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Wu Dejun

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G-DIMENSION OVER LOCAL HOMOMORPHISMS WITH RESPECT TO A SEMI-DUALIZING COMPLEX

Wu Dejun, Lanzhou

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Abstract. We study the G-dimension over local ring homomorphisms with respect to a semi-dualizing complex. Some results that track the behavior of Gorenstein properties over local ring homomorphisms under composition and decomposition are given. As an application, we characterize a dualizing complex for R in terms of the finiteness of the G-dimension over local ring homomorphisms with respect to a semi-dualizing complex.

Keywords: Cohen factorization; Gorenstein dimension; Gorenstein homomorphism; semi-dualizing complex

MSC 2010: 13D02, 13D05, 13D07

1. Introduction

Throughout this paper, all rings are commutative and noetherian. It is well known that Gorenstein homological dimensions are refinements of the classical homological dimensions.

Gorenstein dimension (abbreviation G-dimension), which is a homological invariant for modules, was introduced by Auslander and was deeply studied by Auslander and Bridger in [1]. With that as a start, G-dimension has been studied by a lot of algebraists so far. Two of its main features are that it is a finer invariant than the projective dimension and that it satisfies an equality of the Auslander-Buchsbaum type.

Let C be a semi-dualizing complex for R. Christensen introduced G-dimension with respect to C in [7]. More precisely, for $X \in \mathscr{D}_b^f(R)$ the *G-dimension* of X with

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respect to C is defined to be

$$\text{G-dim}_C X = \begin{cases} \inf C - \inf X^{\dagger C}, & X \in {}_C\mathscr{R}(R), \\ \infty, & X \notin {}_C\mathscr{R}(R) \end{cases}$$

(see [7], 3.11).

Iyengar and Sather-Wagstaff in [11] develop a theory of Gorenstein dimension over local ring homomorphisms. More precisely, let $\varphi\colon R \to S$ be a local ring homomorphism and $X \in \mathscr{D}_b^f(S)$, and let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \hat{S}$ be a Cohen factorization of $\dot{\varphi}$. The *Gorenstein dimension of* X *over* φ is defined by $\text{G-dim}_\varphi X = \text{G-dim}_{R'} \hat{X} - \text{edim}(\dot{\varphi})$.

Motivated by this, it is natural to consider G-dimension over local ring homomorphisms with respect to a semi-dualizing complex. In this paper, G-dimension over local ring homomorphisms with respect to a semi-dualizing complex is studied and the corresponding results are generalized.

Transfer of homological properties along ring homomorphisms is a classical field of study. The following stability result generalizes [11], Theorem 5.1, and [7], Theorem 3.17 (b), (see Theorem 3.15).

Theorem A. *Let C be a semi-dualizing complex for R. Let $\varphi\colon R \to S$ and $\sigma\colon S \to T$ be local ring homomorphisms. If $P \in \mathscr{D}_b^f(T)$ with $\text{pd}_\sigma P$ finite and $X \in \mathscr{D}_b^f(S)$, then we have the equality*

$$\text{G-dim}_{\sigma\varphi}^C(X \otimes_S^{\mathbf{L}} P) = \text{G-dim}_\varphi^C(X) + \text{pd}_\sigma P.$$

In particular, $\text{G-dim}_{\sigma\varphi}^C(X \otimes_S^{\mathbf{L}} P)$ *and* $\text{G-dim}_\varphi^C(X)$ *are simultaneously finite.*

As an application, we have the following result which recovers [11], Theorem 6.1, (see Theorem 4.1).

Theorem B. *Let $(R, \mathfrak{m}, k)$ be a local ring and C a semi-dualizing complex for R. Then the following conditions are equivalent.*

(i) *C is dualizing for R.*

(ii) *For every local ring homomorphism $\varphi\colon R \to S$ and $X \in \mathscr{D}_b^f(S)$, $\text{G-dim}_\varphi^C(X) < \infty$.*

(iii) *There is a local ring homomorphism $\varphi\colon R \to S$ and an ideal I of S such that $I \supseteq \mathfrak{m}S$, and $\text{G-dim}_\varphi^C(S/I) < \infty$.*

2. Preliminaries

The derived category is written as $\mathscr{D}(R)$. If M is an R-complex, then the projective dimension of M is abbreviated as $\operatorname{pd}_R M$. The symbols $\sup M$ and $\inf M$ are used for the supremum and infimum of the set $\{i \in \mathbb{Z};\ \mathrm{H}_i(M) \neq 0\}$, with the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. A complex M is called *homologically bounded above* if $\sup M$ is finite, it is called *homologically bounded below* if $\inf M$ is finite, and it is called *homologically bounded* if it is homologically bounded above and below. The full subcategories $\mathscr{D}_{\sqsubset}(R)$ and $\mathscr{D}_{\sqsupset}(R)$ consist of complexes X with, respectively, $\sup X < \infty$ and $\inf X > -\infty$. We set $\mathscr{D}_b(R) = \mathscr{D}_{\sqsubset}(R) \cap \mathscr{D}_{\sqsupset}(R)$. The full subcategory $\mathscr{P}(R)$ of $\mathscr{D}_b(R)$ consists of complexes of finite projective dimensions. We use the superscript f to denote finite (finitely generated) homology.

We use the standard notation $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes_R^{\mathbf{L}} -$ for the derived Hom and derived tensor product of complexes.

2.1 (Depth)**.** Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex. The *depth* of M is defined as

$$\operatorname{depth}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(k, M).$$

By [9], 1.5.(3), for every R-complex M one has

$$\operatorname{depth}_R M \geqslant -\sup M.$$

If $\sup M = s$ is finite, then equality holds if and only if $\mathfrak{m}$ is an associated prime of the homology module $\mathrm{H}_s(M)$.

2.2 (Auslander-Buchsbaum formula)**.** If R is local and $X \in \mathscr{P}^f(R)$, then we have the equality

$$\operatorname{pd}_R X = \operatorname{depth} R - \operatorname{depth}_R X.$$

3. G-dimension over a local ring homomorphism

In this section, we introduce the G-dimension over a local ring homomorphism with respect to a semi-dualizing complex and study some of its properties. First, we need to recall the following definitions from [7].

Definition 3.1. An R-complex C is said to be *semi-dulizing* for R if and only if $C \in \mathscr{D}_b^f(R)$ and the homothety morphism $\chi_C^R\colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism (see [7], 2.1).

Let R be a local ring. Recall that a *dualizing complex* for R is a semi-dualizing complex with finite injective dimension. For instance, when R is complete, it possesses a dualizing complex.

Definition 3.2. Let C be a semi-dualizing complex for R. For $X \in \mathscr{D}(R)$ the *dagger dual* with respect to C is the complex $X^{\dagger C} = \mathbf{R}\mathrm{Hom}_R(X, C)$, and $-^{\dagger C} = \mathbf{R}\mathrm{Hom}_R(-, C)$ is the corresponding *dagger duality functor.*

An R-complex X is said to be *C-reflexive* if and only if X and the dagger dual $X^{\dagger C}$ belong to $\mathscr{D}_b^f(R)$, and the biduality morphism $\delta_X^C\colon X \to (X^{\dagger C})^{\dagger C}$ is invertible. By ${}_C\mathscr{R}(R)$ we denote the full subcategory of $\mathscr{D}_b^f(R)$ whose objects are the C-reflexive complexes (see [7], 2.7).

Definition 3.3. Let C be a semi-dualizing complex for R. For $X \in \mathscr{D}_b^f(R)$ the *G-dimension* of X with respect to C is defined to be

$$\text{G-dim}_C X = \begin{cases} \inf C - \inf X^{\dagger C}, & X \in {}_C\mathscr{R}(R), \\ \infty, & X \notin {}_C\mathscr{R}(R) \end{cases}$$

(see [7], 3.11).

The change of rings theorems for Gorenstein dimensions of modules have been investigated in [5]. For complexes of modules, change of rings theorem for G-dimension has been given by Christensen (see [6], Theorem 2.3.12). Here we also have the following change of rings theorem for G-dimension over local homomorphism with respect to a semi-dualizing complex C.

Lemma 3.4. *Let R be a local ring and C a semi-dualizing complex for R. Let $\boldsymbol{x} = x_1, x_2, \ldots, x_t$ be an R-sequence and $S = R/(\boldsymbol{x})$. For $X \in \mathscr{D}_b^f(S)$ there is an equality*

$$\text{G-dim}_C X = \text{G-dim}_{C\otimes_R^{\mathbf{L}} S} X + t.$$

In particular, the two dimensions are simultaneously finite.

P r o o f. It follows from [7], Proposition 5.7, that $C \otimes_R^{\mathbf{L}} S$ is semi-dualizing for S. Since it is straightforward to prove that $X \in {}_C\mathscr{R}(R)$ if and only if $X \in {}_{C\otimes_R^{\mathbf{L}} S}\mathscr{R}(R)$, one has $\text{G-dim}_C X$ and that $\text{G-dim}_{C\otimes_R^{\mathbf{L}} S} X$ are simultaneously finite. Now the result follows from [7], Theorem 3.14, and the Auslander-Buchsbaum formula for projective dimension (see (2.2)). □

Here we also need to recall the definition of Cohen factorizations of local homomorphisms from [4].

3.5. Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local ring homomorphism. The *embedding dimension of* φ is

$$\mathrm{edim}(\varphi) := \mathrm{edim}(S/\mathfrak{m}S).$$

A *regular* or *Gorenstein factorization* of φ is a diagram of local ring homomorphisms, $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \hat{S}$, where $\varphi = \varphi'\dot\varphi$, with $\dot\varphi$ flat, the closed fibre $R'/\mathfrak{m}R'$ regular or Gorenstein, respectively, and $\varphi'\colon R' \to S$ surjective.

Let $\hat{S}$ denote the completion of S at its maximal ideal and let $\iota\colon S \to \hat{S}$ be the canonical inclusion. By [4], (1.1), the composition $\dot\varphi = \iota\varphi$ admits a regular factorization $R \to R' \to \hat{S}$ with R' complete. Such a regular factorization is called a *Cohen factorization* of $\dot\varphi$.

In order to introduce the concept of G-dimension over a local ring homomorphism with respect to a semi-dualizing complex, we also need the following result.

Theorem 3.6. *Let C be a semi-dualizing complex for R. Let $\varphi\colon R \to S$ be a local ring homomorphism and $X \in \mathscr{D}_b^f(S)$. If $R \xrightarrow{\dot\varphi_1} R_1 \xrightarrow{\varphi_1'} \hat{S}$ and $R \xrightarrow{\dot\varphi_2} R_2 \xrightarrow{\varphi_2'} \hat{S}$ are Cohen factorizations of $\dot\varphi$, then we have the equality*

$$\text{G-dim}_{C\otimes^{\mathbf{L}}_R R_1}(\hat{X}) - \operatorname{edim}(\dot\varphi_1) = \text{G-dim}_{C\otimes^{\mathbf{L}}_R R_2}(\hat{X}) - \operatorname{edim}(\dot\varphi_2),$$

where $\hat{X} = X \otimes^{\mathbf{L}}_S \hat{S}$.

P r o o f. It follows from [7], Theorem 5.6, that $C \otimes^{\mathbf{L}}_R R_1 = C \otimes_R R_1$ is semi-dualizing for R_1 and $C \otimes^{\mathbf{L}}_R R_2 = C \otimes_R R_2$ is semi-dualizing for R_2. Now by analogy with the proof of [11], Theorem 3.2, using Lemma (3.4) one obtains the result. $\square$

Definition 3.7. Let C be a semi-dualizing complex for R. Let $\varphi\colon R \to S$ be a local ring homomorphism and $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \hat{S}$ a Cohen factorization of $\dot\varphi$. For $X \in \mathscr{D}_b^f(S)$ we define the *Gorenstein dimension of X over φ with respect to C*, $\text{G-dim}_\varphi^C(X)$, as

$$\text{G-dim}_\varphi^C(X) := \text{G-dim}_{C\otimes^{\mathbf{L}}_R R'}(\hat{X}) - \operatorname{edim}(\dot\varphi),$$

where $\hat{X} = X \otimes^{\mathbf{L}}_S \hat{S}$. It follows from [7], Theorem 5.6, that $C \otimes^{\mathbf{L}}_R R' = C \otimes_R R'$ is semi-dualizing for R'. Theorem 3.6 shows that $\text{G-dim}_\varphi^C(X)$ does not depend on the choice of Cohen factorization. Note that $\text{G-dim}_\varphi^C(X) \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, and also that $\text{G-dim}_\varphi^C(X) = -\infty$ if and only if X is acyclic.

The *Gorenstein dimension of φ with respect to C* is defined to be

$$\text{G-dim}_C(\varphi) := \text{G-dim}_\varphi^C(S).$$

Recall the next definition from [11], Definition 4.2.

Definition 3.8. Let $\varphi\colon R \to S$ be a local ring homomorphism and $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \hat{S}$ a Cohen factorization of $\dot{\varphi}$. For $X \in \mathscr{D}_b^f(S)$ the *projective dimension of X over φ*, $\operatorname{pd}_\varphi X$, is defined by

$$\operatorname{pd}_\varphi X = \operatorname{pd}_{R'} \hat{X} - \operatorname{edim}(\dot{\varphi}),$$

where $\hat{X} = X \otimes_S^{\mathbf{L}} \hat{S}$. The *projective dimension of* φ is defined to be

$$\operatorname{pd}(\varphi) = \operatorname{pd}_\varphi S.$$

The following proposition shows that G-dimension with respect to a semi-dualizing complex is a refinement of projective dimension over a local ring homomorphism and recovers [7], Proposition 3.15, and [11], Proposition 4.6.

Proposition 3.9. *Let C be a semi-dualizing complex for R. Let $\varphi\colon R \to S$ be a local ring homomorphism and $X \in \mathscr{D}_b^f(S)$. Then we have the inequality*

$$\operatorname{G-dim}_\varphi^C(X) \leqslant \operatorname{pd}_\varphi X,$$

and equality holds if $\operatorname{pd}_\varphi X < \infty$.

Proof. Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \hat{S}$ be a Cohen factorization of $\dot{\varphi}$. Then we have

$$\begin{aligned}\operatorname{G-dim}_\varphi^C(X) &= \operatorname{G-dim}_{C\otimes_R^{\mathbf{L}} R'}(\hat{X}) - \operatorname{edim}(\varphi)\\ &\leqslant \operatorname{pd}_{R'} \hat{X} - \operatorname{edim}(\varphi)\\ &= \operatorname{pd}_\varphi X\end{aligned}$$

with equality if $\operatorname{pd}_\varphi X < \infty$ (see [7], Proposition 3.15). □

The next theorem is an extension of the Auslander-Bridger formula for G-dimension over a local ring homomorphism (see [11], Theorem 3.5) and for G-dimension with respect to a semi-dualizing complex (see [7], Theorem 3.14), which is a special case by putting $C = R$ and $\varphi = \operatorname{id}_R$ respectively.

Theorem 3.10. *Let C be a semi-dualizing complex for R. Let $\varphi\colon R \to S$ be a local ring homomorphism and $X \in \mathscr{D}_b^f(S)$. If* $\operatorname{G-dim}_\varphi^C(X) < \infty$, *then*

$$\operatorname{G-dim}_\varphi^C(X) = \operatorname{depth} R - \operatorname{depth}_S X.$$

Proof. By analogy with the proof of [11], Theorem 3.5, and this time using [7], Theorem 3.14, one obtains the result. □

Lemma 3.11. *Let $\varphi\colon R \to R'$ be a ring homomorphism of finite flat dimension and C a semi-dualizing complex for R. Assume that R' has a dualizing complex D. Then $\mathbf{R}\mathrm{Hom}_R(C,D)$ is a semi-dualizing complex for R'.*

P r o o f. Note that $\mathbf{R}\mathrm{Hom}_R(C,D) \in \mathscr{D}^f_{\sqsubset}(R')$ by [3], 1.2.2. Since $\mathrm{fd}_R R'$ is finite, one has that $\mathrm{id}_R D$ is finite and so $\mathbf{R}\mathrm{Hom}_R(C,D) \in \mathscr{D}^f_b(R')$. It follows from [6], A.4.24, that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{R'}(D,D) &= \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C,C),D),D)\\ &= \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_R(C,D)\otimes^{\mathbf{L}}_R C,D).\end{aligned}$$

Now the commutative diagram

$$\begin{array}{ccc} R' & \longrightarrow & \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_R(C,D),\mathbf{R}\mathrm{Hom}_R(C,D)) \\ \Big\downarrow{\scriptstyle\simeq} & & \Big\downarrow{\scriptstyle\simeq} \\ \mathbf{R}\mathrm{Hom}_{R'}(D,D) & \xrightarrow{\ \simeq\ } & \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_R(C,D)\otimes^{\mathbf{L}}_R C,D) \end{array}$$

shows that the homothety morphism

$$\chi^{R'}_{\mathbf{R}\mathrm{Hom}_R(C,D)}\colon R' \to \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_R(C,D),\mathbf{R}\mathrm{Hom}_R(C,D))$$

is an isomorphism. Therefore, $\mathbf{R}\mathrm{Hom}_R(C,D)$ is a semi-dualizing complex for R'. $\square$

We proceed by recalling the definition of the C-Aslander class from [7], 4.1.

3.12. Let C be a semi-dualizing complex of R. The objects in the *C-Auslander class* ${}_C\mathscr{A}(R)$ are the homologically bounded R-complexes X such that $C\otimes^{\mathbf{L}}_R X$ is homologically bounded and the natural morphism $X \to \mathbf{R}\mathrm{Hom}_R(C, C\otimes^{\mathbf{L}}_R X)$ is an isomorphism.

We list some stability properties of ${}_C\mathscr{A}(R)$.

Proposition 3.13. *Let C be a semi-dualizing complex for R. Let $\varphi\colon R \to S$ be a local ring homomorphism and X an S-complex. Then the following statements hold.*

(i) *$X \in {}_C\mathscr{A}(R)$ if and only if $X\otimes_S \hat{S} \in {}_{C\otimes_R \hat{R}}\mathscr{A}(\hat{R})$.*
(ii) *If φ is of finite flat dimension, then $X \in {}_C\mathscr{A}(R)$ if and only if $X \in {}_{C\otimes^{\mathbf{L}}_R S}\mathscr{A}(S)$.*
(iii) *If $S \to S'$ is a flat local ring homomorphism, then $X \in {}_C\mathscr{A}(R)$ if and only if $X\otimes^{\mathbf{L}}_S S' \in {}_C\mathscr{A}(R)$.*
(iv) *If $\mathfrak{q} \in \mathrm{Spec} S$ and $\mathfrak{p} = \mathfrak{q}\cap R$, then $X \in {}_C\mathscr{A}(R)$ if and only if $X_{\mathfrak{q}} \in {}_{C_{\mathfrak{p}}}\mathscr{A}(R_{\mathfrak{p}})$.*

Proof. If φ is of finite flat dimension, then $C \otimes_R^{\mathbf{L}} S$ is a semi-dualizing complex for S by [7], Proposition 5.7. Let C be a semi-dualizing complex of R. Then $C_{\mathfrak{p}}$ a semi-dualizing complex of $R_{\mathfrak{p}}$ by [7], Lemma 2.5. Now by analogy with the proof of [3], Proposition 3.7, one obtains the result. □

Proposition 3.14. *Let $\varphi\colon R \to S$ be a local ring homomorphism and $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \hat{S}$ a Cohen factorization of $\dot{\varphi}$, and let $X \in \mathscr{D}_b^f(S)$. Let C be a semi-dualizing complex for R and D a dualizing complex for R'. Then the following conditions are equivalent.*

(i) $\operatorname{G-dim}_{\varphi}^{C}(X) < \infty$.

(ii) $\operatorname{G-dim}_{C\otimes_R^{\mathbf{L}} R'}(\hat{X}) < \infty$.

(iii) $\hat{X} \in {}_{\mathbf{R}\mathrm{Hom}_R(C,D)}\mathscr{A}(R')$.

Proof. (i) $\Leftrightarrow$ (ii) By definition.

(ii) $\Leftrightarrow$ (iii) By Lemma (3.11), $\mathbf{R}\mathrm{Hom}_R(C,D)$ is a semi-dualizing complex for R'. Now the result follows from [7], Theorem 4.7, as

$$C \otimes_R^{\mathbf{L}} R' = \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_R(C,D),D)$$

(see [6], A.4.24). □

The following stability result is one of the main results in this paper which generalizes [11], Theorem 5.1, and [7], [Theorem 3.17 (b)].

Theorem 3.15. *Let C be a semi-dualizing complex for R. Let $\varphi\colon R \to S$ and $\sigma\colon S \to T$ be local ring homomorphisms. If $P \in \mathscr{D}_b^f(T)$ with $\operatorname{pd}_\sigma P$ finite and $X \in \mathscr{D}_b^f(S)$, then we have the equality*

$$\operatorname{G-dim}_{\sigma\varphi}^{C}(X \otimes_S^{\mathbf{L}} P) = \operatorname{G-dim}_{\varphi}^{C}(X) + \operatorname{pd}_\sigma P.$$

In particular, $\operatorname{G-dim}_{\sigma\varphi}^{C}(X \otimes_S^{\mathbf{L}} P)$ *and* $\operatorname{G-dim}_{\varphi}^{C}(X)$ *are simultaneously finite.*

Proof. Note that $X \otimes_S^{\mathbf{L}} P \in \mathscr{D}_b^f(T)$ by [11], Lemma 2.11. Passing to the completions of S and T at their respective maximal ideals, and replacing X and P by $\hat{S} \otimes_S X$ and $\hat{T} \otimes_T P$, respectively, one may assume that S and T are complete. In doing so, one uses the isomorphism

$$(\hat{S} \otimes_S X) \otimes_{\hat{S}}^{\mathbf{L}} (\hat{T} \otimes_T P) \simeq \hat{T} \otimes_T (X \otimes_S^{\mathbf{L}} P).$$

The next step is the reduction to the case where φ and σ are surjective. To achieve this, take Cohen factorizations $R \to R' \to S$ and $R' \to R'' \to T$, and expand to a commutative diagram as in [11], 5.9. Let $X' = S' \otimes_S X$. Since $S' = R'' \otimes_{R'} S$, by construction, $X' \cong R'' \otimes_{R'} X$ and hence $X' \otimes^{\mathbf{L}}_{S'} P \simeq X \otimes^{\mathbf{L}}_S P$. Since $R' \to R''$ is faithfully flat, [7], Corollary 5.11, yields that

$$\text{G-dim}_{C\otimes^{\mathbf{L}}_R R'}(X) = \text{G-dim}_{C\otimes^{\mathbf{L}}_R R''}(X').$$

Also, in conjunction with those in [11], 5.9, we have

$$\begin{aligned}
\text{pd}_\sigma(P) &= \text{pd}_{S'}(P) - \text{edim}(\dot{\varrho}),\\
\text{G-dim}^C_\varphi(X) &= \text{G-dim}_{C\otimes^{\mathbf{L}}_R R''}(X') - \text{edim}(\dot{\varphi}),\\
\text{G-dim}^C_{\sigma\varphi}(X \otimes^{\mathbf{L}}_S P) &= \text{G-dim}_{C\otimes^{\mathbf{L}}_R R''}(X' \otimes^{\mathbf{L}}_{S'} P) - \text{edim}(\dot{\varrho}) - \text{edim}(\dot{\varphi}).
\end{aligned}$$

Therefore, it suffices to verify the equality for the diagram $R'' \to S' \to T$ and complexes X' and P. This places us in the situation where $R \to S$ is surjective and then the equality we seek is

$$\text{G-dim}_C(X \otimes^{\mathbf{L}}_S P) = \text{G-dim}_C(X) + \text{pd}_S(P).$$

It suffices to prove that $\text{G-dim}_C(X \otimes^{\mathbf{L}}_S P)$ and $\text{G-dim}_C(X)$ are simultaneously finite. For, when they are both finite, one has

$$\begin{aligned}
\text{G-dim}_C(X \otimes^{\mathbf{L}}_S P) &= \operatorname{depth} R - \operatorname{depth}_R(X \otimes^{\mathbf{L}}_S P)\\
&= \operatorname{depth} R - \operatorname{depth}_S(X \otimes^{\mathbf{L}}_S P)\\
&= \operatorname{depth} R - \operatorname{depth}_S X - \operatorname{depth}_S P + \operatorname{depth} S\\
&= \operatorname{depth} R - \operatorname{depth}_R X + \text{pd}_S P\\
&= \text{G-dim}_C(X) + \text{pd}_S P
\end{aligned}$$

where the first and the last equalities follow by the Auslander-Bridger formula (see [7], Theorem 3.14), the second by [11], Lemma 2.8, the third a consequence of [10], Theorem 4.1, while the forth is a consequence of [11], Lemma 2.8, and (2.2).

The rest of the proof is dedicated to proving that $\text{G-dim}_C(X)$ and $\text{G-dim}_C(X \otimes^{\mathbf{L}}_S P)$ are simultaneously finite. This is tantamount to proving that

$$X \in {}_C\mathscr{R}(R) \Leftrightarrow X \otimes^{\mathbf{L}}_S P \in {}_C\mathscr{R}(R).$$

First, note that $X \in \mathscr{D}^f_b(R)$ if and only if $X \otimes^{\mathbf{L}}_S P \in \mathscr{D}^f_b(R)$ by [11], Theorem 2.9. Secondly, since $\text{pd}_S \mathbf{R}\text{Hom}_R(P, S) = -\inf P$ is finite, we have the equalities

$$\begin{aligned}
\mathbf{R}\text{Hom}_R(X \otimes^{\mathbf{L}}_S P, C) &= \mathbf{R}\text{Hom}_S(P, \mathbf{R}\text{Hom}_R(X, C))\\
&= \mathbf{R}\text{Hom}_S(P, S) \otimes^{\mathbf{L}}_S \mathbf{R}\text{Hom}_R(X, C),
\end{aligned}$$

where the first equality follows by adjointness and the second by tensor evaluation (see, for example, [6], A.4.21 and A.4.23). Hence $\mathbf{R}\mathrm{Hom}_R(X, C) \in \mathscr{D}_b^f(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(X \otimes_S^{\mathbf{L}} P, C) \in \mathscr{D}_b^f(R)$ by virtue of [11], Theorem 2.9.

Finally, since $\mathrm{pd}_S P$ is finite, one has that

$$\delta_P^S\colon\ P \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(P, S), S)$$

is an isomorphism. Also, since $\mathrm{pd}_S \mathbf{R}\mathrm{Hom}_R(P, S) = -\inf P$ is finite, we have the equalities

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C) \otimes_S^{\mathbf{L}} P \\
&\qquad = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C) \otimes_S^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(P, S), S) \\
&\qquad = \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(P, S), \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)) \\
&\qquad = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(P, S) \otimes_S^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(X, C), C).
\end{aligned}$$

Now the commutative diagram

$$\begin{array}{ccc}
X \otimes_S^{\mathbf{L}} P & \xrightarrow{(\delta_X^C) \otimes_S^{\mathbf{L}} P} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C) \otimes_S^{\mathbf{L}} P \\
\| & & \downarrow \simeq \\
 & & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C) \otimes_S^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_S(P, S), C) \\
\| & & \simeq \uparrow \\
X \otimes_S^{\mathbf{L}} P & \xrightarrow{\delta^C_{(X \otimes_S^{\mathbf{L}} P)}} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X \otimes_S^{\mathbf{L}} P, C), C)
\end{array}$$

shows that δ_X^C is an isomorphism if and only if $\delta^C_{(X \otimes_S^{\mathbf{L}} P)}$ is an isomorphism. This completes the proof. □

The next result is just the special case arising by taking $X = S$ and $P = T$ in Theorem 3.15 and it generalizes [11], Theorem 5.2, by putting $C = R$.

Corollary 3.16. *Let C be a semi-dualizing complex for R. Let $\varphi\colon\ R \to S$ and $\sigma\colon\ S \to T$ be local homomorphisms with $\mathrm{pd}(\sigma)$ finite. Then*

$$\text{G-}\dim_C(\sigma\varphi) = \text{G-}\dim_C(\varphi) + \mathrm{pd}(\sigma).$$

In particular, $\text{G-}\dim_C(\sigma\varphi)$ *is finite if and only if* $\text{G-}\dim_C(\varphi)$ *is finite.*

The next stability result generalizes [11], Theorem 5.6, and [7], Theorem 3.17 (a).

Corollary 3.17. *Let C be a semi-dualizing complex for R. Let $\varphi\colon R \to S$ be a local ring homomorphism and $P \in \mathscr{D}_b^f(S)$ with $\operatorname{pd}_S P$ finite. For $X \in \mathscr{D}_b^f(S)$ we have the equality*

$$\text{G-dim}_{\varphi}^{C}(\mathbf{R}\mathrm{Hom}_S(P, X)) = \text{G-dim}_{\varphi}^{C}(X) - \inf P.$$

Thus, $\text{G-dim}_{\varphi}^{C}(X)$ *and* $\text{G-dim}_{\varphi}^{C}(\mathbf{R}\mathrm{Hom}_S(P, X))$ *are simultaneously finite.*

Proof. By analogy with the proof of [11], Theorem 5.7, and this time using Theorem 3.15, one obtains the result. □

4. Some applications

In this section, we characterize R and a dualizing complex for R in terms of the finiteness of G-dimension over local ring homomorphisms with respect to a semi-dualizing complex.

Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local ring homomorphism. Recall that φ is *Gorenstein* (see [2]), or more precisely, *Gorenstein at* $\mathfrak{n}$, if $\operatorname{fd}_R \varphi < \infty$ and $\mu_R^{i+\operatorname{depth} R} = \mu_S^{i+\operatorname{depth} S}$ for all $i \in \mathbb{Z}$. By [2], 4.2, a flat ring homomorphism is Gorenstein if and only if the ring $S/\mathfrak{m}S$ is Gorenstein.

Applying the next result to $C = R$ we recover [11], Theorem 6.1.

Theorem 4.1. *Let $(R, \mathfrak{m}, k)$ be a local ring and C a semi-dualizing complex for R. Then the following conditions are equivalent.*

(i) *C is dualizing for R.*

(ii) *For every local ring homomorphism $\varphi\colon R \to S$ and $X \in \mathscr{D}_b^f(S)$,* $\text{G-dim}_{\varphi}^{C}(X) < \infty$.

(iii) *There is a local ring homomorphism $\varphi\colon R \to S$ and an ideal I of S such that $I \supseteq \mathfrak{m}S$, and* $\text{G-dim}_{\varphi}^{C}(S/I) < \infty$.

Proof. (i) ⇒ (ii) Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \hat{S}$ be a Cohen factorization of $\dot{\varphi}$. Then $\dot{\varphi}$ is Gorenstein. By [3], 2.11, one has that $C \otimes_R^{\mathbf{L}} R'$ is dualizing for R'. It follows from [7], Proposition 8.4, that ${}_{C \otimes_R^{\mathbf{L}} R'}\mathscr{R}(R') = \mathscr{D}_b^f(R')$ and so $\hat{X} \in {}_{C \otimes_R^{\mathbf{L}} R'}\mathscr{R}(R')$. Therefore, $\text{G-dim}_{C \otimes_R^{\mathbf{L}} R'}(\hat{X}) < \infty$. Thus $\text{G-dim}_{\varphi}^{C}(X) < \infty$.

(ii) ⇒ (iii) It is trivial.

(iii) ⇒ (i) Let $R \to R' \to \hat{S}$ be a Cohen factorization. Composing with the surjection $\hat{S} \xrightarrow{\pi} \hat{S}/I\hat{S}$ gives a diagram $R \to R' \to \hat{S}/I\hat{S}$ that is also a Cohen factorization. Since $\text{G-dim}_{C \otimes_R^{\mathbf{L}} R'}(\hat{S}/I\hat{S})$ is finite, so is $\text{G-dim}_C(\pi\dot{\varphi})$. The composition $\pi\dot{\varphi}$ factors through the residue field k of R, giving the commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \hat{S}/I\hat{S}. \\ & \searrow \qquad \nearrow & \\ & k & \end{array}$$

The map $k \to \hat{S}/I\hat{S}$ has finite projective dimension as k is a field. Therefore, Corollary (3.16) implies that the surjection $R \to k$ has finite G-dimension with respect to C. Thus, C is dualizing for R by [7], Proposition 8.4. $\square$

The next result generalizes [11], Theorem 6.2.

Theorem 4.2. *Let C be a semi-dualizing complex for R and $\varphi\colon R \to S$ a local ring homomorphism such that S is Gorenstein. Then the following conditions are equivalent.*
(i) *C is dualizing for R.*
(ii) $\text{G-dim}_C(\varphi)$ *is finite.*
(iii) *There exists a complex $P \in \mathscr{D}_b^f(S)$ such that $\text{pd}_S P$ is finite and $\text{G-dim}_\varphi^C(P)$ is finite.*

P r o o f. (i) $\Rightarrow$ (ii) By Theorem 4.1.

(ii) $\Leftrightarrow$ (iii) By Theorem 3.15.

(ii) $\Rightarrow$ (i) Let $\text{G-dim}_C(\varphi)$ be finite. In particular, $\text{G-dim}(\varphi)$ is finite. By [11], Theorem 6.2, R is Gorenstein. Hence the result follows from [7], Corollary 8.6. $\square$

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Author's address: Wu Dejun, Department of Applied Mathematics, Lanzhou University of Technology, The Langongping Road No. 287, Lanzhou, 730 050, Gansu, China, e-mail: wudj2007@gmail.com, wudj@lut.cn.