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GENERALIZING A THEOREM OF SCHUR

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Abstract. In a letter written to Landau in 1935, Schur stated that for any integer $t > 2$, there are primes $p_1 < p_2 < \ldots < p_t$ such that $p_1 + p_2 > p_t$. In this note, we use the Prime Number Theorem and extend Schur's result to show that for any integers $t \geqslant k \geqslant 1$ and real $\varepsilon > 0$, there exist primes $p_1 < p_2 < \ldots < p_t$ such that

$$p_1 + p_2 + \ldots + p_k > (k - \varepsilon)p_t.$$

Keywords: Prime Number Theorem; Schur

MSC 2010: 11N05, 11A41

1. Introduction

In a letter written to Landau in 1935, Schur [1] stated the following fact.

Proposition 1.1. *For any integer $t > 2$, there exist primes $p_1 < p_2 < \ldots < p_t$ such that $p_1 + p_2 > p_t$.*

In 1936, E. Lehmer [1] used Proposition 1.1 to prove a theorem concerning the size of the coefficients of the cyclotomic polynomials. She did not, however, include a proof of Proposition 1.1, but merely referenced Schur's letter. The first publication of a proof of Proposition 1.1 occurred in an article written by Jiro Suzuki [2], in which he used Proposition 1.1 to prove that every integer appears as a coefficient in some cyclotomic polynomial. In this brief note, we use the Prime Number Theorem to present a generalization of Schur's original result.

2. The generalization

In this section we present the generalization of Proposition 1.1, but first we need a lemma.

Lemma 2.1. *Let $t \geqslant 1$ be an integer. Then, for any real number $\varepsilon > 0$, there exist primes $p_1 < p_2 < \ldots < p_t$ such that*

$$p_1 + p_2 + \ldots + p_t > (t - \varepsilon)p_t.$$

Proof. By way of contradiction, assume there exists some $\varepsilon > 0$ and some integer $t \geqslant 1$ such that, for any set of t primes $p_1 < p_2 < \ldots < p_t$, we have

$$p_1 + p_2 + \ldots + p_t \leqslant (t - \varepsilon)p_t.$$

Then, clearly $t \geqslant 2$, $\varepsilon < t$, and

$$\frac{tp_1}{t - \varepsilon} < p_t. \tag{2.1}$$

Now, if for some real number n there exist primes $p_1 < p_2 < \ldots < p_t$ such that

$$\Big(\frac{t}{t-\varepsilon}\Big)^{n-1} \leqslant p_1 < p_2 < \ldots < p_t \leqslant \Big(\frac{t}{t-\varepsilon}\Big)^{n},$$

then

$$p_1\Big(\frac{t}{t-\varepsilon}\Big) \geqslant \Big(\frac{t}{t-\varepsilon}\Big)^{n} \geqslant p_t,$$

contradicting (2.1). Hence, for any real number n, there are fewer than t primes between $(t/(t-\varepsilon))^{n-1}$ and $(t/(t-\varepsilon))^{n}$. It follows that $\pi((t/(t-\varepsilon))^{n}) < nt$ for all real numbers n, where $\pi(x)$ is the number of primes less than or equal to x. Therefore,

$$\frac{\pi\Big(\Big(\frac{t}{t-\varepsilon}\Big)^{n}\Big)\log\Big(\Big(\frac{t}{t-\varepsilon}\Big)^{n}\Big)}{\Big(\frac{t}{t-\varepsilon}\Big)^{n}} < \frac{nt\log\Big(\Big(\frac{t}{t-\varepsilon}\Big)^{n}\Big)}{\Big(\frac{t}{t-\varepsilon}\Big)^{n}} \tag{2.2}$$

for all real numbers n. As n approaches infinity, the right-hand side of (2.2) approaches 0, but the Prime Number Theorem implies that the limit of the left-hand side of (2.2) is 1. This contradiction completes the proof. □

Theorem 2.2. *For any integers $t \geqslant k \geqslant 1$ and real $\varepsilon > 0$, there exist primes $p_1 < p_2 < \ldots < p_t$ such that*

$$p_1 + p_2 + \ldots + p_k > (k - \varepsilon)p_t.$$

Proof. By Lemma 2.1, we have that there exist primes $p_1 < p_2 < \ldots < p_t$ such that

$$p_1 + p_2 + \ldots + p_t > (t - \varepsilon)p_t. \tag{2.3}$$

The case $k = t$ is Lemma 2.1, so assume that $k < t$. Then, since

$$p_{k+1} + p_{k+2} + \ldots + p_t \leqslant (t - k)p_t, \tag{2.4}$$

we can subtract the left and right-hand sides of (2.4) from the left and right-hand sides of (2.3) respectively, preserving the inequality in (2.3), and the theorem is established. □

Remark 2.3. Note that the special case of Theorem 2.2 with $t > k = 2$ and $\varepsilon = 1$ is simply Proposition 1.1.

References

[1] *E. Lehmer*: On the magnitude of the coefficients of the cyclotomic polynomial. Bull. Am. Math. Soc. *42* (1936), 389–392.

[2] *J. Suzuki*: On coefficients of cyclotomic polynomials. Proc. Japan Acad., Ser. A *63* (1987), 279–280.

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