Dong Qiu; Chongxia Lu; Shuai Deng; Liang Wang

On the hyperspace of bounded closed sets under a generalized Hausdorff stationary fuzzy metric

*Kybernetika*, Vol. 50 (2014), No. 5, 758–773

Persistent URL: http://dml.cz/dmlcz/144105

KYBERNETIKA — VOLUME *50* (2014), NUMBER 5, PAGES 758–773

# ON THE HYPERSPACE OF BOUNDED CLOSED SETS UNDER A GENERALIZED HAUSDORFF STATIONARY FUZZY METRIC

Dong Qiu, Chongxia Lu, Shuai Deng and Liang Wang

In this paper, we generalize the classical Hausdorff metric with t-norms and obtain its basic properties. Furthermore, for a given stationary fuzzy metric space with a t-norm without zero divisors, we propose a method for constructing a generalized Hausdorff fuzzy metric on the set of the nonempty bounded closed subsets. Finally we discuss several important properties as completeness, completion and precompactness.

*Keywords:* Hausdorff metric, hyperspace, triangular norms, stationary fuzzy metric

*Classification:* 46S40, 03E72, 54A40

## 1. INTRODUCTION

It is well known that the Hausdorff metric is very important concept not only in general topology but also in other areas of Mathematics and Computer Science, such as convex analysis and optimization, fractals, mathematical economics, image computing, etc. (see [2, 15, 27, 29]). As a natural generalization of the concept of set, fuzzy sets was introduced initially by Zadeh [39]. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application [3, 6, 16, 17, 18, 19, 23, 24, 28, 32, 37, 38]. Various concepts of fuzzy metrics on ordinary set were considered in [7, 9, 13, 20, 26].

In [35] J. Rodríguez-López and S. Romaguera introduced and discussed a suitable notion for the Hausdorff fuzzy metric of a given fuzzy metric space (in the sense of George and Veeramani) on the set of its nonempty compact subsets. In particular, they explored several properties of the Hausdorff fuzzy metric. They also pointed out that in general the Hausdorff fuzzy metric does not work on the set of bounded closed subsets. As is known, in finite dimensional metric space a set is compact iff it is bounded and closed; in infinite dimensional metric space a set is compact, then it is also bounded and closed but not vice versa.

The stationary fuzzy metric space was introduced by V. Gregori and S. Romaguera in [13] and it has been studied in [14, 31, 33, 34]. In this paper, we will generalize the Hausdorff metrics with t-norms and investigate some basic properties of this generalized

DOI: 10.14736/kyb-2014-5-0758

Hausdorff metric. Furthermore, for a given stationary fuzzy metric space with a t-norm without zero divisors, we will propose a method for constructing a generalized Hausdorff fuzzy metric on the set of the nonempty bounded closed subsets. And then we will discuss several important properties as completeness, completion and precompactness for the hyperspace of bounded closed sets under the generalized Hausdorff stationary fuzzy metric.

## 2. PRELIMINARIES

We start this section by recalling some pertinent concepts.

**Definition 2.1.** (Klement et al. [25]) A triangular norm (or t-norm for short) is a binary operation $*$ on the unit interval $[0,1]$, i. e., a function $* : [0,1]^2 \to [0,1]$, such that for all $a, b, c, d \in [0,1]$ the following four axioms are satisfied:

| | | | |
|---|---|---|---|
| (i) | $a * 1 = a$; | | (boundary condition) |
| (ii) | $a * b \leq c * d$ | whenever $a \leq c$ and $b \leq d$; | (monotonicity) |
| (iii) | $a * b = b * a$; | | (commutativity) |
| (iv) | $a * (b * c) = (a * b) * c$; | | (associativity). |

A t-norm $*$ is said to be continuous if it is a continuous function in $[0,1]^2$; a t-norm $*$ is called a t-norm without zero divisors if $a * b > 0$ whenever $a, b \in (0,1]$. The following are examples of t-norms: $a \wedge b = \min(a,b)$; $a *_P b = a \cdot b$, where $a \cdot b$ denotes the usual multiplication for all $a, b \in [0,1]$.

**Definition 2.2.** (Qiu et al. [34]) A stationary fuzzy pseudo-metric space is an ordered triple $(X, M, *)$ such that $X$ is an arbitrary nonempty set, $*$ is a continuous t-norm and $M$ is a fuzzy set of $X \times X$ satisfying the following conditions, for all $x, y, z \in X$:

(i) $M(x,x) = 1$ for all $x \in X$;

(ii) $M(x,y) = M(y,x)$;

(iii) $M(x,y) \geq M(x,z) * M(z,y)$.

If $(X, M, *)$ is a stationary fuzzy pseudo-metric space, we will say that $(M, *)$ is a stationary fuzzy pseudo-metric on $X$.

**Definition 2.3.** (Gregori and Romaguera [13]) A stationary fuzzy metric space is an ordered triple $(X, M, *)$ such that $X$ is an arbitrary nonempty set, $*$ is a continuous t-norm and $M$ is a fuzzy set of $X \times X$ satisfying the following conditions, for all $x, y, z \in X$:

(i) $M(x,y) > 0$;

(ii) $M(x,y) = 1$ iff $x = y$;

(iii) $M(x,y) = M(y,x)$;

(iv) $M(x,y) \geq M(x,z) * M(z,y)$.

If $(X, M, *)$ is a stationary fuzzy metric space, we will say that $(M, *)$ is a stationary fuzzy metric on $X$.

**Example 2.1.** Let $(X, d)$ be a metric space. Denote by $a \cdot b$ the usual multiplication for all $a, b \in [0, 1]$, and define $M_d$ on $X \times X$ by

$$M_d(x, y) = \frac{1}{1 + d(x, y)},$$

for all $x, y \in X$. Then $(M_d, \cdot)$ is a stationary fuzzy metric on $X$ which will be called a standard stationary fuzzy metric.

Since a stationary fuzzy metric is a special fuzzy metric, just like fuzzy metrics in [9], we can prove that every stationary fuzzy metric $(M, *)$ on $X$ generates a topology $\tau_M$ on $X$ which has as a base the family of sets of the form $\{B_M(x, \varepsilon) : x \in X, 0 < \varepsilon < 1\}$, where $B_M(x, \varepsilon) = \{y \in X : M(x, y) > 1 - \varepsilon\}$ for all $\varepsilon \in (0, 1)$. A sequence $\{x_i\}_{i \in \mathbb{N}}$ in a stationary fuzzy metric space $(X, M, *)$ is said to be Cauchy if $\lim_{i,j \to \infty} M(x_i, x_j) = 1$; a sequence $\{x_i\}_{i \in \mathbb{N}}$ in $X$ converges to $x$ if $\lim_{i \to \infty} M(x_i, x) = 1$ [13].

**Definition 2.4.** Let $(X, M, *)$ be a stationary fuzzy metric space and $A \subset X$. If for all $\varepsilon \in (0, 1)$, $B_M(x, \varepsilon) \bigcap (A - \{x\}) \neq \emptyset$, then $x$ is an accumulation point of $A$. The set of all accumulation points of $A$ is called the derived set of $A$ and denoted by $A^\circ$. The union of $A$ and $A^\circ$ is called the closure of $A$ and denoted by $\overline{A}$. If $A^\circ \subset A$, then $A$ is a closed set of $X$.

**Definition 2.5.** Let $(X, M, *)$ be a stationary fuzzy metric space and $A \subset X$. If there exists $x_0 \in X, r \in (0, 1)$ such that $A \subset B_M(x_0, r)$, then we say $A$ is a bounded subset of $X$; if $X$ itself is a bounded set we will say $(X, M, *)$ is a bounded stationary fuzzy metric space.

**Definition 2.6.** (Gregori and Romaguera [12]) Let $(X, M, *)$, $(\widetilde{X}, \widetilde{M}, \widetilde{*})$ be two stationary fuzzy metric spaces. If there exists an isometry $f : X \to \widetilde{X}$, i. e., $\widetilde{M}(f(x), f(y)) = M(x, y)$ for all $x, y \in X$, then $(X, M, *)$ and $(\widetilde{X}, \widetilde{M}, \widetilde{*})$ are said to be isometric, and $f$ is called an isometric mapping.

**Definition 2.7.** (Gregori and Romaguera [13]) Let $(X, M, *)$ be a stationary fuzzy metric space, if there exists a complete stationary fuzzy metric space $(\widetilde{Y}, \widetilde{M}, \widetilde{*})$ such that $(X, M, *)$ is isometrically isometric to a dense subspace $(\widetilde{X}, \widetilde{M}, \widetilde{*})$ of $(\widetilde{Y}, \widetilde{M}, \widetilde{*})$, we say that $(X, M, *)$ is a completeable stationary fuzzy metric space.

Given a stationary fuzzy metric space $(X, M, *)$, we shall denote by $\mathcal{P}(X)$, $\mathcal{P}_0(X)$ and $\mathcal{CB}(X)$, the powerset, the set of nonempty subsets and the set of nonempty bounded closed subsets of $X$, respectively.

Let $B$ be a nonempty subset of a stationary fuzzy metric space $(X, M, *)$. For all $x \in X$, let

$$M(x, B) = \sup_{y \in B} M(x, y) = M(B, x).$$

For the empty index set $\emptyset$, we will make the convention that for $a_x \in [0,1]$,

$$\sup_{x\in\emptyset} a_x = 0 \text{ and } \inf_{x\in\emptyset} a_x = 1.$$

It follows that $M(x,\emptyset) = M(\emptyset, x) = 0$.

**Definition 2.8.** Let $(X, M, *)$ be a stationary fuzzy metric space. For all $A, B \in \mathcal{P}(X)$, we define a function $H_M^* : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,1]$ by

$$H_M^*(A,B) = \inf_{x\in A} M(x,B) * \inf_{y\in B} M(y,A) = M(A,B) * M(B,A),$$

where $M(A,B) = \inf_{x\in A} M(x,B)$.

## 3. MAIN RESULTS

In this section we will establish our main theorems.

**Proposition 3.1.** Let $(X, M, *)$ be a stationary fuzzy metric space. Then for all $A, B, C \in \mathcal{P}_0(X)$ it holds

(1) $M(A,B) = 1$ iff $A \subset \overline{B}$ iff $\overline{A} \subset \overline{B}$ iff $M(\overline{A}, \overline{B}) = 1$;

(2) $M(x,B) \geq M(x,x') * M(x',B)$ for all $x, x' \in X$;

(3) $M(A,B) = M(A,\overline{B}) = M(\overline{A}, \overline{B})$;

(4) $M(C,A) \geq M(B,A) * M(C,B)$;

(5) $H_M^*(A,B) = 1$ iff $\overline{A} = \overline{B}$.

P r o o f. (1) On the one hand, let $M(A,B) = \inf_{x\in A}\sup_{y\in B} M(x,y) = 1$. Then, for each $x \in A$, we have $\sup_{y\in B} M(x,y) = M(x,B) = 1$, which implies that for all $n \in \mathbb{N}$ there exists $y_n \in B$ such that $M(x,y_n) > 1 - \dfrac{1}{n}$, i.e., $\lim_{n\to\infty} M(x,y_n) = 1$. Hence, we get $A \subset \overline{B}$.

On the other hand, let $A \subset \overline{B}$. Then, for each $x \in A \subset \overline{B}$, there exists a sequence $\{y_n\} \subset B$ converging to $x$, which implies $\lim_{n\to\infty} M(x,y_n) = 1$.

Then for all $x \in A$, we can get that $\sup_{y\in B} M(x,y) = 1$. Thus we have $M(A,B) = \inf_{x\in A}\sup_{y\in B} M(x,y) = 1$. Consequently, $M(A,B) = 1$ iff $A \subset \overline{B}$. It is obvious that $A \subset \overline{B}$ iff $\overline{A} \subset \overline{B}$. By a similar proof, we have $\overline{A} \subset \overline{B}$ iff $M(\overline{A}, \overline{B}) = 1$.

(2) For all $x, x' \in X$ and $y \in B \subset X$, we have

$$M(x,B) = \sup_{y\in B} M(x,y) \geq M(x,y) \geq M(x,x') * M(x',y).$$

Since $*$ is continuous, we get

$$M(x,B) \geq M(x,x') * \sup_{y\in B} M(x',y) = M(x,x') * M(x',B).$$

(3) We will only prove $M(A,B) = M(A,\overline{B})$. Since by an analogous proof, we can also obtain $M(A,\overline{B}) = M(\overline{A},\overline{B})$ .

For any given $x \in A$, let $\alpha = M(x,\overline{B}) = \sup_{y\in\overline{B}} M(x,y)$. Then for each $n \in \mathbb{N}$, there exists $y_n \in \overline{B}$, such that $M(x,y_n) > \alpha - \dfrac{1}{n}$.

For each $n \in \mathbb{N}$, there is a sequence $\{y_n^{(m)}\} \subset B$ converging to $y_n$, which implies that for each $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$, such that $M(y_n, y_n^{(m)}) > 1 - \dfrac{1}{n}$, whenever $m \geq m_n$. Now, for the sequence $\{y_n^{(m_n)}\}_n$, by monotonicity of $*$, we get that

$$M(x, y_n^{(m_n)}) \geq M(x,y_n) * M(y_n, y_n^{(m_n)}) \geq \Big(\alpha - \frac{1}{n}\Big) * \Big(1 - \frac{1}{n}\Big).$$

Since $M(x,B) = \sup_{y\in B} M(x,y) \geq M(x, y_n^{(m_n)})$. Thus $M(x,B) \geq \Big(\alpha - \dfrac{1}{n}\Big) * \Big(1 - \dfrac{1}{n}\Big)$. By taking limits as $n \to \infty$, we obtain

$$M(x,B) \geq \lim_{n\to\infty}\Big(\alpha - \frac{1}{n}\Big) * \lim_{n\to\infty}\Big(1 - \frac{1}{n}\Big) = \alpha * 1 = \alpha.$$

In addition, we have that

$$M(x,B) = \sup_{y\in B} M(x,y) \leq \sup_{y\in\overline{B}} M(x,y) = M(x,\overline{B}).$$

Consequently, we get $M(x,B) = M(x,\overline{B})$, for all $x \in A$. Eventually, we obtain

$$M(A,B) = \inf_{x\in A} M(x,B) = \inf_{x\in A} M(x,\overline{B}) = M(A,\overline{B}).$$

(4) For each $x_0 \in A$, $y_0 \in B$, $z_0 \in C$, we have $M(x_0,z_0) \geq M(x_0,y_0) * M(y_0,z_0)$. By the continuity of $*$, we obtain that

$$M(A,y_0) * M(y_0,z_0) = \sup_{x\in A} M(x,y_0) * M(y_0,z_0) \leq \sup_{x\in A} M(x,z_0) = M(A,z_0)$$

and

$$M(B,A) * M(B,z_0) = \inf_{y\in B} M(A,y) * \sup_{y\in B} M(y,z_0) \leq M(A,z_0).$$

Then

$$M(B,A) * \inf_{z\in C} M(B,z) \leq M(B,A) * M(B,z_0) \leq M(A,z_0),$$

that is, $M(B,A) * M(C,B) \leq M(A,z_0)$. Consequently,

$$M(B,A) * M(C,B) \leq \inf_{z\in C} M(A,z) = M(C,A).$$

(5) Since

$$H_M^*(A,B) = M(A,B) * M(B,A) = 1$$

and

$$M(A,B) * M(B,A) \le M(A,B) \wedge M(B,A) \le 1,$$

we obtain that $M(A,B) = M(B,A) = 1$. It follows from (1) that $H_M^*(A,B) = 1$ iff $\overline{A} = \overline{B}$. □

In fact, the conclusions of Proposition 3.1 still hold on $\mathcal{P}(X)$.

**Proposition 3.2.** Let $(X, M, *)$ be a stationary fuzzy metric space. Then for all $A, B, C \in \mathcal{P}(X)$, we have the following conclusions:

(1) $M(A,B) = 1$ iff $A \subset \overline{B}$ iff $\overline{A} \subset \overline{B}$ iff $M(\overline{A}, \overline{B}) = 1$;

(2) $M(x,B) \ge M(x,x') * M(x',B)$ for all $x, x' \in X$;

(3) $M(A,B) = M(A,\overline{B}) = M(\overline{A},\overline{B})$;

(4) $M(A,C) \ge M(A,B) * M(B,C)$;

(5) $H_M^*(A,B) = 1$ iff $\overline{A} = \overline{B}$.

Proof. From Proposition 3.1, it is true for all nonempty sets. Here we need to prove this proposition holds in those cases that at least one of the concerned sets is empty. We only prove (1). Similarly, the others can be proved.

(1) If $A=\emptyset$, $B\neq\emptyset$, then $M(\emptyset,B)=\inf_{x\in\emptyset} M(x,B)=1$; if $A\neq\emptyset$, $B=\emptyset$, then $M(A,\emptyset)=\inf_{x\in A} M(x,\emptyset)=0$; if $A=\emptyset$, $B=\emptyset$, then $M(\emptyset,\emptyset)=\inf_{x\in\emptyset}\sup_{y\in\emptyset} M(x,y)=1$. □

**Theorem 3.1.** Let $(X, M, *)$ be a stationary fuzzy metric space, then $(\mathcal{P}(X), H_M^*, *)$ is a stationary fuzzy pseudo-metric space.

Proof. (a) By the definition we have $H_M^*(\emptyset,\emptyset) = \inf_{x\in\emptyset} M(x,\emptyset) * \inf_{y\in\emptyset} M(\emptyset,y) = 1 * 1 = 1$, and $H_M^*(A,A) = M(A,A) * M(A,A) = 1 * 1 = 1$.

(b) Symmetry follows from commutativity of t-norm $*$.

(c) Since

$$\begin{array}{rcl} H_M^*(A,C) & = & M(A,C) * M(C,A), \\ H_M^*(A,B) & = & M(A,B) * M(B,A), \\ H_M^*(B,C) & = & M(B,C) * M(C,B), \end{array}$$

from (4) of Proposition 3.2, we obtain

$$M(A,C) \ge M(A,B) * M(B,C) \text{ and } M(C,A) \ge M(B,A) * M(C,B).$$

Hence by the commutativity and associativity of t-norm, we have

$$M(A,C) * M(C,A) \ge M(A,B) * M(B,A) * M(B,C) * M(C,B),$$

i. e., $H_M^*(A,C) \geq H_M^*(A,B) * H_M^*(B,C)$.

Consequently, $(\mathcal{P}(X), H_M^*, *)$ is a stationary fuzzy pseudo-metric space. □

**Theorem 3.2.** Let $(X, M, *)$ be a stationary fuzzy metric space and $A \subset X$, where $*$ is a t-norm without zero divisors. Then the following conditions are equivalent.

(1) $A$ is a bounded subset of $X$.

(2) There exists $r \in (0,1)$ such that for all $x, y \in A$ we have $M(x,y) > 1-r$.

(3) For each $x \in X$, there exists $r_x \in (0,1)$ such that $A \subset B_M(x, r_x)$.

Proof. (1) ⇒ (2) Suppose $A$ is a bounded subset of $X$. Then there exists $x_0 \in X$, $r_1 \in (0,1)$ such that $A \subset B_M(x_0, r_1) = \{y : M(x_0, y) > 1 - r_1\}$. Thus for all $x, y \in A$, we have

$$M(x_0, x) > 1 - r_1, M(x_0, y) > 1 - r_1.$$

Consequently, since $*$ is without zero divisors, we obtain that

$$\begin{aligned} M(x,y) &\geq M(x,x_0) * M(x_0,y) \\ &= M(x_0,x) * M(x_0,y) \\ &\geq (1-r_1) * (1-r_1) > 1-r \end{aligned}$$

where $r \in (1-(1-r_1)*(1-r_1), 1)$.

(2) ⇒ (3) Since (2) holds, then there exists $r_1 \in (0,1)$ such that $M(x,y) > 1-r_1$ for all $x, y \in A$. Fix $x_0 \in A$, for any $y \in A$, we have $M(y, x_0) > 1 - r_1$. Thus for each $x \in X$, we obtain

$$M(x,y) \geq M(x,x_0) * M(x_0,y) \geq M(x,x_0) * (1-r_1).$$

Since $M(x, x_0) \in (0,1]$ and $*$ is a t-norm without zero divisors, there exists $r_x \in (0,1)$ such that

$$M(x, x_0) * (1 - r_1) > 1 - r_x,$$

i. e., $y \in B_M(x, r_x)$. By the arbitrariness of $y$, we have $A \subset B_M(x, r_x)$.

The implication (3) ⇒ (1) is obvious. □

**Proposition 3.3.** Let $(X, M, *)$ be a stationary fuzzy metric space, where $*$ is a t-norm without zero divisors. If $A, B \subset X$ are any two bounded subsets of $X$, then $A \cup B$ is a bounded subset of $X$.

Proof. Fix $x_0 \in A, y_0 \in B, z_0 \in X$. For any $x \in A, y \in B$, we have

$$M(x,y) \geq M(x,x_0) * M(x_0,z_0) * M(z_0,y_0) * M(y_0,y).$$

Since $A, B$ are bounded subsets of $X$, there exists $r_A, r_B \in (0,1)$ such that for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$, $M(x_1, x_2) > 1 - r_A$ and $M(y_1, y_2) > 1 - r_B$. Hence we have

$$M(x,y) > (1-r_A) * (1-r_B) * M(x_0,z_0) * M(y_0,z_0) > 0.$$

Let $1-r_1 = (1-r_A)*(1-r_B)*M(x_0,z_0)*M(y_0,z_0) \in (0,1)$ then we obtain $M(x,y) > 1-r_1$.

In addition, if $x,y \in A-B$, we have $M(x,y) > 1-r_A$; if $x,y \in B-A$, we have $M(x,y) > 1-r_B$. Thus let $r = \max\{r_1, r_A, r_B\} \in (0,1)$. Then for any $x,y \in A \cup B$, we have $M(x,y) > 1-r$ which implies $A \cup B$ is a bounded subset of $X$. □

**Theorem 3.3.** Let $(X, M, *)$ be a stationary fuzzy metric space, where $*$ is a t-norm without zero divisors. Then $(\mathcal{CB}(X), H_M^*, *)$ is a stationary fuzzy metric space.

Proof. Let $A, B, C \in \mathcal{CB}(X)$. By Proposition 3.3, we have $A \cup B \in \mathcal{CB}(X)$ which means there exists $r \in (0,1)$ such that for all $x \in A, y \in B$, $M(x,y) > 1-r$. Hence for any $x \in A$, we can get that

$$M(x,B) = \sup_{y \in B} M(x,y) > 1-r > 0.$$

Thus we obtain

$$M(A,B) = \inf_{x \in A} M(x,B) \geq (1-r) > 0.$$

Similarly,

$$M(B,A) = \inf_{y \in A} M(y,A) \geq (1-r) > 0.$$

Since $*$ is a t-norm without zero divisors, thus we have $H_M^*(A,B) = M(A,B) * M(B,A) > 0$.

By (5) of Proposition 3.1, we have $H_M^* = 1$ iff $A = B$; by the definition of $H_M^*$, we get

$$H_M^*(A,B) = M(A,B) * M(B,A) = M(B,A) * M(A,B) = H_M^*(B,A).$$

In addition, by (4) of Proposition 3.1, we have

$$M(A,C) \geq M(A,B) * M(B,C) \text{ and } M(C,A) \geq M(B,A) * M(C,B).$$

which implies

$$M(A,C) * M(C,A) \geq M(A,B) * M(B,A) * M(B,C) * M(C,B),$$

i. e.,

$$H_M^*(A,C) \geq H_M^*(A,B) * H_M^*(B,C).$$

Consequently, $(\mathcal{CB}(X), H_M^*, *)$ is a stationary fuzzy metric space. □

The following example shows that the condition: "$*$ is a t-norm without zero divisors" in the above theorem is essential.

**Example 3.1.** Let $X = \mathbb{R}$ and let $*$ be the Lukasiewicz t-norm which is defined by $a * b = \max\{0, a+b-1\}$ for all $a, b \in [0,1]$ [25]. It is easy to see that Lukasiewicz t-norm is not a t-norm without zero divisors. We define a fuzzy set $M$ on $X \times X$ by

$$M(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 0.5, & \text{if } x \neq y, \end{cases}$$

for all $x, y \in X$.

It is obvious that $M$ satisfies (i), (ii) and (iii) of Definition 2.3. Next, we will show that $M$ satisfies Condition (iv) of Definition 2.3.

Let $x, y, z \in X$. If $x = y$, then $M(x,y) = 1$. Since $M(x,z) \le 1$ and $M(z,y) \le 1$, we have

$$M(x,z) * M(z,y) = \max\{0, M(x,z) + M(z,y) - 1\} \le 1 = M(x,y).$$

Thus without loss of generality, suppose $x \neq y$ and $x \neq z$. Then $M(x,y) = M(x,z) = 0.5$ and $M(x,z) * M(z,y) = \max\{0, M(x,z) + M(z,y) - 1\} = 0$. Hence $M(x,y) \ge M(x,z) * M(z,y)$. Consequently, $(X, M, *)$ is a stationary fuzzy metric space. In addition, since $B_M(x, 0.2) = \{x\}$ for all $x \in X$, we have $\tau_M$ is a discrete topology on $X$.

However $(\mathcal{CB}(X), H_M^*, *)$ is not a stationary fuzzy metric space. In fact, for any nonempty subset $A$, we have $A \subseteq B_M(0, 0.5)$, which implies $\mathcal{CB}(X) = \mathcal{P}_0(X)$. Let $A = [-1, 0]$ and $B = [1, 2]$. Thus $A, B \in \mathcal{CB}(X)$. Since $M(x,y) = 0.5$ for any $x \in A$ and $y \in B$, we have $M(A,B) = M(B,A) = 0.5$. Hence

$$H_M^*(A,B) = M(A,B) * M(B,A) = \max\{0, 0.5 + 0.5 - 1\} = 0,$$

which implies that $H_M^*$ does not satisfy Condition (i) of Definition 2.3.

Let us recall that if $(X, \mathcal{U})$ is a uniform space, then the Hausdorff–Bourbaki uniformity $H_{\mathcal{U}}$ (of $\mathcal{U}$) on $\mathcal{P}(X)$, has a base the family of sets of the form

$$H_U = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X) : B \subset U(A), A \subset U(B)\}$$

where $U \in \mathcal{U}$ [8].

The restriction of $H_{\mathcal{U}}$ to $\mathcal{CB}(X) \times \mathcal{CB}(X)$ will also be denoted by $H_{\mathcal{U}}$. On the other hand, if $(X, M, *)$ is a stationary fuzzy metric space, then $\{U_\varepsilon : \varepsilon \in (0,1)\}$ is a base for the uniformity $\mathcal{U}_M$ on $X$ compatible with $\tau_M$, where

$$U_\varepsilon = \{(x,y) \in X \times X : M(x,y) > 1 - \varepsilon\}$$

for all $\varepsilon \in (0,1)$. $\mathcal{U}_M$ is called the uniformity induced by $(M, *)$. In particular, $\mathcal{U}_{H_M^*}$ is the uniformity induced by the Hausdorff stationary fuzzy metric of $(M, *)$. We have the following useful result.

**Theorem 3.4.** Let $(X, M, *)$ be a stationary fuzzy metric space, where $*$ is a t-norm without zero divisors. Then the Hausdorff–Bourbaki uniformity $H_{\mathcal{U}_M}$ coincides with the uniformity $\mathcal{U}_{H_M^*}$ on $\mathcal{CB}(X)$.

Proof. For any $\varepsilon \in (0,1)$, by the continuity of $*$, there exists $\varepsilon_1 \in (0, \varepsilon)$ such that

$$(1-\varepsilon_1) * (1-\varepsilon_1) > 1-\varepsilon.$$

For any

$$(A,B) \in \{(A,B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : B \subset U_{\varepsilon_1}(A), A \subset U_{\varepsilon_1}(B)\},$$

and $y \in B$, since $B \subset U_{\varepsilon_1}(A)$, there exists $x_y \in A$ satisfies $M(x_y, y) > 1-\varepsilon_1$. For all $y \in B$, we have

$$M(A,y) = \sup_{x\in A} M(x,y) \geq M(x_y,y) > 1-\varepsilon_1.$$

Hence,

$$\inf_{y\in B} M(A,y) \geq 1-\varepsilon_1.$$

Similarly,

$$\inf_{x\in A} M(x,B) \geq 1-\varepsilon_1.$$

Thus we obtain

$$H_M^*(A,B) = \inf_{y\in B} M(y,A) * \inf_{x\in A} M(x,B) \geq (1-\varepsilon_1) * (1-\varepsilon_1) > 1-\varepsilon.$$

Consequently,

$$(A,B) \in \{(A,B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : H_M^*(A,B) > 1-\varepsilon\}$$

i. e.,

$$\begin{aligned} &\{(A,B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : B \subseteq U_{\varepsilon_1}(A), A \subseteq U_{\varepsilon_1}(B)\} \\ &\subseteq \{(A,B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : H_M^*(A,B) > 1-\varepsilon\}. \end{aligned}$$

Conversely, for any

$$(A,B) \in \{(A,B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : H_M^*(A,B) > 1-\varepsilon\},$$

since

$$H_M^*(A,B) = \inf_{y\in B} M(y,A) * \inf_{x\in A} M(y,B) > 1-\varepsilon$$

and

$$\inf_{y\in B} M(y,A) * \inf_{x\in A} M(x,B) \leq \inf_{y\in B} M(y,A) \wedge \inf_{x\in A} M(x,B),$$

we have

$$\inf_{y\in B} M(y,A) \wedge \inf_{x\in A} M(x,B) > 1-\varepsilon.$$

Thus we can get that

$$\inf_{y\in B} M(y,A) > 1-\varepsilon, \inf_{x\in A} M(x,B) > 1-\varepsilon.$$

Since $\inf_{x\in A} M(x,B) > 1-\varepsilon$, for every $x \in A$, we have

$$M(x,B) = \sup_{y\in B} M(x,y) > 1-\varepsilon.$$

Then there exists $y_x \in B$ such that $M(x,y_x) > 1-\varepsilon$, i. e., $A \subset U_\varepsilon(B)$. Similarly $B \subset U_\varepsilon(A)$. Thus

$$(A,B) \in \{(A,B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : B \subset U_\varepsilon(A), A \subset U_\varepsilon(B)\},$$

i. e.,

$$\{(A,B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : H_M^*(A,B) > 1-\varepsilon\}$$
$$\subseteq \{(A,B) \in \mathcal{CB}(X) \times \mathcal{CB}(X) : B \subseteq U_\varepsilon(A), A \subseteq U_\varepsilon(B)\}.$$

We conclude that $H_{\mathcal{U}_M} = \mathcal{U}_{H_M^*}$ on $\mathcal{CB}(X)$. □

**Theorem 3.5.** Let $(X,M,*)$ be a stationary fuzzy metric space, where $*$ is a t-norm without zero divisors. Then $(\mathcal{CB}(X), H_M^*, *)$ is complete iff $(X,M,*)$ is complete.

Proof. By Theorem 2 of [36], we have $(\mathcal{CB}(X), H_M^*, *)$ is complete iff $(\mathcal{CB}(X), \mathcal{U}_{H_M^*})$ is complete. Since, by Theorem 3.4, $H_{\mathcal{U}_M} = \mathcal{U}_{H_M^*}$ on $\mathcal{CB}(X)$, it follows from [4] that $(\mathcal{CB}(X), \mathcal{U}_{H_M^*})$ is complete iff $(X, \mathcal{U}_M)$ is complete. Thus $(\mathcal{CB}(X), H_M^*, *)$ is complete iff $(X,M,*)$ is complete. □

Analogous arguments yield the following result.

**Theorem 3.6.** Let $(X,M,*)$ be a stationary fuzzy metric space, where $*$ is a t-norm without zero divisors. Then $(\mathcal{CB}(X), H_M^*, *)$ is precompact iff $(X,M,*)$ is precompact.

**Lemma 3.1.** Let $(Y,M,*)$ be a stationary fuzzy metric space, where $*$ is a t-norm without zero divisors. If $X$ is a dense subset of $Y$, then $\mathcal{CB}(X)$ is a dense subset of $(\mathcal{CB}(Y), H_M^*, *)$.

Proof. For each $\varepsilon \in (0,1)$, by the continuity of $*$, there exists $\varepsilon_1$ such that

$$(1-\varepsilon_1) * (1-\varepsilon_1) > 1-\varepsilon.$$

For each $A \in \mathcal{CB}(Y)$, we have

$$A \subset \bigcup_{x\in A} B_M(x,\varepsilon_1).$$

Since $X$ is a dense subset of $Y$, for every $x \in A$, there exists $y_x \in B_M(x,\varepsilon_1) \cap X$. Let $C = \{y_x : x \in A\}$. Since $A \in \mathcal{CB}(Y)$, there exists an $r \in (0,1)$ such that $M(x_1,x_2) > 1-r$ for any $x_1, x_2 \in A$. Then we have

$$M(y_{x_1}, y_{x_2}) \geq M(y_{x_1}, x_1) * M(x_1,x_2) * M(x_2, y_{x_2}) > (1-\varepsilon_1) * (1-r) * (1-\varepsilon_1).$$

Let

$$r' = 1 - [(1-\varepsilon_1) * (1-r) * (1-\varepsilon_1)] \in (0,1).$$

Hence, we obtain $M(y_{x_1}, y_{x_2}) > 1 - r'$, i. e., $C$ is a bounded subset of $X$.

Let $\overline{C}$ be the closure of $C$ in $X$. Now we verify that

$$\overline{C} \in B_{H_M^*}(A,\varepsilon) = \{B \in \mathcal{P}(Y) : (A,B) \in \mathcal{CB}(X) \times \mathcal{P}(Y), H_M^*(A,B) > 1-\varepsilon\}.$$

Since for any $x \in A$, we have

$$M(A, y_x) = \sup_{x \in A} M(x, y_x) \geq M(x, y_x) > 1 - \varepsilon_1.$$

Thus, we obtain

$$M(C, A) = \inf_{y_x \in C} M(y_x, A) \geq 1 - \varepsilon_1.$$

Similarly, we can get

$$M(A, C) = \inf_{x \in A} M(x, C) \geq 1 - \varepsilon_1.$$

Therefore, by Proposition 3.1, we have

$$H_M^*(A, \overline{C}) = H_M^*(A, C) = M(C, A) * M(A, C) \geq (1-\varepsilon_1) * (1-\varepsilon_1) > 1 - \varepsilon$$

i. e., $\overline{C} \in \mathcal{CB}(X) \cap B_{H_M^*}(A,\varepsilon)$. We conclude that $\mathcal{CB}(X)$ is a dense subset of $(\mathcal{CB}(Y), H_M^*, *)$.

**Lemma 3.2.** (Gregori and Romaguera [13]) A stationary fuzzy metric space $(X, M, *)$ is completable iff $\lim_{n\to\infty} M(x_n, y_n) > 0$ for each pair of Cauchy sequence $\{x_n\}_n, \{y_n\}_n$ in $X$.

**Theorem 3.7.** Let $(X, M, *)$ be a stationary fuzzy metric space, where $*$ is a t-norm without zero divisors. Then $(\mathcal{CB}(X), H_M^*, *)$ is completable iff $(X, M, *)$ is completable.

Proof. If $(X, M, *)$ is completable, then there exists a complete stationary fuzzy metric space $(\widetilde{Y}, \widetilde{M}, \widetilde{*})$, such that $(X, M, *)$ is isometric to a dense subspace $(\widetilde{X}, \widetilde{M}, \widetilde{*})$ of $(\widetilde{Y}, \widetilde{M}, \widetilde{*})$, where $\widetilde{*}$ is a t-norm without zero divisors. By Lemma 3.1, we know that $(\mathcal{CB}(\widetilde{X}), H_{\widetilde{M}}^{\widetilde{*}}, \widetilde{*})$ is a dense subspace of a stationary fuzzy metric space $(\mathcal{CB}(\widetilde{Y}), H_{\widetilde{M}}^{\widetilde{*}}, \widetilde{*})$. Since $(\widetilde{Y}, \widetilde{M}, \widetilde{*})$ is complete, by Theorem 3.5, we have $(\mathcal{CB}(\widetilde{Y}), H_{\widetilde{M}}^{\widetilde{*}}, \widetilde{*})$ is complete. Because $(X, M, *)$ and $(\widetilde{X}, \widetilde{M}, \widetilde{*})$ are isometric, there exists an isometry mapping $f$ from $X$ to $\widetilde{X}$.

According to the classical extension principle, from mapping $f$, we can induce the following mappings

$$f : \mathcal{P}(X) \longrightarrow \mathcal{P}(\widetilde{X}), A \longmapsto f(A) \in \mathcal{P}(\widetilde{X});$$
$$f^{-1} : \mathcal{P}(\widetilde{X}) \longrightarrow \mathcal{P}(X), B \longmapsto f^{-1}(B) \in \mathcal{P}(X).$$

Now we shall prove that $f$ is the isometry mapping from $\mathcal{CB}(X)$ to $\mathcal{CB}(\widetilde{X})$.

Let $A \in \mathcal{CB}(X)$. Since $f$ is a homeomorphic mapping, we have that $f(A)$ is a closed set of $\widetilde{X}$. For each $\widetilde{x}, \widetilde{y} \in f(A)$, there exists a unique $x, y \in A$ such that $\widetilde{x} = f(x), \widetilde{y} = f(y)$, which implies $\widetilde{M}(\widetilde{x}, \widetilde{y}) = \widetilde{M}(f(x), f(y)) = M(x, y)$. Since $A \in \mathcal{CB}(X)$, there exists $r \in (0,1)$ such that for all $x, y \in A$, we have $M(x,y) > 1 - r$, Consequently $\widetilde{M}(\widetilde{x}, \widetilde{y}) > 1 - r$, i. e., $f(A) \in \mathcal{CB}(\widetilde{X})$.

For any $\widetilde{A} \in \mathcal{CB}(\widetilde{X})$ we have $f^{-1}(\widetilde{A}) = \{x \in X : f(x) \in \widetilde{A}\}$. Then for each $x, y \in f^{-1}(\widetilde{A})$, there exists a unique $\widetilde{x}, \widetilde{y} \in \widetilde{A}$ such that $\widetilde{x} = f(x)$, $\widetilde{y} = f(y)$, which implies

$$M(x,y) = M(f^{-1}(\widetilde{x}), f^{-1}(\widetilde{y})) = \widetilde{M}(f(f^{-1}(\widetilde{x})), f(f^{-1}(\widetilde{y}))) = \widetilde{M}(\widetilde{x}, \widetilde{y}).$$

Since $\widetilde{A} \in \mathcal{CB}(\widetilde{X})$, there exists $r \in (0,1)$ such that for all $\widetilde{x}, \widetilde{y} \in \widetilde{A}$, we have $\widetilde{M}(\widetilde{x}, \widetilde{y}) > 1 - r$. Hence $M(x,y) > 1 - r$, i. e., $f^{-1}(\widetilde{A}) \in \mathcal{CB}(X)$.

For any $A, B \in \mathcal{CB}(X)$, we have

$$H^{\widetilde{*}}_{\widetilde{M}}(f(A), f(B)) = \inf_{x \in A} \sup_{y \in B} \widetilde{M}(f(x), f(y)) * \inf_{y \in B} \sup_{x \in A} \widetilde{M}(f(y), f(x))$$

and

$$H^*_M(A,B) = \inf_{x \in A} \sup_{y \in B} M(x,y) * \inf_{y \in B} \sup_{x \in A} M(y,x).$$

Because for any $x, y \in X, \widetilde{M}(f(x), f(y)) = M(x,y)$, we obtain that

$$H^{\widetilde{*}}_{\widetilde{M}}(f(A), f(B)) = H^*_M(A,B).$$

From what we have proved above, we can get that $f$ is the isometry mapping from $\mathcal{CB}(X)$ to $\mathcal{CB}(\widetilde{X})$ and $(\mathcal{CB}(X), H^*_M, *)$ to $(\mathcal{CB}(\widetilde{X}), H^{\widetilde{*}}_{\widetilde{M}}, \widetilde{*})$ are isometric. Consequently, $(\mathcal{CB}(X), H^*_M, *)$ is completeable.

Conversely, suppose $(\mathcal{CB}(X), H^*_M, *)$ is completeable. For any $\varepsilon \in (0,1)$, by the continuity of $*$, there exists $\varepsilon_1 \in (0,1)$ such that

$$1 - \varepsilon < (1 - \varepsilon_1) * (1 - \varepsilon_1).$$

Let $\{x_n\}$ be a Cauchy sequence of $X$. For any $\varepsilon_1 \in (0,1)$, there exist $N \in N^+$, such that

$$M(x_n, x_m) > 1 - \varepsilon_1$$

whenever $n, m > N$. Hence, for the sequence $\{\{x_n\}\} \subset \mathcal{CB}(X)$ we have

$$H^*_M(\{x_n\}, \{x_m\}) = M(x_n, x_m) * M(x_m, x_n) \geq (1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon,$$

whenever $n, m > N$. Thus $\{\{x_n\}\}$ is a Cauchy sequence of $\mathcal{CB}(X)$. Because $(\mathcal{CB}(X), H^*_M, *)$ is completeable, by Lemma 3.2, for any Cauchy sequences $\{x_n\}$, $\{y_n\}$ of $X$, we have

$$\lim_{n \to \infty} H^*_M(\{x_n\}, \{y_n\}) > 0,$$

i. e.,

$$\lim_{n \to \infty} M(x_n, y_n) * \lim_{n \to \infty} M(x_n, y_n) > 0.$$

In addition, since $*$ is a t-norm without zero divisors, thus we can get that $a > 0$, whenever $a * a > 0$. It follows that $\lim_{n \to \infty} M(x_n, y_n) > 0$. By Lemma 3.2, we conclude that $(X, M, *)$ is completeable. □

## 4. CONCLUSIONS

We have generalized the classical Hausdorff metrics with triangular norms and proposed a method for constructing a generalized Hausdorff fuzzy metric on the set of the nonempty bounded closed subsets of a given stationary fuzzy metric space. We also have discussed several important properties as completeness, completion and precompactness for this hyperspace. In fuzzy functional analysis, many researchers have been working on the fixed point theory in the space of compact fuzzy sets equipped with the supremum metric [1, 5, 10, 11, 21, 22, 30, 40]. The approach of this paper, however, enables us to use fixed point theory to study the fuzzy mappings from the new perspective. So, we assume our results would provide a mathematical background for ongoing work in the problems of those related fields.

## ACKNOWLEDGEMENT

This work was supported by The National Natural Science Foundation of China (Grant no. 11201512 and 61472056) and The Natural Science Foundation Project of CQ CSTC (cstc2012jjA00001).

(Received October 15, 2013)

## REFERENCES

[1] H. Adibi, Y. J. Cho, D. O'Regan, and R. Saadati: Common fixed point theorems in $L$-fuzzy metric spaces. Appl. Math. Comput. *182* (2006), 820–828.

[2] J. P. Aubin and I. Ekeland: Applied Nonlinear Analysis. Wiley, New York 1984.

[3] D. Azé, J. N. Corvellec, and R. E. Lucchetti: Variational pairs and applications to stability in nonsmooth analysis. Nonlinear Anal. Theory Methods Appl. *49* (2002), 643–670.

[4] G. Beer: Topologies on Closed and Closed Convex Sets. Kluwer Academic Publishers, Dordrecht 1993.

[5] S. S. Chang, Y. J. Cho, B. S. Lee, J. S. Jung, and S. M. Kang: Coincidence point and minimization theorems in fuzzy metric spaces. Fuzzy Sets and Systems *88* (1997), 119–128.

[6] Y. J. Cho and N. Petrot: Existence theorems for fixed fuzzy points with closed $\alpha$-cut sets in complete metric spaces. Fuzzy Sets and Systems *26* (2011), 115–124.

[7] Z. K. Deng: Fuzzy pseudo-metric spaces. J. Math. Anal. Appl. *86* (1982), 74–95.

[8] R. Engelking: General Topology. PWN-Polish Science Publishers, Warsaw 1977.

[9] A. George and P. Veeramani: On some results in fuzzy metric spaces. Fuzzy Sets and Systems *64* (1994), 395–399.

[10] B. M. Ghil and Y. K. Kim: Continuity of functions defined on the space of fuzzy sets. Inform. Sci. *157* (2003), 155–165.

[11] V. Gregori and A. Sapena: On fixed point theorem in fuzzy metric spaces. Fuzzy Sets and Systems *125* (2002), 245–252.

[12] V. Gregori and S. Romaguera: On completion of fuzzy metric spaces. Fuzzy Sets and Systems *130* (2002), 399–404.

[13] V. Gregori and S. Romaguera: Characterizing completable fuzzy metric spaces. Fuzzy Sets and Systems *144* (2004), 411–420.

[14] V. Gregori, S. Morillas, and A. Sapena: Examples of fuzzy metrics and applications. Fuzzy Sets and Systems *170* (2011), 95–111.

[15] F. Hausdorff: Set Theory. Chelsea, New York 1957.

[16] N.V. Hop: Solving fuzzy (stochastic) linear programming problems using superiority and inferiority measures. Inform. Sci. *177* (2007), 1977–1991.

[17] N. V. Hop: Solving linear programming problems under fuzziness and randomness environment using attainment values. Inform. Sci. *177* (2007), 2971–2984.

[18] S. Y. Joo and Y. K. Kim: The Skorokhod topology on space of fuzzy numbers. Fuzzy Sets and Systems *111* (2000), 497–501.

[19] S. Y. Joo and Y. K. Kim: Topological properties on the space of fuzzy sets. J. Math. Anal. Appl. *246* (2000), 576–590.

[20] O. Kaleva and S. Seikkala: On fuzzy metric spaces. Fuzzy Sets and Systems *12* (1984), 215–229.

[21] O. Kaleva: On the convergence of fuzzy sets. Fuzzy Sets and Systems *17* (1985), 53–65.

[22] O. Kaleva: Fuzzy differential equations. Fuzzy Sets and Systems *24* (1987), 301–317.

[23] Y. K. Kim: Compactness and convexity on the space of fuzzy sets. J. Math. Anal. Appl. *264* (2001), 122–132.

[24] Y. K. Kim: Compactness and convexity on the space of fuzzy sets II. Nonlinear Anal. Theory Methods Appl. *57* (2004), 639–653.

[25] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000.

[26] I. Kramosil and J. Michálek: Fuzzy metric and statistical metric spaces. Kybernetika *11* (1975), 326–334.

[27] G. Matheron: Random Sets and Integral Geometry. Wiley, New York 1975.

[28] B.S. Mordukhovich and Y. Shao: Fuzzy calculus for coderivatives of multifunctions. Nonlinear Anal. Theory Methods Appl. *29* (1997), 605–626.

[29] S. B. Nadler Jr: Multi-valued contraction mappings. Pacific J. Math. *30* (1969), 475–487.

[30] M. L. Puri and D. A. Ralescu: Fuzzy random variables. J. Math. Anal. Appl. *114* (1986), 409–422.

[31] D. Qiu and W. Zhang: On Decomposable Measures Induced by Metrics. J. Appl. Math. Volume 2012, Article ID 701206, 8 pages.

[32] D. Qiu and W. Zhang: The strongest t-norm for fuzzy metric spaces. Kybernetika *49* (2013), 141–148.

[33] D. Qiu, W. Zhang, and C. Li: On decomposable measures constructed by using stationary fuzzy pseudo-ultrametrics. Int. J. Gen. Syst. *42* (2013), 395–404.

[34] D. Qiu, W. Zhang, and C. Li: Extension of a class of decomposable measures using fuzzy pseudometrics. Fuzzy Sets and Systems *222* (2013), 33–44.

[35] J. Rodríguez-López and S. Romaguera: The Hausdorff fuzzy metric on compact sets. Fuzzy Sets and Systems *147* (2004), 273–283.

[36] S. Romaguera and M. Sanchis: On fuzzy metric groups. Fuzzy Sets and Systems *124* (2001), 109–115.

[37] F. G. Shi and C. Y. Zheng: Metrization theorems in L-topological spaces. Fuzzy Sets and Systems *149* (2005), 455–471.

[38] E. Trillas: On the use of words and fuzzy sets. Inf. Sci. *176* (2006), 1463–1487.

[39] L. A. Zadeh: Fuzzy sets. Inform. Control *8* (1965), 338–353.

[40] W. Zhang, D. Qiu, Z. Li, and G. Xiong: Common fixed point theorems in a new fuzzy metric space. J. Appl. Math. Volume 2012, Article ID 890678, 18 pages.

*Dong Qiu, School of Computer Science, Chongqing University, Shapingba, Chongqing, 400044, P. R. China and College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Nanan, Chongqing, 400065. P. R. China.*
*e-mail: dongqiumath@163.com*

*Chongxia Lu, College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Nanan, Chongqing, 400065. P. R. China.*
*e-mail: lcx19882012@163.com*

*Shuai Deng, College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Nanan, Chongqing, 400065. P. R. China.*
*e-mail: 15123126186@163.com*

*Liang Wang, College of Automation, Chongqing University of Posts and Telecommunications, Nanan, Chongqing, 400065. P. R. China.*
*e-mail: wangliang77@163.com*