projection is a post-processing technique introduced by Wang [21] for the standard Galerkin method. The basic idea is to project the approximate solution to another finite dimensional space on a different, but coarser mesh. The difference in the two mesh sizes can be used to achieve a superconvergence after the post-processing procedure.

Now, we introduce other two partitions K_{ϱ_i} with mesh sizes ϱ_i, where $h \ll \varrho_i$ $(i = 1, 2)$. Assume that the ϱ_i and h have the relationship

$$\varrho_i = h^{\sigma_i}$$

with $\sigma_i \in (0,1)$. The parameter σ_i will play an important role later in achieving a superconvergence for the stabilized finite element approximation (u_h, p_h). Let U_{ϱ_1} and V_{ϱ_2} be any two finite dimensional spaces which consist of piecewise polynomials of degree s and t, respectively, associated with the partitions K_{ϱ_1} and K_{ϱ_2}.

Subsequently, define Q_{ϱ_1} and R_{ϱ_2} to be the L^2-projectors from $L^2(\Omega)^2$ onto the spaces U_{ϱ_1} and V_{ϱ_2}, respectively. Roughly speaking, the post-processing of the stabilized finite element approximation (u_h, p_h) is simply given by their L^2-projections:

$$
\begin{aligned}
(Q_{\varrho_1} u, v) &= (u, v), \quad \forall\, u \in L^2(\Omega)^2,\ v \in U_{\varrho_1},\\
(R_{\varrho_2} p, q) &= (p, q), \quad \forall\, p \in L^2(\Omega),\ q \in V_{\varrho_2}.
\end{aligned}
$$

Here we assume that all eigenvalues have ascent and their geometric multiplicity is one. So, an argument similar to that in [3], [12] yields the following theorem based on the results of [17].

Theorem 4.1. *Under the assumptions of Theorem* 3.1, *if* ϱ_1, σ_1, *and* h *satisfy* $\varrho_1 = O(h^{\sigma_1})$ *with* $\sigma_1 = 2/(s+1)$, *then*

$$\|u - Q_{\varrho_1} u_h\|_0 \leqslant ch^2(\|u\|_{s+1} + \|p\|_1)$$

and

$$\|\nabla_{\varrho_1}(u - Q_{\varrho_1} u_h)\|_0 \leqslant ch^{2s/(1+s)}(\|u\|_{s+1} + \|p\|_1),$$

where ∇_{ϱ_1} *is defined element-wise over the partition* K_{ϱ_1}. *Furthermore, if* ϱ_2, σ_2, *and* h *satisfy* $\varrho_2 = O(h^{\sigma_2})$ *with* $\sigma_2 = 2/(t+2)$, *then*

$$\|p - R_{\varrho_2} p_h\|_0 \leqslant ch^{2(t+1)/(t+2)}(\|u\|_2 + \|p\|_{t+1}).$$

Using the above results, we propose an eigenvalue approximation based on projection method. The scheme is as follows:

$$\hat{\lambda}_h = B_h((Q_{\varrho_1} u_h, R_{\varrho_2} p_h); (Q_{\varrho_1} u_h, R_{\varrho_2} p_h)). \tag{4.1}$$

Lemma 4.1 ([10])**.** *Let* $(u, p; \lambda)$ *be an eigenvalue pair of* (2.1). *For any* $w \in X \backslash \{0\}$ *and* $s \in M$,

$$\frac{B((w,s);(w,s))}{(w,w)} - \lambda = \frac{B((w-u, s-p);(w-u, s-p))}{(w,w)} - \lambda \frac{(w-u, w-u)}{(w,w)}.$$

Theorem 4.2. *Under the assumptions of Theorem* 3.1, *let* $\hat{\lambda}_h$ *be defined by* (4.1). *Then*

$$|\lambda - \hat{\lambda}_h| \leqslant c(h^{4s/(1+s)} + h^{4(t+1)/(t+2)}).$$

P r o o f. In view of Lemma 4.1, we see that

$$\begin{aligned}(4.2)\quad \hat{\lambda}_h - \lambda &\\ = \frac{B((Q_{\varrho 1}u_h - u, R_{\varrho 2}p_h - p); (Q_{\varrho 1}u_h - u, R_{\varrho 2}p_h - p)) - G(R_{\varrho 2}p_h, R_{\varrho 2}p_h)}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)} &\\ - \lambda\frac{(Q_{\varrho 1}u_h - u, Q_{\varrho 1}u_h - u)}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)} - \frac{B_h((Q_{\varrho 1}u_h, R_{\varrho 2}p_h); (Q_{\varrho 1}u_h, R_{\varrho 2}p_h))}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)} &\\ + B_h((Q_{\varrho 1}u_h, R_{\varrho 2}p_h); (Q_{\varrho 1}u_h, R_{\varrho 2}p_h)).&\end{aligned}$$

Moreover, due to Theorem 4.1, we have

$$(4.3)\qquad \frac{B((Q_{\varrho 1}u_h - u, R_{\varrho 2}p_h - p); (Q_{\varrho 1}u_h - u, R_{\varrho 2}p_h - p))}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)} \leqslant ch^{4s/(1+s)}(\|u\|_{s+1} + \|p\|_1)^2,$$

$$(4.4)\qquad \lambda\frac{(Q_{\varrho 1}u_h - u, Q_{\varrho 1}u_h - u)}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)} \leqslant ch^4(\|u\|_{s+1} + \|p\|_1)^2.$$

Utilizing (3.1)–(3.3), we arrive at

$$\begin{aligned}(4.5)\quad \frac{G(R_{\varrho 2}p_h, R_{\varrho 2}p_h)}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)} &= \frac{(\|R_{\varrho 2}p_h - p\|_0 + \|p - \Pi_h p\|_0 + \|\Pi_h p - \Pi_h(R_{\varrho 2}p_h)\|_0)^2}{\|Q_{\varrho 1}u_h\|_0^2}\\ &\leqslant c(h + h^{2(t+1)/(t+2)})^2(\|u\|_2 + \|p\|_{t+1})^2.\end{aligned}$$

Moreover, by the definition of the L^2-projection and the fact that $\|u_h\|_0 = 1$, we obtain

$$\begin{aligned}B_h((Q_{\varrho 1}u_h, R_{\varrho 2}p_h); (Q_{\varrho 1}u_h, R_{\varrho 2}p_h)) &- \frac{B_h((Q_{\varrho 1}u_h, R_{\varrho 2}p_h); (Q_{\varrho 1}u_h, R_{\varrho 2}p_h))}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)}\\ &= B_h((Q_{\varrho 1}u_h, R_{\varrho 2}p_h); (Q_{\varrho 1}u_h, R_{\varrho 2}p_h))\Big(1 - \frac{1}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)}\Big),\end{aligned}$$

and

$$\begin{aligned}\Big(\frac{1}{(Q_{\varrho 1}u_h, Q_{\varrho 1}u_h)} - 1\Big) &= \frac{1}{\|Q_{\varrho 1}u_h\|_0^2}(1 - \|Q_{\varrho 1}u_h\|_0^2)\\ &= \frac{1}{\|Q_{\varrho 1}u_h\|_0^2}(\|u_h\|_0^2 - \|Q_{\varrho 1}u_h\|_0^2)\\ &= \frac{1}{\|Q_{\varrho 1}u_h\|_0^2}(u_h + Q_{\varrho 1}u_h, u_h - Q_{\varrho 1}u_h)\end{aligned}$$

$$= \frac{1}{\|Q_{\varrho 1} u_h\|_0^2}(u_h - Q_{\varrho 1} u_h, u_h - Q_{\varrho 1} u_h)$$
$$= \frac{1}{\|Q_{\varrho 1} u_h\|_0^2}\|u_h - Q_{\varrho 1} u_h\|_0^2 \leqslant ch^4(\|u\|_{s+1} + \|p\|_1)^2.$$

The above inequality is deduced by Theorems 3.1, 4.1, and the triangle inequality.

Thus, the above inequality and (4.3)–(4.5) lead directly to

$$|\lambda - \hat{\lambda}_h| \leqslant c(h^{4s/(1+s)} + h^{4(t+1)/(t+2)}).$$

□

5. Numerical experiments

In this section we present numerical experiments to check the numerical theory developed in the previous sections and exhibit the superconvergence results of the stabilized mixed finite element approximation for the Stokes eigenvalue problem by L^2-projection. The stabilized method is characterized by using linear polynomial functions for both the velocity and the pressure field. The stabilized term is defined by local Gauss integration [15] as

$$G(p_h, q) = \sum_{K \in K_h} \left\{ \int_{K,2} p_h q \, \mathrm{d}x - \int_{K,1} p_h q \, \mathrm{d}x \right\}, \quad \forall\, p_h,\ q \in M_h,$$

where $\int_{K,i} g(x)\,\mathrm{d}x$ indicates a local Gauss integral over K that is exact for polynomials of degree $i = 1, 2$. In particular, the trial function $p_h \in M_h$ must be projected to a piecewise constant space W_h defined below when $i = 1$ for any $q \in M_h$.

In the given experiment, the pressure and velocity are approximated by the lowest equal-order finite element pairs defined with respect to the same uniform triangulation. And the algorithms are implemented using public domain finite element software [7] with some of our additional codes. From Theorem 3.1, we know that the stabilized finite element solutions $(u_h, p_h; \lambda_h)$ have the optimal error estimates. Moreover, in order to achieve superconvergence for the numerical solutions, an L^2-projection is applied. The key of this technique is to project a finite element space onto other finite element space based on a high order of polynomials on the coarse mesh. In Table 1, we present the superconvergence results for the stabilized finite element numerical solutions by Theorems 3.1 and 4.2.

FEM type	FEM solutions	σ_1	σ_2	λ-rate
$P_1 - P_1$	λ_h	—	—	2
$P_2 - P_1$	$\hat{\lambda}_h$	2/3	2/3	8/3

Table 1. Superconvergence results of $P_1 - P_1$ by L^2-projection

Let the computation be carried out in the region $\Omega = \{(x,y)\colon\ 0 < x, y < 1\}$. We consider the Stokes eigenvalue problem in the case of the viscidity $\nu = 1$, and it will be numerically solved by the stabilized mixed method on uniform mesh. Here, we just consider the first eigenvalue of the Stokes eigenvalue problem for the sake of simplicity. The exact solution of this problem is unknown. Thus, we take the numerical solution by the standard Galerkin method ($P_2 - P_1$ element) computed on a very fine mesh (6742 grid points) as the "exact" solution for the purpose of comparison. Here, we take $\lambda = 52.3447$ as the first exact eigenvalue.

The results of $P_1 - P_1$ and superconvergence results of $P_2 - P_1$ by projecting $P_1 - P_1$ to $P_2 - P_1$ are tabulated in Table 2. And the convergence rates are reported in Figure 1. From Table 2 and Figure 1, we can see that the numerical results support the theoretical analysis well.

$\frac{1}{h}$	λ_h	$\frac{\lvert\lambda - \lambda_h\rvert}{\lambda}$	$\hat{\lambda}_h$	$\frac{\lvert\lambda - \hat{\lambda}_h\rvert}{\lambda}$
4	72.316	0.381474	70.591	0.3485111
8	57.395	0.096432	54.597	0.0429883
12	54.608	0.043190	52.848	0.0095694
16	53.620	0.024319	52.515	0.0031972

Table 2. The results of $P_1 - P_1$ and superconvergence results of $P_2 - P_1$ by projecting $P_1 - P_1$ to $P_2 - P_1$

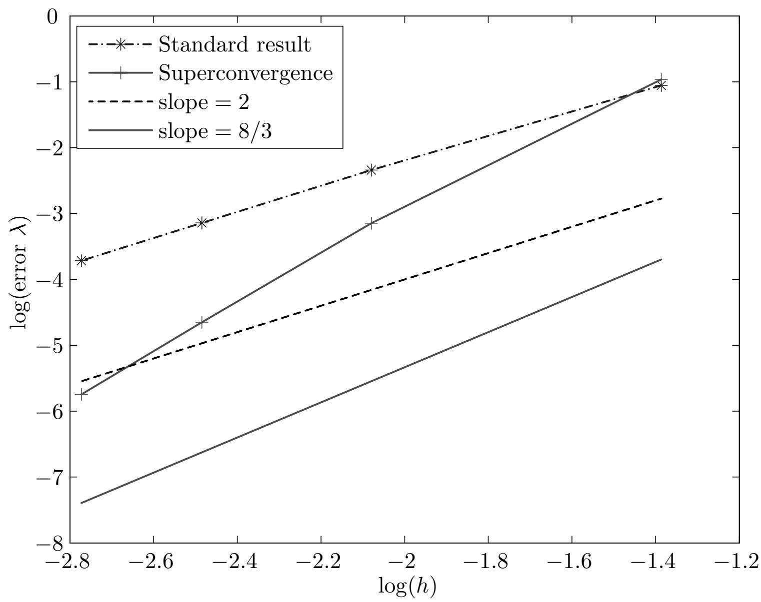

Figure 1. The rate analysis for the eigenvalue.

References

[1] *P. Bochev, C. R. Dohrmann, M. D. Gunzburger*: Stabilization of low-order mixed finite elements for the Stokes equations. SIAM J. Numer. Anal. *44* (2006), 82–101 (electronic).

[2] *H. Chen, S. Jia, H. Xie*: Postprocessing and higher order convergence for the mixed finite element approximations of the Stokes eigenvalue problems. Appl. Math., Praha *54* (2009), 237–250.

[3] *W. Chen, Q. Lin*: Approximation of an eigenvalue problem associated with the Stokes problem by the stream function-vorticity-pressure method. Appl. Math., Praha *51* (2006), 73–88.

[4] *H. Chen, J. Wang*: An interior estimate of superconvergence for finite element solutions for second-order elliptic problems on quasi-uniform meshes by local projections. SIAM J. Numer. Anal. *41* (2003), 1318–1338 (electronic).

[5] *S. H. Chou, X. Ye*: Superconvergence of finite volume methods for the second-order elliptic problem. Comput. Methods Appl. Mech. Eng. *196* (2007), 3706–3712.

[6] *M. Cui, X. Ye*: Superconvergence of finite volume methods for the Stokes equations. Numer. Methods Partial Differ. Equations *25* (2009), 1212–1230.

[7] *F. Hecht, O. Pironneau, A. Le Hyaric, K. Ohtsuka*: FREEFEM++, version 2.3-3, 2008. Software avaible at http://www.freefem.org.

[8] *B. O. Heimsund, X. C. Tai, J. P. Wang*: Superconvergence for the gradient of finite element approximations by L^2 projections. SIAM J. Numer. Anal. *40* (2002), 1263–1280.

[9] *P. Z. Huang, Y. N. He, X. L. Feng*: Numerical investigations on several stabilized finite element methods for the Stokes eigenvalue problem. Math. Probl. Eng. *2011* (2011), Article ID 745908, pp. 14.

[10] *P. Z. Huang, Y. N. He, X. L. Feng*: Two-level stabilized finite element method for Stokes eigenvalue problem. Appl. Math. Mech., Engl. Ed. *33* (2012), 621–630.

[11] *P. Z. Huang, T. Zhang, X. L. Ma*: Superconvergence by L^2-projection for a stabilized finite volume method for the stationary Navier-Stokes equations. Comput. Math. Appl. *62* (2011), 4249–4257.

[12] *S. Jia, H. Xie, X. Yin, S. Gao*: Approximation and eigenvalue extrapolation of Stokes eigenvalue problem by nonconforming finite element methods. Appl. Math., Praha *54* (2009), 1–15.

[13] *J. Li*: Penalty finite element approximations for the Stokes equations by L^2 projection. Math. Methods Appl. Sci. *32* (2009), 470–479.

[14] *J. Li, Y. N. He*: Superconvergence of discontinuous Galerkin finite element method for the stationary Navier-Stokes equations. Numer. Methods Partial Differ. Equations *23* (2007), 421–436.

[15] *J. Li, Y. N. He*: A stabilized finite element method based on two local Gauss integrations for the Stokes equations. J. Comput. Appl. Math. *214* (2008), 58–65.

[16] *J. Li, Y. N. He, J. H. Wu*: A local superconvergence analysis of the finite element method for the Stokes equations by local projections. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *74* (2011), 6499–6511.

[17] *J. Li, L. Q. Mei, Z. X. Chen*: Superconvergence of a stabilized finite element approximation for the Stokes equations using a local coarse mesh L^2 projection. Numer. Methods Partial Differ. Equations *28* (2012), 115–126.

[18] *J. Li, J. Wang, X. Ye*: Superconvergence by L^2-projections for stabilized finite element methods for the Stokes equations. Int. J. Numer. Anal. Model. *6* (2009), 711–723.

[19] *H. P. Liu, N. N. Yan*: Enhancing finite element approximation for eigenvalue problems by projection method. Comput. Methods Appl. Mech. Eng. *233/236* (2012), 81–91.

[20] *C. Lovadina, M. Lyly, R. Stenberg*: A posteriori estimates for the Stokes eigenvalue problem. Numer. Methods Partial Differ. Equations *25* (2009), 244–257.

[21] *J. Wang*: Superconvergence analysis for finite element solutions by the least-squares surface fitting on irregular meshes for smooth problems. J. Math. Study *33* (2000), 229–243.

[22] *J. Wang, X. Ye*: Superconvergence of finite element approximations for the Stokes problem by projection methods. SIAM J. Numer. Anal. *39* (2001), 1001–1013 (electronic).

[23] *X. Ye*: Superconvergence of nonconforming finite element method for the Stokes equations. Numer. Methods Partial Differ. Equations *18* (2002), 143–154.

[24] *X. Yin, H. Xie, S. Jia, S. Gao*: Asymptotic expansions and extrapolations of eigenvalues for the Stokes problem by mixed finite element methods. J. Comput. Appl. Math. *215* (2008), 127–141.

Author's address: *Pengzhan Huang*, College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, P. R. China, e-mail: `hpzh007@gmail.com`, `hpzh007@yahoo.cn`.