Abdelhamid Ainouz

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HOMOGENIZATION OF A DUAL-PERMEABILITY PROBLEM IN TWO-COMPONENT MEDIA WITH IMPERFECT CONTACT

Abdelhamid Ainouz, Algiers

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Abstract. In this paper, we study the macroscopic modeling of a steady fluid flow in an ε-periodic medium consisting of two interacting systems: fissures and blocks, with permeabilities of different order of magnitude and with the presence of flow barrier formulation at the interfacial contact. The homogenization procedure is performed by means of the two-scale convergence technique and it is shown that the macroscopic model is a one-pressure field model in a one-phase flow homogenized medium.

Keywords: porous media; homogenization; two scale convergence

MSC 2010: 35B27, 76S05

1. Introduction

The study of fluid flows in porous media is a subject of practical interest in many engineering areas, such as geomechanics, material sciences, and water resources management. Some types of naturally porous rocks, like aquifers or petroleum reservoirs, are usually described as a dual-permeability (or a double porosity) medium, that is a two-component structure: one related to blocks, and the other related to fractures.

When a porous medium is composed by two or more different constituents, a precise mathematical modeling is required. Actually, due to the complexity of microstructures, any mathematical modeling used to determine fluid flows through heterogeneous porous media must take into account the rapid spatial variation of the phenomenological parameters. Furthermore, numerical modeling of such systems yields at the local scale a huge number of discretized equations, so computations will be fastidious and intractable. It is then important to study fluid flows in porous media at the microscopic scale and to describe their behavior at the macroscopic scale. Roughly speaking, it consists in the passage from microscopic scale to the macroscopic one by tending to zero a small parameter, usually denoted ε, which is the ratio between the two characteristic scales, see [6], [11]. We remark here that the fact that homogenization in double-porosity phases can lead to effective fluid flow behavior was observed by many authors in various problems [1], [2], [5], [9], [10]. For example, in [5], a microscopic model consisting of the usual equations describing Darcy flow in a reservoir with highly discontinuous porosity and permeability coefficients, was addressed. It was rigorously proved that the macroscopic (homogenized) equation is a double porosity model of single phase flow. Also for poroelastic heterogeneous media, various effective double porosity models of composites made of a mixture of two poroelastic solids saturated by a compressible Newtonian fluid have been derived. In [10], the homogenization of a compact bone poroelasticity model, describing interactions between deformation of the bone tissue and induced flow, is addressed. The double-porous structure consists of the Havers-Volkmann channels (the primary porosity) and the canaliculi (the dual porosity). The macroscopic model is derived by means of periodic unfolding method and it describes the deformation-induced Darcy flow in the primary porosities whereas the micro-flow in the double porosity is responsible for the fading memory effects via the macroscopic poro-visco-elastic constitutive law. In [1], [2], Barenblatt-Biot consolidation models for flows in periodic porous elastic media are derived by using the two-scale convergence technique. The micro-structures consist of fluid flows of slightly compressible viscous fluids through two-component poro-elastic media separated by periodic interfacial barriers, described by the Biot model of consolidation with the Deresiewicz-Skalak interface boundary condition.

In this paper, we shall deal with the homogenization of a steady fluid flow in media made of two interacting porous systems with a high contrast of permeabilities. In fact, for such a configuration, it is well-known that the hydraulic conductivity in the fractures system is higher at the local scale than the hydraulic conductivity in the block matrix [5], [7]. The family of the corresponding micro-models that we shall study is described by an elliptic system of two partial differential equations in a two-medium description, with Darcy's law in each phase and with contrasting permeabilities, plus exchange terms representing the interfacial coupling that results from the interaction, at the micro-scale, between the two phases, see (2.1a)–(2.1e) below. The macro-model is derived by means of the two-scale convergence method [3]. It is shown that the overall behavior of fluid flow in such media behaves as a single porosity model with an average permeability and obeys a single equation of elliptic type, meaning that no dual-permeability effects occur at the macro-scale description, see (2.15) below. Besides that, the derived model presents an extra source surface density on the exterior boundary, which essentially arises from the fact that (1) blocks have low permeability when compared to the fissures, (2) non null and regular source density on the blocks and (3) the interface contact between the two constituents is assumed imperfect.

The paper is organized as follows: Section 2 is devoted to the problem setting of the micro-model and the statement of the main result. In Section 3, we shall be concerned with the derivation of the homogenized model via the two scale convergence method.

2. Setting of the problem and the main result

We consider Ω a bounded and smooth domain of $\mathbb{R}^N$ ($N \geqslant 2$) and $Y =]0,1[^N$ the generic cell of periodicity. Let $Y_1, Y_2 \subset Y$ be two open disjoint subsets of Y such that $Y = Y_1 \cup Y_2 \cup \Gamma$, where $\Gamma = \partial Y_1 \cap \partial Y_2$, assumed to be a smooth submanifold. We denote ν the unit normal of Γ, outward to Y_1. For $i = 1, 2$, let χ_i denote the characteristic function of Y_i, extended by Y-periodicity to $\mathbb{R}^N$. For $\varepsilon > 0$, we set

$$\Omega_i^\varepsilon = \Big\{x \in \Omega\colon\ \chi_i\Big(\frac{x}{\varepsilon}\Big) = 1\Big\} \quad \text{and} \quad \Gamma^\varepsilon = \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon.$$

To avoid some unnecessary technical computations, we assume that the dual porosities do not meet the boundary $\partial\Omega$, that is $\overline{\Omega_2^\varepsilon} \subset \Omega$ so that $\Gamma^\varepsilon = \partial\Omega_2^\varepsilon$ and $\partial\Omega_1^\varepsilon = \partial\Omega \cup \Gamma^\varepsilon$ (see Figure 1 below). Let $Z_i = \bigcup_{k\in\mathbb{Z}^N} (Y_i + k)$. As in [3], we also assume that Z_1 is smooth and a connected open subset of $\mathbb{R}^N$. Note that Z_2 may not be connected. Also, Z_1 and Z_2 are the primary and dual porosities, respectively.

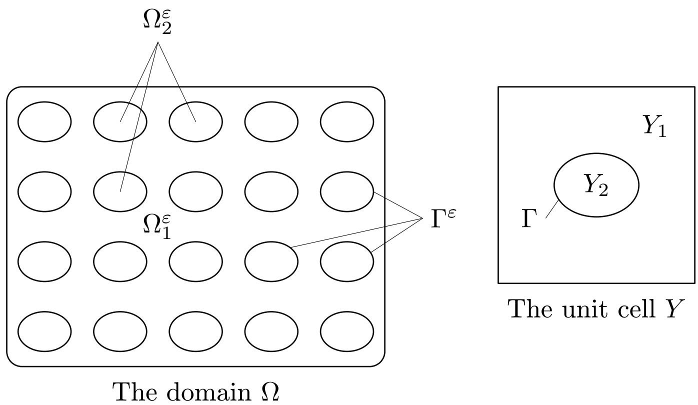

Figure 1. An example of a periodic two-component medium considered in this paper.

Let A (resp. B) denote the permeability of the medium Z_1 (resp. Z_2). Let f_i be a measurable function representing the internal source density of the fluid flow in Ω_i^ε. Finally, let ϑ be the *non-rescaled* hydraulic permeability of the thin layer Γ^ε. We shall assume the followings:

(H1) A (resp. B) is continuous on $\mathbb{R}^N$, Y-periodic and satisfies the ellipticity condition:

$$A\xi \cdot \xi \geqslant C|\xi|^2 \quad (\text{resp. } B\xi \cdot \xi \geqslant C|\xi|^2) \quad \forall \xi \in \mathbb{R}^N,$$

where, here and in what follows, C denotes various positive constants which are independent of ε;

(H2) $f_1, f_2 \in L^2(\Omega)$;

(H3) ϑ is a continuous function on $\mathbb{R}^N$, Y-periodic and bounded from below:

$$\vartheta(y) \geqslant C > 0, \quad y \in \mathbb{R}^N.$$

Remark 2.1. It should be noticed that in the hypothesis (H1), the continuity is not necessary. Indeed, one can take $A, B \in L^\infty(\mathbb{R}^N)$ and the main result of this paper remains unchanged.

To deal with periodic homogenization with micro-structures, we shall denote for $x \in \mathbb{R}^N$,

$$\chi_i^\varepsilon(x) = \chi_i\Big(\frac{x}{\varepsilon}\Big), \quad A^\varepsilon(x) = A\Big(\frac{x}{\varepsilon}\Big), \quad B^\varepsilon(x) = B\Big(\frac{x}{\varepsilon}\Big), \quad \text{and} \quad \vartheta^\varepsilon(x) = \varepsilon\vartheta\Big(\frac{x}{\varepsilon}\Big).$$

The micro-model that we shall study in this paper is given by the following set of equations:

$$-\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f_1 \quad \text{in } \Omega_1^\varepsilon, \tag{2.1a}$$

$$-\varepsilon^2 \operatorname{div}(B^\varepsilon \nabla v^\varepsilon) = f_2 \quad \text{in } \Omega_2^\varepsilon, \tag{2.1b}$$

$$A^\varepsilon \nabla u^\varepsilon \cdot \nu^\varepsilon = -\vartheta^\varepsilon(u^\varepsilon - v^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \tag{2.1c}$$

$$\varepsilon^2 B^\varepsilon \nabla v^\varepsilon \cdot n^\varepsilon = -\vartheta^\varepsilon(v^\varepsilon - u^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \tag{2.1d}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{2.1e}$$

where ν^ε and n^ε stand for the unit normal of Γ^ε outward to Ω_1^ε and Ω_2^ε, respectively. Here, Ω_1^ε represents the fissured region with permeability A^ε and Ω_2^ε the block region with permeability $\varepsilon^2 B^\varepsilon$. The physical quantities u^ε and v^ε are respectively the fluid flow pressures in Ω_1^ε and Ω_2^ε. As in Arbogast, Douglas, and Hornung [5], we have chosen a particular scaling of the permeability coefficients in (2.1b). This means that both terms $\int_{\Omega_1^\varepsilon} |\nabla u^\varepsilon|^2$ and $\varepsilon^2 \int_{\Omega_2^\varepsilon} |\nabla v^\varepsilon|^2$ have the same order of magnitude and thus lead to a balance in dissipation potential. Equations (2.1a) and (2.1b) express the conservation of mass of fluid with Darcy's law in Ω_1^ε and Ω_2^ε, respectively. Conditions (2.1c) and (2.1d) express flux continuity across Γ^ε and the imperfect contact between the block and the fissures along Γ^ε with permeability given by ϑ^ε, see [8]. Transmission condition (2.1d) is known in the literature as Deresiewicz-Skalak condition. Finally, (2.1e) is the homogeneous Dirichlet condition on the exterior boundary of Ω.

Let $H^\varepsilon = (H^1(\Omega_1^\varepsilon) \cap H_0^1(\Omega)) \times H^1(\Omega_2^\varepsilon)$. The space H^ε is equipped with the norm:

$$\|(\varphi,\psi)\|_{H^\varepsilon}^2 = \|\nabla\varphi\|_{L^2(\Omega_1^\varepsilon)}^2 + \varepsilon^2\|\nabla\psi\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon\|\varphi-\psi\|_{L^2(\Gamma^\varepsilon)}^2.$$

The weak formulation of (2.1a)–(2.1e) is as follows: find $(u^\varepsilon, v^\varepsilon) \in H^\varepsilon$, such that for all $(\varphi,\psi) \in H^\varepsilon$, we have

$$\begin{aligned}(2.2)\qquad &\int_{\Omega_1^\varepsilon} A\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon \nabla\varphi \,\mathrm{d}x + \varepsilon^2\int_{\Omega_2^\varepsilon} B\Big(\frac{x}{\varepsilon}\Big)\nabla v^\varepsilon\nabla\psi\,\mathrm{d}x \\ &\quad + \varepsilon\int_{\Gamma^\varepsilon}\vartheta\Big(\frac{x}{\varepsilon}\Big)(u^\varepsilon - v^\varepsilon)(\varphi-\psi)\,\mathrm{d}s^\varepsilon = \int_{\Omega_1^\varepsilon} f_1\varphi\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f_2\psi\,\mathrm{d}x,\end{aligned}$$

where $\mathrm{d}x$ and $\mathrm{d}s^\varepsilon$ denote, respectively, the Lebesgue measure on $\mathbb{R}^N$ and the Hausdorff measure on Γ^ε. Next, we state the existence and uniqueness result of the weak formulation (2.2).

Theorem 2.1. *Let the assumptions* (H1)–(H3) *be fulfilled. Then, for any sufficiently small* $\varepsilon > 0$, *there exists a unique couple* $(u^\varepsilon, v^\varepsilon) \in H^\varepsilon$, *solution of the weak problem* (2.2), *such that*

$$(2.3)\qquad \|(u^\varepsilon, v^\varepsilon)\|_{H^\varepsilon} \leqslant C.$$

Proof. We shall use the Lax-Milgram lemma. Let us denote

$$\begin{aligned} a^\varepsilon((\varphi,\psi),(\eta,\varsigma)) &= \int_{\Omega_1^\varepsilon} A^\varepsilon\nabla\varphi\nabla\eta\,\mathrm{d}x + \varepsilon^2\int_{\Omega_2^\varepsilon} B^\varepsilon\nabla\psi\nabla\varsigma\,\mathrm{d}x \\ &\quad + \varepsilon\int_{\Gamma^\varepsilon}\vartheta\Big(\frac{x}{\varepsilon}\Big)(\varphi-\psi)(\eta-\varsigma)\,\mathrm{d}s^\varepsilon, \\ L^\varepsilon((\varphi,\psi)) &= \int_{\Omega_1^\varepsilon} f_1\varphi\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f_2\psi\,\mathrm{d}x,\end{aligned}$$

where $(\varphi,\psi),(\eta,\varsigma) \in H^\varepsilon$. Therefore, the weak formulation (2.2) is equivalent to: find $(u^\varepsilon, v^\varepsilon) \in H^\varepsilon$ such that for all $(\varphi,\psi)\in H^\varepsilon$ we have

$$(2.4)\qquad a^\varepsilon((u^\varepsilon,v^\varepsilon),(\varphi,\psi)) = L^\varepsilon((\varphi,\psi)).$$

The coerciveness and the continuity of the form $a^\varepsilon(\cdot,\cdot)$ follow immediately from (H1) and (H3). It remains to prove the continuity of L^ε. First, from (H2), we easily see that for all $(\varphi,\psi)\in H^\varepsilon$,

$$(2.5)\qquad |L^\varepsilon((\varphi,\psi))| \leqslant M(f_1,f_2)\bigg(\Big(\int_{\Omega_1^\varepsilon}|\varphi|^2\,\mathrm{d}x\Big)^{1/2} + \Big(\int_{\Omega_2^\varepsilon}|\psi|^2\,\mathrm{d}x\Big)^{1/2}\bigg),$$

where

$$M(f_1,f_2)=\max\biggl(\Bigl(\int_\Omega |f_1|^2\,\mathrm{d}x\Bigr)^{1/2},\Bigl(\int_\Omega |f_2|^2\,\mathrm{d}x\Bigr)^{1/2}\biggr)$$

is a constant independent of ε. Next, following an idea of H. Ene and D. Polisevski [9], we know that there exists $C>0$ such that for all $\varphi=(\varphi,\psi)\in H^\varepsilon$

$$\int_{\Omega_1^\varepsilon}|\varphi|^2\,\mathrm{d}x\leqslant C\int_{\Omega_1^\varepsilon}|\nabla\varphi|^2\,\mathrm{d}x, \tag{2.6}$$

$$\int_{\Omega_2^\varepsilon}|\psi|^2\,\mathrm{d}x\leqslant C\Bigl(\varepsilon^2\int_{\Omega_2^\varepsilon}|\nabla\psi|^2\,\mathrm{d}x+\varepsilon\int_{\Gamma^\varepsilon}|\psi|^2\,\mathrm{d}s^\varepsilon\Bigr), \tag{2.7}$$

$$\varepsilon\int_{\Gamma^\varepsilon}|\varphi|^2\,\mathrm{d}s^\varepsilon\leqslant C\Bigl(\varepsilon^2\int_{\Omega_1^\varepsilon}|\nabla\varphi|^2\,\mathrm{d}x+\int_{\Omega_1^\varepsilon}|\varphi|^2\,\mathrm{d}x\Bigr). \tag{2.8}$$

The inequalities (2.6) and (2.7) are Poincaré's inequality and (2.8) is the trace inequality. These are obtained by the change of variable: $x=\varepsilon(k+y)$, $k\in\{k\in\mathbb{Z}^N\colon\,\varepsilon(k+y)\subset\Omega_i^\varepsilon\}$, $y\in Z_i$, $i=1,2$, and using Poincaré's inequality and the trace theorem on the reference cell Y_i. As ε is sufficiently small, say $\varepsilon<1$, we have from (2.8)

$$\varepsilon\int_{\Gamma^\varepsilon}|\varphi|^2\,\mathrm{d}s^\varepsilon\leqslant C\Bigl(\int_{\Omega_1^\varepsilon}|\nabla\varphi|^2\,\mathrm{d}x+\int_{\Omega_1^\varepsilon}|\varphi|^2\,\mathrm{d}x\Bigr). \tag{2.9}$$

Using (2.6) in (2.9), we get

$$\varepsilon\int_{\Gamma^\varepsilon}|\varphi|^2\,\mathrm{d}s^\varepsilon\leqslant C\Bigl(\int_{\Omega_1^\varepsilon}|\nabla\varphi|^2\,\mathrm{d}x\Bigr). \tag{2.10}$$

Next, from (2.7), we have

$$\int_{\Omega_2^\varepsilon}|\psi|^2\,\mathrm{d}x\leqslant C\Bigl(\varepsilon^2\int_{\Omega_2^\varepsilon}|\nabla\psi|^2\,\mathrm{d}x+\varepsilon\int_{\Gamma^\varepsilon}|\varphi-\psi|^2\,\mathrm{d}s^\varepsilon+\varepsilon\int_{\Gamma^\varepsilon}|\varphi|^2\,\mathrm{d}s^\varepsilon\Bigr). \tag{2.11}$$

Now, combining (2.10) and (2.11) gives

$$\int_{\Omega_2^\varepsilon}|\psi|^2\,\mathrm{d}x\leqslant C\Bigl(\int_{\Omega_1^\varepsilon}|\nabla\varphi|^2\,\mathrm{d}x+\varepsilon^2\int_{\Omega_2^\varepsilon}|\nabla\psi|^2\,\mathrm{d}x+\varepsilon\int_{\Gamma^\varepsilon}|\varphi-\psi|^2\,\mathrm{d}s^\varepsilon\Bigr),$$

which means that

$$\int_{\Omega_2^\varepsilon}|\psi|^2\,\mathrm{d}x\leqslant C\|(\varphi,\psi)\|^2_{H^\varepsilon}. \tag{2.12}$$

Observe that (2.6) yields

$$\int_{\Omega_1^\varepsilon} |\varphi|^2 \, \mathrm{d}x \leqslant C\|(\varphi,\psi)\|_{H^\varepsilon}^2. \tag{2.13}$$

Using (2.5), (2.12), and (2.13) we deduce that

$$|L^\varepsilon((\varphi,\psi))| \leqslant C\|(\varphi,\psi)\|_{H^\varepsilon}. \tag{2.14}$$

Thus, L^ε is continuous on H^ε. Note that the constant C appearing in (2.5) is independent of ε.

By Lax-Milgram's lemma, we conclude that there exists a unique solution $(u^\varepsilon, v^\varepsilon) \in H^\varepsilon$ to the weak formulation (2.4). Finally, putting $(\varphi,\psi) = (u^\varepsilon, v^\varepsilon)$ in (2.4), using the uniform coerciveness of $a^\varepsilon(\cdot,\cdot)$ and the continuity of L^ε yields the uniform estimate

$$\|(\varphi,\psi)\|_{H^\varepsilon} \leqslant C,$$

where again C is independent of ε. This concludes the proof of the theorem. □

Now, we are ready to state the main result of the paper:

Theorem 2.2. *Let $(u^\varepsilon, v^\varepsilon) \in H^\varepsilon$ be the solution of the weak system* (2.2)*. Assume that $f_2 \in H^1(\Omega)$. Let $U^\varepsilon = \chi_1(x/\varepsilon)u^\varepsilon + \chi_2(x/\varepsilon)v^\varepsilon$ denote the overall pressure. Then, up to a subsequence, there exists a unique $U \in H^1(\Omega)$, such that U^ε converges weakly in $H^1(\Omega)$ to U. Furthermore, U is the unique solution to the homogenized model:*

$$\begin{cases} -\operatorname{div}(\tilde{A}\nabla U) = F & \text{in } \Omega, \\ U = G & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where $\tilde{A}$, F and G are given in (3.17)–(3.18).

Remark 2.2. Observe that we need more regularity on f_2. Namely, we require that $f_2 \in H^1(\Omega)$ so that the function G defined by (3.18) is in $H^1(\Omega)$ and which gives $F \in H^{-1}(\Omega)$. See also Remark 3.1 below.

The remainder of this paper is devoted to the proof of this theorem. To this aim, we shall apply in the next section the two-scale convergence technique.

3. Proof of Theorem 2.2

In this section, we shall derive the homogenized system (2.15). To do so, we shall first begin with some notations. We define $\mathcal{C}_{\#}(Y)$ to be the space of all continuous functions on $\mathbb{R}^N$ which are Y-periodic. Let $\mathcal{C}^{\infty}_{\#}(Y) = \mathcal{C}^{\infty}(\mathbb{R}^N) \cap \mathcal{C}_{\#}(Y)$ and let $L^2_{\#}(Y)$ (resp. $L^2_{\#}(Y_i)$, $i = 1, 2$) to be the space of all functions belonging to $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ (resp. $L^2_{\mathrm{loc}}(Z_i)$) which are Y-periodic, and $H^1_{\#}(Y)$ (resp. $H^1_{\#}(Y_i)$) to be the space of those functions together with their derivatives belonging to $L^2_{\#}(Y)$ (resp. $L^2_{\#}(Z_i)$). Next, we recall the definition of the two-scale convergence [3].

Definition 3.1. A sequence (w^{ε}) in $L^2(\Omega)$ two-scale converges to $w \in L^2(\Omega \times Y)$ (we write $w^{\varepsilon} \overset{2-s}{\rightharpoonup} w$) if, for any admissible test function $\varphi \in L^2(\Omega; \mathcal{C}_{\#}(Y))$,

$$\lim_{\varepsilon \to 0} \int_{\Omega} w^{\varepsilon}(x)\varphi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_{\Omega \times Y} w(x, y)\varphi(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

The following result will be of use, see [3], [4].

Theorem 3.1.

(1) *Let (w^{ε}) be a uniformly bounded sequence in $H^1(\Omega)$ (resp. $H^1_0(\Omega)$). Then there exists $w \in H^1(\Omega)$ (resp. $H^1_0(\Omega)$) and $w_1 \in L^2(\Omega; H^1_{\#}(Y)/\mathbb{R})$ such that, up to a subsequence, $w^{\varepsilon} \overset{2-s}{\rightharpoonup} w$ and $\nabla w^{\varepsilon} \overset{2-s}{\rightharpoonup} \nabla w + \nabla_y w_1$.*

(2) *Let (w^{ε}) be a sequence of functions in $H^1(\Omega)$ such that*

$$\|w^{\varepsilon}\|_{L^2(\Omega)} + \varepsilon\|\nabla w^{\varepsilon}\|_{L^2(\Omega)^N} \leqslant C.$$

Then, there exist a subsequence of (w^{ε}), still denoted by (w^{ε}), and $w_0(x, y) \in L^2(\Omega; H^1_{\#}(Y))$ such that $w^{\varepsilon} \overset{2-s}{\rightharpoonup} w_0$ and $\varepsilon\nabla w^{\varepsilon} \overset{2-s}{\rightharpoonup} \nabla_y w_0$ and for every $\varphi \in \mathcal{D}(\Omega; \mathcal{C}^{\infty}_{\#}(Y))$ we have

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma^{\varepsilon}} w^{\varepsilon}\varphi^{\varepsilon}\,\mathrm{d}s^{\varepsilon} = \int_{\Omega \times \Gamma} w_0\varphi\,\mathrm{d}x\,\mathrm{d}s, \quad \varphi^{\varepsilon}(x) = \varphi\Big(x, \frac{x}{\varepsilon}\Big),$$

where $\mathrm{d}s$ is the Hausdorff measure on Γ.

Now, we turn our attention to determining the limiting problem (2.15). Thanks to the a priori estimates (2.3) and using Theorem 3.1, there exists a subsequence of $(u^{\varepsilon}, v^{\varepsilon})$, solution of (2.2), still denoted $(u^{\varepsilon}, v^{\varepsilon})$, and there exist

$$u \in H^1_0(\Omega),\ u_1 \in L^2(\Omega; H^1_{\#}(Y)/\mathbb{R}) \quad \text{and} \quad v_0 \in L^2(\Omega; H^1_{\#}(Y_2))$$

such that

$$\text{(3.1)} \qquad \chi_1^\varepsilon u^\varepsilon \overset{2-s}{\rightharpoonup} \chi_1 u, \quad \chi_2^\varepsilon v^\varepsilon \overset{2-s}{\rightharpoonup} \chi_2 v_0,$$
$$\chi_1^\varepsilon \nabla u^\varepsilon \overset{2-s}{\rightharpoonup} \chi_1(\nabla u + \nabla_y u_1), \quad \varepsilon\chi_2^\varepsilon \nabla v^\varepsilon \overset{2-s}{\rightharpoonup} \chi_2 \nabla_y v_0,$$

and for any $\psi \in \mathcal{D}(\Omega; \mathcal{C}_\#(Y))$

$$\text{(3.2)} \qquad \lim_{\varepsilon\to 0}\int_{\Gamma^\varepsilon} \varepsilon(u^\varepsilon - v^\varepsilon)\psi^\varepsilon \,\mathrm{d}s^\varepsilon = \int_{\Omega\times\Gamma}(u - v_0)\psi \,\mathrm{d}x\,\mathrm{d}s, \ \psi^\varepsilon(x) = \psi(x, x/\varepsilon).$$

For more details, we refer the reader to [3], Proposition 1.14 i) and ii) and [4], Proposition 2.6.

Now, let $\varphi \in \mathcal{D}(\Omega)$ and $\varphi_1,\ \psi \in \mathcal{D}(\Omega; \mathcal{C}_\#^\infty(Y))$. Set $\varphi^\varepsilon(x) = \varphi(x) + \varepsilon\varphi_1(x, x/\varepsilon)$ and $\psi^\varepsilon(x) = \psi(x, x/\varepsilon)$. Taking $\varphi = \varphi^\varepsilon$ and $\psi = \psi^\varepsilon$ in (2.2), we obtain

$$\text{(3.3)} \qquad \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u^\varepsilon \Big(\nabla\varphi + \nabla_y\varphi_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \varepsilon B^\varepsilon \nabla v^\varepsilon \nabla_y \psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x$$
$$+ \int_{\Gamma^\varepsilon} \vartheta^\varepsilon(u^\varepsilon - v^\varepsilon)(\varphi - \psi^\varepsilon)\,\mathrm{d}s^\varepsilon + \varepsilon R^\varepsilon = \int_{\Omega_1^\varepsilon} f_1\varphi\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f_2\psi\,\mathrm{d}x,$$

where

$$R^\varepsilon = \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla_x \varphi_1\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x + \varepsilon\int_{\Omega_2^\varepsilon} B^\varepsilon \nabla v^\varepsilon \nabla_x\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x$$
$$+ \int_{\Gamma^\varepsilon} \vartheta^\varepsilon(u^\varepsilon - v^\varepsilon)\varphi_1\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}s^\varepsilon.$$

According to the assumptions (H1)–(H3), ${}^tA\nabla\varphi$, ${}^tA\nabla_y\varphi_1$, ${}^tB\nabla_x\psi$, and ${}^tB\nabla_y\psi$ are admissible test functions. Therefore, in view of (3.1)–(3.2), there hold the following limits:

$$\text{(3.4)} \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla v^\varepsilon(\nabla\varphi + (\nabla_y\varphi_1)^\varepsilon)\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega\times Y_1} A(\nabla u + \nabla_y u_1)(\nabla\varphi + \nabla_y\varphi_1)\,\mathrm{d}x\,\mathrm{d}y,$$

$$\text{(3.5)} \qquad \int_{\Omega_2^\varepsilon} \varepsilon B^\varepsilon \nabla v^\varepsilon \nabla_y\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega\times Y_2} B\nabla v_0 \nabla_y\psi\,\mathrm{d}x\,\mathrm{d}y,$$

$$\text{(3.6)} \qquad \int_{\Gamma^\varepsilon} \vartheta^\varepsilon(u^\varepsilon - v^\varepsilon)(\varphi - \psi^\varepsilon)\,\mathrm{d}s^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega\times\Gamma} \vartheta(u - v_0)(\varphi - \psi)\,\mathrm{d}x\,\mathrm{d}s,$$

where we have denoted $(\nabla_y\varphi_1)^\varepsilon(x) = (\nabla_y\varphi_1)(x, x/\varepsilon)$. Moreover, using (2.3), it is easy to see that $R^\varepsilon = O(1)$. Thus, by (3.4)–(3.6) and passing to the limit in (2.2), we get the two-scale variational formulation:

$$\text{(3.7)} \qquad \int_{\Omega\times Y_1} A(\nabla u + \nabla_y u_1)(\nabla\varphi + \nabla_y\varphi_1)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y_2} B(y)\nabla_y v_0 \nabla_y\psi\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{\Omega\times\Gamma} \vartheta(y)(u - v_0)(\varphi - \psi)\,\mathrm{d}x\,\mathrm{d}s = \int_{\Omega\times Y_1} f_1\varphi\,\mathrm{d}x + \int_{\Omega\times Y_2} f_2\psi\,\mathrm{d}x.$$

By a density argument, the equation (3.7) still holds true for any $(\varphi, \varphi_1, \varphi_2) \in H_0^1(\Omega) \times L^2(\Omega; H^1_\#(Y_1)/\mathbb{R}) \times L^2(\Omega; H^1_\#(Y_2))$. Now, integrating by parts in (3.7) yields the following two-scale homogenized system:

$$-\mathrm{div}_y(A(\nabla u + \nabla_y u_1)) = 0 \quad \text{a.e. in } \Omega \times Y_1, \tag{3.8}$$

$$-\mathrm{div}_y(B\nabla_y v_0) = f_2 \quad \text{a.e. in } \Omega \times Y_2, \tag{3.9}$$

$$-\mathrm{div}\left(\int_{Y_1} A(\nabla u + \nabla_y u_1)\,\mathrm{d}y\right) + \int_\Gamma \vartheta(y)[u - v_0]\,\mathrm{d}s = f_1 \quad \text{a.e. in } \Omega, \tag{3.10}$$

$$(A(\nabla u + \nabla_y u_1)) \cdot \nu = 0 \quad \text{a.e. on } \Omega \times \Gamma, \tag{3.11}$$

$$B\nabla_y v_0 \cdot v = -\vartheta(u - v_0) \quad \text{a.e. on } \Omega \times \Gamma, \tag{3.12}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{3.13}$$

Let us first note that equations (3.8) and (3.11) lead to the following relation:

$$u_1(x,y) = \sum_{j=1}^N \frac{\partial u}{\partial x_j}(x)\omega_j(y) + u^*(x), \tag{3.14}$$

where, for $1 \leqslant j \leqslant N$, $\omega_j \in H^1_\#(Y_1)/\mathbb{R}$ is the unique solution to the following cell problem:

$$\begin{cases} -\mathrm{div}_y(A(\nabla_y \omega_j + e_j)) = 0 & \text{a.e. in } Y_1, \quad (e_j) \text{ is the canonical basis of } \mathbb{R}^N, \\ A(\nabla_y \omega_j + e_j) \cdot \nu = 0 & \text{a.e. on } \Gamma, \end{cases}$$

and $u^*(x)$ is any additive function independent of y. Similarly, from (3.9) and (3.12) we see that v_0 can be written as

$$v_0(x,y) - u(x) = \alpha(y) f_2(x), \ (x,y) \in \Omega \times Y_2, \tag{3.15}$$

where $\alpha \in H^1_\#(Y_2)$ is the unique solution of the following problem:

$$\begin{cases} -\mathrm{div}_y(B\nabla_y \alpha) = 1 & \text{in } Y_2, \\ B\nabla_y \alpha \cdot \nu + \vartheta\alpha = 0 & \text{on } \Gamma. \end{cases} \tag{3.16}$$

In the sequel, we shall denote for convenience

$$\tilde{A} = (\widetilde{a_{ij}})_{1 \leqslant i,j \leqslant N}, \quad \widetilde{a_{ij}} = \int_{Y_1} A(\nabla_y \omega_i + e_i) \cdot (\nabla_y \omega_j + e_j)\,\mathrm{d}y, \tag{3.17}$$

$$f^* = |Y_1| f_1 + |Y_2| f_2, \quad G = \left(\int_{Y_2} \alpha\right) f_2, \quad F = f^* + \mathrm{div}(\tilde{A}\nabla G). \tag{3.18}$$

Let us mention that, in view of (H1), $\tilde{A}$ is symmetric and positive definite, see [6]. Observe also that f^* lies in $L^2(\Omega)$ and since $f_2 \in H^1(\Omega)$, G is in $H^1(\Omega)$. Therefore, $F \in H^{-1}(\Omega)$. Inserting (3.14)–(3.15) into (3.10) yields the elliptic equation:

$$-\mathrm{div}(\tilde{A}\nabla u) = f^*. \tag{3.19}$$

Now, with (3.15) in mind, the overall pressure $U^\varepsilon = \chi_1(x/\varepsilon)u^\varepsilon + \chi_2(x/\varepsilon)v^\varepsilon$ two scale converges to $u + \chi_2\alpha f_2$. Consequently, U^ε converges weakly in $L^2(\Omega)$ to $U = u + G$ which is the unique solution of the homogenized model:

$$\begin{cases} -\mathrm{div}(\tilde{A}\nabla U) = F & \text{in } \Omega,\ U \in H^1(\Omega), \\ U = G & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

The proof of Theorem 2.2 is then achieved.

Remark 3.1. If $f_2 \in H^1(\Omega)$ is no longer satisfied, say f_2 is only in $L^2(\Omega)$, then as already mentioned by G. Allaire in [3], Remark 4.5, the solution U does not satisfy the required Dirichlet boundary condition. It is then preferable to write U as a sum of two terms: u and $\int_{Y_2} v_0 \,\mathrm{d}y$. Thus, the homogenized problem consists of two equations: (3.15), (3.19) with the homogeneous boundary condition $u = 0$ on $\partial\Omega$.

Remark 3.2. In view of (H2), we see that we simply choose the source densities f_1 and f_2 independent of ε and defined a.e. on the whole domain Ω whereas, throughout this paper, f_1 and f_2 are only used on the subregions Ω_1^ε and Ω_2^ε respectively. In fact, we can consider the case, where source densities are defined on their respective regions as well, without modifying substantially the homogenized model (3.20) except for the averaged source density f^* defined in (3.18). More precisely, if $f_i = f_i^\varepsilon$ a.e. in Ω_i^ε $(i = 1, 2)$, where $f_i^\varepsilon \in L^2(\Omega_i^\varepsilon)$ with $\|f_i^\varepsilon\|_{L^2(\Omega_i^\varepsilon)} \leqslant C$, then using the extension by zero to Ω of f_i^ε, we see that $\|\chi_i(x/\varepsilon)f_i^\varepsilon\|_{L^2(\Omega)} \leqslant C$. Denoting by f_i^0 the two scale limit of $\chi_i(x/\varepsilon)f_i^\varepsilon$, the weak limit of $\chi_i(x/\varepsilon)f_i^\varepsilon$ is then given by $F_i(x) = \int_{Y_i} f_i^0(x, y)\,\mathrm{d}y$ instead of $|Y_i|f_i(x)$ (see the r.h.s. of two-scale variational formulation (3.7)) and f^* should be given by $F_1 + F_2$ instead of $|Y_1|f_1 + |Y_2|f_2$.

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Author's address: *Abdelhamid Ainouz*, Laboratory AMNEDP, Faculty of Mathematics, University of Sciences and Technology Houari Boumedienne, POB 32 El Alia, Algiers, Algeria, e-mail: `aainouz@usthb.dz`.