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PORTFOLIO OPTIMIZATION FOR PENSION PLANS UNDER HYBRID STOCHASTIC AND LOCAL VOLATILITY

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Abstract. Based upon an observation that it is too restrictive to assume a definite correlation of the underlying asset price and its volatility, we use a hybrid model of the constant elasticity of variance and stochastic volatility to study a portfolio optimization problem for pension plans. By using asymptotic analysis, we derive a correction to the optimal strategy for the constant elasticity of variance model and subsequently the fine structure of the corrected optimal strategy is revealed. The result is a generalization of Merton's strategy in terms of the stochastic volatility and the elasticity of variance.

Keywords: pension plan; portfolio optimization; constant elasticity of variance; stochastic volatility; asymptotic analysis

MSC 2010: 90C39, 90C59, 90C90, 91G10

1. Introduction

As the prospects of elderly society become real in the world, there is a strong trend toward much interest in the retirement plan. Among many types of the retirement plan, we focus on the defined contribution (DC) pension plan in this paper. In the DC pension plan, benefits vary depending on the return from the investments, while the contribution is fixed based upon a percentage of the salary. So, whereas the payout of a defined benefit (DB) pension plan is fixed, a DC pension plan depends on portfolio performance. Therefore, it is important to choose the optimal strategy for DC pension plans. We consider a portfolio optimization problem for DC pension plans with a power utility function (CRRA), when the stock is assumed to follow a hybrid stochastic and local volatility model.

Our goal is to find the optimal strategy to maximize the expected value of a terminal utility function. Two important initial works on portfolio optimization are given by Merton [24] using the dynamic programming method, and by Cox and Huang [8] using the martingale method. The interest of this paper belongs to the first type. On the other hand, there is much literature about DC pension plans. See Boulier et al. [3], Haberman and Vigna [19], Devolder et al. [12] and Deelstra et al. [11] to name a few. But these studies have assumed to take the geometric Brownian motion with constant volatility (the Black-Scholes model) for an underlying asset price model. This is not an appropriate model in many practical senses. In particular, the constant volatility assumption has been proved to be too restrictive to reflect the real financial phenomena characterized by smile or skew of the implied volatilities of the risky asset prices as shown by, for example, Rubinstein [28] and Jackwerth and Rubinstein [21]. One alternative model in this point of view is a local volatility model, where volatility is a function of underlying asset price and time. Jump-diffusion and Levy models are other alternative ones. The most well-known local volatility model is the constant elasticity variance (in brief, CEV) model introduced by Cox and Ross [9], which captures the implied volatility skew to some extent. The CEV model has been successfully applied to many option pricing problems. See for example Beckers [2], Cox [7], Boyle and Tian [4], Yuen et al. [30], and Davydov and Linetsky [10]. For retirement plan problems, Xiao et al. [29] examined the optimal portfolio for DC pension plans under the CEV model using a Legendre transform and dual theory. Moreover, Gao [17] used a power transform and the variable change technique to find an explicit solution for the utility function. In the CEV model, however, volatility is perfectly correlated either positively or negatively with the underlying asset price. There is no clear evidence that there is a perfect definite correlation all the time. In fact, there are empirical studies showing the random nature of the correlation. See Harvey [20] and Ghysels et al. [18].

Based on the observation above, it is necessary to incorporate stochastic volatility driven by a hidden process into the CEV model. In this paper, we choose a model introduced by Choi et al. [6], in which the volatility of underlying risky asset is given by the product of a function of a stochastic process and a constant power of the underlying asset price. So, this model is thought of as a hybrid stochastic volatility and constant elasticity of variance model. We call it the SVCEV model. We assume that the hidden process driving the stochastic volatility is given by a fast mean-reverting Ornstein-Uhlenbeck (OU) process. As shown in Fouque et al. [14], the choice of the fast mean-revering OU process provides us with an analytic advantage since it is related to the averaging principle and ergodic theorem or, more directly, asymptotic diffusion limit theory of stochastic differential equations with a small parameter. This theory has been initiated by Khas'minskii [22] and developed by Papanicolaou et al. [27], Asch et al. [1], Kim [23] and Cerrai [5]. See, for example, Noh and Kim [25] for an application to optimization problem.

The results of overcoming the shortfalls of either local or stochastic volatility models may produce an issue of practical application. In practice, the hybrid volatility structure of the model makes calibration of its parameters more difficult. One possible solution for this type of problem would be to find a proxy by which the hybrid volatility could be estimated, as the stochastic volatility can be estimated by GARCH volatility in practice. Or, as chosen in this paper, the exact strategy can by approximated by a strategy with the unobservable stochastic volatility component averaged out.

The structure of this paper is as follows. In Section 2, a stochastic portfolio optimization problem is formulated based upon the SVCEV model. In Section 3, the Hamilton-Jacobi-Bellman equation for the optimization problem is derived by using the dynamic programming method and subsequently the value function is obtained in a split form of variables. Section 4 achieves an asymptotic result on the optimal strategy and compares it with the ones corresponding to the Black-Scholes (Merton's strategy) and CEV models. In Section 5, a practical solution is obtained in the sense that the strategy does not depend upon the unobservable (stochastic volatility) variable. Section 6 presents numerical results to illustrate the correction effect produced by the hybrid model. Section 7 provides concluding remarks.

2. Model formulation

In this section, we present a financial market setup and formulate a stochastic optimization problem. We consider a market structure that consists of a risk-free asset (treasury bond or bank account) whose price dynamics is given by the ordinary differential equation (ODE)

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t, \tag{2.1}$$

and a risky asset (stock) whose price evolves according to the stochastic differential equations (SDEs)

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma(Y_t)S_t^{\theta}\,\mathrm{d}W_t^{s}, \tag{2.2}$$

$$\mathrm{d}Y_t = \alpha(m - Y_t)\,\mathrm{d}t + \beta\,\mathrm{d}W_t^{y}, \tag{2.3}$$

where W^s and W^y are one-dimensional Brownian motions whose correlation structure is given by $\mathrm{d}\langle W^s, W^y\rangle_t = \varrho\,\mathrm{d}t$. Here, r is a constant interest rate, μ is the instantaneous mean return rate of the risky asset, and θ is the elasticity parameter. The SDE (2.3) is a mean-reverting Ornstein-Uhlenbeck equation, where α denotes the rate of the mean-reversion and β is the volatility of the Ornstein-Uhlenbeck process Y_t. The volatility $\sigma(y)$ is assumed to be a smooth, bounded and positive function. Let $\mathcal{F}_t$ denote the filtration generated by W^s and W^y.

We consider a DC pension plan with the assumption of the constant contribution rate and the unit wage. Let X_t be the pension wealth function, and π_t and $1-\pi_t$ be the proportion of the pension wealth invested in the risky asset and risk-free asset, respectively, at time $t \in [0,T]$. Assume that π_t is $\mathcal{F}_t$-measurable and adapted such that $\int_0^T \pi_t^2 \,\mathrm{d}t < \infty$ almost surely for all $T > 0$. Then the dynamics of X_t is given by the SDE

$$\mathrm{d}X_t = \pi_t X_t \frac{\mathrm{d}S_t}{S_t} + (1-\pi_t) X_t \frac{\mathrm{d}B_t}{B_t} + c\,\mathrm{d}t, \quad 0 < t < T,$$

with contribution rate c, which, from (2.1)–(2.3), leads to the SDE

$$\mathrm{d}X_t = [(r + (\mu - r)\pi_t)X_t + c]\,\mathrm{d}t + \pi_t \sigma(Y_t) S_t^\theta X_t \,\mathrm{d}W_t^s, \quad 0 < t < T.$$

The interest of investors is to find a strategy π_t^* which maximizes the expectation of the utility function given by $E[U(X_T)|S_t = s, X_t = x, Y_t = y]$. Here, the joint process (S_t, X_t, Y_t) is a Markov process and $U(\cdot)$ is required to be strictly concave. In this paper, we consider the well-known utility function, namely, the power utility function (CRRA) given by

$$U(x) = \frac{x^p}{p}, \quad p \neq 0,\ p < 1. \tag{2.4}$$

Here, p is called the CRRA coefficient.

3. Hamilton-Jacobi-Bellman equation

In this section, we derive the Hamilton-Jacobi-Bellman (HJB) equation for the optimization problem for $E[U(X_T)|S_t = s, X_t = x, Y_t = y]$ in terms of the utility function given by (2.4).

We define the value function by

$$V(t,s,x,y) := \sup_\pi E[U(X_T)|S_t = s, X_t = x, Y_t = y], \quad 0 < t < T. \tag{3.1}$$

Then the Hamilton-Jacobi-Bellman equation associated with the optimization problem is given by the partial differential equation (PDE)

$$\begin{aligned} V_t + (rx + c)V_x + \alpha(m - y)V_y + \frac{1}{2}\beta^2 V_{yy} + \mu s V_s \\ + \frac{1}{2}\sigma^2(y) s^{2(\theta+1)} V_{ss} + \varrho\sigma(y)\beta s^{\theta+1} V_{sy} \\ + \sup_\pi \Big[\frac{1}{2}\pi^2\sigma^2(y) s^{2\theta} x^2 V_{xx} + \pi((\mu - r)xV_x \\ + \sigma^2(y) s^{2\theta+1} x V_{sx} + \varrho\beta\sigma(y) s^\theta x V_{xy})\Big] = 0 \end{aligned} \tag{3.2}$$

with the final condition $V(T,s,x,y) = U(x)$. Here, each V with a subscript denotes the partial derivative with respect to the corresponding variable. See Øksendal [26] or Fleming and Soner [13] for general reference on the derivation of the HJB equation.

From (3.2) it is immediately observed that the first order maximizing condition for the optimal strategy π^* is given by

$$\pi^* = -\frac{(\mu - r)V_x + \sigma^2(y)s^{2\theta+1}V_{sx} + \varrho\beta\sigma(y)s^\theta V_{xy}}{x\sigma^2(y)s^{2\theta}V_{xx}}. \tag{3.3}$$

Putting (3.3) into (3.2), we obtain the PDE

$$\begin{aligned} V_t + (rx + c)V_x + \alpha(m - y)V_y + \frac{1}{2}\beta^2 V_{yy} + \mu s V_s \\ + \frac{1}{2}\sigma^2(y)s^{2(\theta+1)}V_{ss} + \varrho\sigma(y)\beta s^{\theta+1}V_{sy} \\ - \frac{[(\mu - r)V_x + \sigma^2(y)s^{2\theta+1}V_{sx} + \varrho\beta\sigma(y)s^\theta V_{xy}]^2}{2\sigma^2(y)s^{2\theta}V_{xx}} = 0. \end{aligned} \tag{3.4}$$

Now, we need to solve the nonlinear PDE (3.4) for the value function V and plug the result into (3.3) to obtain the optimal strategy.

For the power utility function (2.4) we conjecture a solution to the PDE (3.4) with the following form:

$$V(t,s,x,y) = g(t,s,y)\frac{(x - a(t))^p}{p} \tag{3.5}$$

for some $a(t)$, which will be chosen later, and $g(t,s,y)$ which satisfy the final conditions $a(T) = 0$ and $g(T,s,y) = 1$. Then the PDE (3.4) becomes a PDE for g in the form

$$\begin{aligned} &\Big[g_t + \mu s g_s + \alpha(m - y)g_y + \frac{1}{2}\sigma^2(y)s^{2(\theta+1)}g_{ss} + \frac{1}{2}\beta^2 g_{yy} + rpg \\ &+ \varrho\sigma(y)\beta s^{\theta+1}g_{sy} + \frac{p(\mu - r)^2 g}{2\sigma^2(y)s^{2\theta}(1 - p)} + \frac{p\sigma^2(y)s^{2(\theta+1)}g_s^2}{2g(1 - p)} + \frac{p\varrho^2\beta^2 g_y^2}{2g(1 - p)} \\ &+ \frac{p(\mu - r)sg_s}{1 - p} + \frac{p\sigma(y)\varrho\beta s^{\theta+1}g_s g_y}{g(1 - p)} + \frac{p(\mu - r)\varrho\beta g_y}{\sigma(y)s^\theta(1 - p)}\Big](x - a)^p \\ &+ pg[c + ra - a_t](x - a)^{p-1} = 0. \end{aligned} \tag{3.6}$$

Now, we choose the function $a(t)$ in (3.5) such that

$$c + ra - a_t = 0 \tag{3.7}$$

is satisfied. Then from the PDE (3.6) we observe that the PDE for the function g is separated to take the form

$$
\begin{aligned}
(3.8)\quad & g_t + \mu s g_s + \alpha(m-y)g_y + \frac{1}{2}\sigma^2(y)s^{2(\theta+1)}g_{ss} + \frac{1}{2}\beta^2 g_{yy} + rpg \\
&+ \varrho\sigma(y)\beta s^{\theta+1}g_{sy} + \frac{p(\mu-r)^2}{2\sigma^2(y)s^{2\theta}(1-p)}g + \frac{p\sigma^2(y)s^{2(\theta+1)}}{2g(1-p)}g_s^2 + \frac{p\varrho^2\beta^2}{2g(1-p)}g_y^2 \\
&+ \frac{p(\mu-r)s}{1-p}g_s + \frac{p\sigma(y)\varrho\beta s^{\theta+1}}{g(1-p)}g_s g_y + \frac{p(\mu-r)\varrho\beta}{\sigma(y)s^{\theta}(1-p)}g_y = 0.
\end{aligned}
$$

Therefore, we have split the PDE for the value function V into two equations for $a(t)$ and $g(t,s,y)$ given by (3.7) and (3.8), respectively. Once these equations with the final conditions $a(T) = 0$ and $g(T,s,y) = 1$ are solved, the value function V can be determined by (3.5). Note that $a(t)$ solving (3.7) with $a(T) = 0$ is given by $a(t) = -c(1-\mathrm{e}^{-r(T-t)})/r$.

4. Corrected Merton strategies

Equation (3.8) is a nonlinear PDE for which it is hard to obtain the full exact solution. So, we approximate the solution as an expansion starting from the classical Merton solution based on the asymptotic analysis developed by Fouque et al. [14], [15]. Basically, the solution of the nonlinear PDE is approximated by the Merton value function plus some correction terms which are given by solutions of an ODE or a linear PDE. Of course, the ODE and linear PDE are much better to deal with than the nonlinear PDE in terms of computing.

The pivotal role in this approach is played by the ergodic Markov property of Y_t as the solution of the Ornstein-Uhlenbeck SDE (2.3). It is characterized by the infinitesimal generator $\mathcal{A}$ given by

$$
\mathcal{A} = \alpha\mathcal{L}_Y,
$$
$$
\mathcal{L}_Y := \nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \quad \nu := \frac{\beta}{\sqrt{2\alpha}}.
$$

Then the probability density function Φ of the long-run (invariant) distribution of Y_t is given by $\Phi(y) = (2\pi\nu^2)^{-1/2}\mathrm{e}^{-(y-m)^2/(2\nu^2)}$. The function Φ is also a solution of the integral equation $\int_{\mathbb{R}}\Phi(y)\mathcal{A}g(y)\,\mathrm{d}y = 0$ for all g in $\mathbb{C}^2(\mathbb{R})$ (the space of twice continuously differentiable functions on $\mathbb{R}$).

From now on, we assume that the parameter α (mean reversion rate) is large while ν is fixed as a moderate constant in the sense that, in terms of a small parameter ε, we have $\alpha = 1/\varepsilon$ and $\beta = \nu\sqrt{2/\varepsilon}$ $(\nu \sim O(1))$. Here, $\alpha = 1/\varepsilon$ means fast mean reversion or small correlation time of the process Y_t while $\beta = \nu\sqrt{2/\varepsilon}$ ($\nu \sim O(1)$) implies a fixed magnitude of variance of Y_t. By choosing α as a large parameter, Y_t becomes a fast mean-reverting Ornstein-Uhlenbeck process. If α goes to infinity, the process Y_t becomes effectively a constant m, which leads our model to an effective CEV model in the long run, so that one can take a perturbation approach of Fouque et al. [15] to the existing CEV model in terms of the scale parameter ε.

In terms of the small parameter ε, the PDE (3.8) now becomes

$$\begin{aligned} g_t &+ \Big(\mu s + \frac{p(\mu - r)s}{1-p}\Big) g_s + \frac{1}{2}\sigma^2(y) s^{2(\theta+1)} g_{ss} + \frac{1}{\varepsilon}\mathcal{L}_Y g \\ &+ \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}} \varrho\sigma s^{\theta+1} g_{sy} + \frac{p\sigma^2(y)s^{2(\theta+1)}}{2g(1-p)} g_s^2 + p\Big(r + \frac{(\mu-r)^2}{2\sigma^2(y)s^{2\theta}(1-p)}\Big) g \\ &+ \frac{1}{\varepsilon}\frac{p\varrho^2\nu^2}{g(1-p)} g_y^2 + \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}} \frac{p\sigma(y)\varrho s^{\theta+1}}{g(1-p)} g_s g_y + \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}} \frac{p(\mu-r)\varrho}{\sigma(y)s^\theta(1-p)} g_y = 0. \end{aligned} \tag{4.1}$$

Let us expand g as $g = g_0 + \sqrt{\varepsilon} g_1 + \varepsilon g_2 + \varepsilon\sqrt{\varepsilon} g_3 + \ldots$ and plug it into (4.1) and compare powers of ε. First, the term in $1/\varepsilon$ gives

$$\mathcal{L}_Y g_0 + \frac{p\nu^2\varrho^2}{(1-p)g_0} g_{0,y}^2 = 0,$$

which is a nonlinear ODE in the y variable but it is equivalent to

$$\nu^2 \frac{g_{0,yy}}{g_{0,y}} + (m - y) + \frac{p\nu^2\varrho^2}{1-p}\frac{g_{0,y}}{g_0} = 0. \tag{4.2}$$

Integrating the expression (4.2) yields

$$g_0^{1+\tilde{c}} = c_1(t,s)\int_0^y \mathrm{e}^{(m-z)^2/(2\nu^2)}\,\mathrm{d}z + c_2(t,s), \tag{4.3}$$

$$\tilde{c} := \frac{p}{1-p}\varrho^2 \tag{4.4}$$

for some functions c_1 and c_2 of time t and risky asset price s. Since (4.3) says that g_0 grows unreasonably fast with respect to y unless $c_1 = 0$, we assume that $c_1 = 0$. Consequently, g_0 is independent of y and thus it has the form $g_0 = g_0(t,s)$.

Next, comparing the $1/\sqrt{\varepsilon}$ terms in the PDE (4.1) gives $\mathcal{L}_Y g_1 = 0$ due to the fact that g_0 does not depend on y. Again, this implies that g_1 is also independent of y, i.e., $g_1 = g_1(t,s)$.

The y-independence of g_0 and g_1 obtained above is a very useful result to obtain the explicit form of g_0, g_1, etc. in the following theorems.

Theorem 1. *If g_0 takes the form $g_0(t,s) = A^{1-p}\mathrm{e}^{(1-p)Bz}$ with $z = s^{-2\theta}$ for some functions $A(t)$ and $B(t)$ satisfying the boundary conditions $A(T) = 1$ and $B(T) = 0$, then $A(t)$ and $B(t)$ are given by*

$$A(t) = \mathrm{e}^{(\lambda_1\theta(2\theta+1)+rp/(1-p))(T-t)}\Big(\frac{\lambda_2-\lambda_1}{\lambda_2-\lambda_1\mathrm{e}^{2\theta^2(\lambda_1-\lambda_2)(T-t)}}\Big)^{2\theta+1/(2\theta)}, \tag{4.5}$$

$$B(t) = \overline{\sigma}^{-2}\frac{1-\mathrm{e}^{2\theta^2(\lambda_1-\lambda_2)(T-t)}}{1/\lambda_1-(1/\lambda_2)\mathrm{e}^{2\theta^2(\lambda_1-\lambda_2)(T-t)}}, \tag{4.6}$$

$$\lambda_{1,2} := \frac{(\mu-rp)\pm\sqrt{(\mu-rp)^2-(\overline{\sigma}/\breve{\sigma})^2p(\mu-r)^2}}{2\theta(1-p)}, \tag{4.7}$$

where $\overline{\sigma}$ and $\breve{\sigma}$ are defined by

$$\overline{\sigma} := \langle\sigma^2\rangle^{1/2}, \tag{4.8}$$

$$\breve{\sigma} := \Big\langle\frac{1}{\sigma^2}\Big\rangle^{-1/2}. \tag{4.9}$$

Proof. From the $O(1)$ terms of (4.1), the y-independence of g_0 and g_1 yields

$$\mathcal{L}_Y g_2 + g_{0,t} + \Big(\mu s + \frac{p(\mu-r)s}{1-p}\Big)g_{0,s} + \frac{1}{2}\sigma^2(y)s^{2(\theta+1)}g_{0,ss} + \frac{1}{2}\sigma^2(y)\frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2 + p\Big(r + \frac{(\mu-r)^2}{2\sigma^2(y)s^{2\theta}(1-p)}\Big)g_0 = 0. \tag{4.10}$$

We observe that if the two variables t and s are fixed, (4.10) is a Poisson equation of the type $\mathcal{L}_Y g_2(t,s,y) + f(t,s,y) = 0$ for some function f. From the Fredholm alternative theorem (cf. Fredholm [16]), the necessary condition for the Poisson equation to have a solution is the solvability (centering) condition $\langle f\rangle = \int_{\mathbb{R}} f(t,s,y)\Phi(y)\,\mathrm{d}y = 0$. Thus we obtain

$$g_{0,t} + \Big(\mu s + \frac{p(\mu-r)s}{1-p}\Big)g_{0,s} + \frac{1}{2}\overline{\sigma}^2 s^{2(\theta+1)}g_{0,ss} + \frac{1}{2}\overline{\sigma}^2\frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2 + p\Big(r + \frac{1}{2\breve{\sigma}^2}\frac{(\mu-r)^2}{s^{2\theta}(1-p)}\Big)g_0 = 0, \tag{4.11}$$

where $\overline{\sigma}$ and $\breve{\sigma}$ are defined by (4.8) and (4.9), respectively. Finally, by applying the variable change and power transformation $g_0(t,s) = f(t,z)^{1-p}$ and $z = s^{-2\theta}$ to the PDE (4.11), we obtain the result of Theorem 4.1. □

Next, we obtain a linear PDE for the first correction term g_1. The y-independence of g_1 and the centering condition for a Poisson equation are the main tools to derive the equation.

Theorem 2. *Given the explicit solution g_0 from Theorem 1, g_1 is given by the solution of the linear PDE*

$$
\begin{aligned}
(4.12)\qquad & g_{1,t} + \Big(\mu s + \frac{p(\mu-r)s}{1-p} + \mathcal{S}(t,s)\Big) g_{1,s} + \frac{1}{2}\overline{\sigma}^2 s^{2(\theta+1)} g_{1,ss} \\
& + p\Big(r + \frac{1}{2\breve{\sigma}^2}\frac{(\mu-r)^2}{(1-p)s^{2\theta}} + \mathcal{J}(t,s)\Big) g_1 = \mathcal{H}(t,s),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{S}(t,s) &= \frac{p\overline{\sigma}^2 s^{2(\theta+1)}}{1-p}\frac{g_{0,s}}{g_0}, \\
\mathcal{J}(t,s) &= -\frac{\overline{\sigma}^2 s^{2(\theta+1)}}{2(1-p)}\Big(\frac{g_{0,s}}{g_0}\Big)^2, \\
\mathcal{H}(t,s) &= \frac{\sqrt{2}\nu\varrho s^{\theta+1}p}{(1-p)g_0} g_{0,s}\mathcal{H}_1(t,s) + \frac{\sqrt{2}\nu p(\mu-r)\varrho}{s^{\theta}(1-p)}\mathcal{H}_2(t,s) - \sqrt{2}\nu\varrho s^{\theta+1}\mathcal{H}_3(t,s), \\
\mathcal{H}_1(t,s) &:= \frac{p(\mu-r)^2}{2(1-p)s^{2\theta}} g_0\langle\sigma\psi'\rangle + \frac{1}{2}\Big(s^{2(\theta+1)}g_{0,ss} + \frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2\Big)\langle\sigma\varphi'\rangle, \\
\mathcal{H}_2(t,s) &:= \frac{p(\mu-r)^2}{2(1-p)s^{2\theta}} g_0\Big\langle\frac{\psi'}{\sigma}\Big\rangle + \frac{1}{2}\Big(s^{2(\theta+1)}g_{0,ss} + \frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2\Big)\Big\langle\frac{\varphi'}{\sigma}\Big\rangle, \\
\mathcal{H}_3(t,s) &:= -\frac{p(\mu-r)^2\langle\sigma\psi'\rangle}{2(1-p)}\Big(\frac{g_{0,s}}{s^{2\theta}} - \frac{2\theta g_0}{s^{2\theta+1}}\Big) \\
&\quad - \frac{1}{2}\langle\sigma\varphi'(y)\rangle\Big[2(\theta+1)s^{2\theta+1}g_{0,ss} + s^{2(\theta+1)}g_{0,sss} \\
&\quad + \frac{p}{1-p}\Big(\frac{2(\theta+1)s^{2\theta+1}}{g_0}g_{0,s}^2 - \frac{s^{2(\theta+1)}}{g_0^2}g_{0,s}^3 + \frac{2s^{2(\theta+1)}}{g_0}g_{0,s}g_{0,ss}\Big)\Big].
\end{aligned}
$$

Here, $\varphi(y)$ and $\psi(y)$ are defined by the solutions of

$$
\begin{aligned}
(4.13)\qquad & \mathcal{L}_Y\varphi = \sigma^2(y) - \overline{\sigma}^2, \\
(4.14)\qquad & \mathcal{L}_Y\psi = \frac{1}{\sigma^2(y)} - \frac{1}{\breve{\sigma}^2},
\end{aligned}
$$

respectively.

Proof. Subtracting (4.11) from (4.10) leads to g_2 which is given by the solution of the ODE

$$
\begin{aligned}
(4.15)\qquad \mathcal{L}_Y g_2 = & -\frac{1}{2}(\sigma^2(y) - \overline{\sigma}^2)s^{2(\theta+1)}g_{0,ss} \\
& - \frac{1}{2}(\sigma(y)^2 - \overline{\sigma}^2)\frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2 - \frac{1}{2}\Big(\frac{1}{\sigma(y)^2} - \frac{1}{\breve{\sigma}^2}\Big)\frac{p(\mu-r)^2}{(1-p)s^{2\theta}}g_0.
\end{aligned}
$$

Then in terms of φ and ψ defined by the solutions of (4.13) and (4.14), respectively, the solution g_2 of the PDE (4.15) can be expressed as

$$g_2 = -\frac{p(\mu-r)^2}{2(1-p)s^{2\theta}} g_0\psi(y) - \frac{1}{2}\Big(s^{2(\theta+1)}g_{0,ss} + \frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2\Big)\varphi(y) + k(s,t) \tag{4.16}$$

for some constant $k(s,t)$ independent of y.

On the other hand, applying the y-independence of g_1 to the order $\sqrt{\varepsilon}$ terms of (4.1), one can obtain

$$\begin{aligned}
&\mathcal{L}_Y g_3 + g_{1,t} + \Big(\mu s + \frac{p(\mu-r)s}{1-p}\Big)g_{1,s} + \frac{1}{2}\sigma^2(y)s^{2(\theta+1)}g_{1,ss} \\
&\quad + \frac{p\sigma^2(y)s^{2(\theta+1)}}{2(1-p)g_0}\Big(2g_{0,s}g_{1,s} - \frac{{g_{0,s}}^2}{g_0}g_1\Big) + p\Big(r + \frac{(\mu-r)^2}{2\sigma^2(y)s^{2\theta}(1-p)}\Big)g_1 \\
&\quad - \Big(\frac{\sqrt{2}\nu p\sigma(y)\varrho s^{\theta+1}}{(1-p)g_0}g_{0,s} + \frac{\sqrt{2}\nu p(\mu-r)\varrho}{\sigma(y)s^\theta(1-p)}\Big) \\
&\quad \times \Big(\frac{p(\mu-r)^2}{2(1-p)s^{2\theta}}g_0\psi'(y) + \frac{1}{2}\varphi'(y)\Big(s^{2(\theta+1)}g_{0,ss} + \frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2\Big)\Big) \\
&\quad - \sqrt{2}\nu\varrho\sigma(y)s^{\theta+1}\Big\{\frac{p(\mu-r)^2\psi'(y)}{2(1-p)}\Big(\frac{g_{0,s}}{s^{2\theta}} - \frac{2\theta g_0}{s^{2\theta+1}}\Big) \\
&\quad + \frac{1}{2}\varphi'(y)\Big[2(\theta+1)s^{2\theta+1}g_{0,ss} + s^{2(\theta+1)}g_{0,sss} \\
&\quad + \frac{p}{1-p}\Big(\frac{2(\theta+1)s^{2\theta+1}}{g_0}g_{0,s}^2 - \frac{s^{2(\theta+1)}}{g_0^2}g_{0,s}^3 + \frac{2s^{2(\theta+1)}}{g_0}g_{0,s}g_{0,ss}\Big)\Big]\Big\} = 0.
\end{aligned} \tag{4.17}$$

Then the centering condition for the Poisson equation (4.17) yields the PDE

$$\begin{aligned}
&g_{1,t} + \Big(\mu s + \frac{p(\mu-r)s}{1-p}\Big)g_{1,s} + \frac{1}{2}\overline{\sigma}^2 s^{2(\theta+1)}g_{1,ss} \\
&+ \frac{p\overline{\sigma}^2 s^{2\theta+2}}{2(1-p)g_0}\Big(2g_{0,s}g_{1,s} - \frac{g_{0,s}^2}{g_0}g_1\Big) + p\Big(r + \frac{1}{2\breve{\sigma}^2}\frac{(\mu-r)^2}{(1-p)s^{2\theta}}\Big)g_1 \\
&- \frac{\sqrt{2}\nu p\varrho s^{\theta+1}}{(1-p)g_0}g_{0,s}\Big(\frac{p(\mu-r)^2}{2(1-p)s^{2\theta}}g_0\langle\sigma\psi'\rangle + \frac{1}{2}\langle\sigma\varphi'\rangle\Big(s^{2(\theta+1)}g_{0,ss} + \frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2\Big)\Big) \\
&+ \frac{\sqrt{2}\nu p(\mu-r)\varrho}{s^\theta(1-p)}\Big(\frac{p(\mu-r)^2}{2(1-p)s^{2\theta}}g_0\Big\langle\frac{1}{\sigma}\psi'\Big\rangle + \frac{1}{2}\Big\langle\frac{1}{\sigma}\varphi'\Big\rangle\Big(s^{2(\theta+1)}g_{0,ss} + \frac{ps^{2(\theta+1)}}{(1-p)g_0}g_{0,s}^2\Big)\Big) \\
&+ \sqrt{2}\nu\varrho s^{\theta+1}\Big\{-\frac{p(\mu-r)^2\langle\sigma\psi'\rangle}{2(1-p)}\Big(\frac{g_{0,s}}{s^{2\theta}} - \frac{2\theta g_0}{s^{2\theta+1}}\Big) - \frac{1}{2}\langle\sigma\varphi'\rangle\Big[2(\theta+1)s^{2\theta+1}g_{0,ss} \\
&+ s^{2(\theta+1)}g_{0,sss} + \frac{p}{1-p}\Big(\frac{2(\theta+1)s^{2\theta+1}}{g_0}g_{0,s}^2 - \frac{s^{2(\theta+1)}}{g_0^2}g_{0,s}^3 + \frac{2s^{2(\theta+1)}}{g_0}g_{0,s}g_{0,ss}\Big)\Big]\Big\} = 0.
\end{aligned} \tag{4.18}$$

By substituting (4.16) into (4.18), we obtain the result (4.12). □

Next, we consider the optimal strategy π^* invested in the stock. By plugging (3.5) into (3.3), we have

(4.19)

$$\begin{aligned}\pi^* &= -\frac{(\mu-r)V_x+\sigma^2(y)s^{2\theta+1}V_{xs}+\varrho\beta\sigma(y)s^{\theta}V_{xy}}{x\sigma^2(y)s^{2\theta}V_{xx}} \\ &= \frac{(\mu-r)g(x-a(t))^{p-1}+\sigma^2(y)s^{2\theta+1}g_s(x-a(t))^{p-1}+\varrho\beta\sigma(y)s^{\theta}g_y(x-a(t))^{p-1}}{x\sigma^2(y)s^{2\theta}g(1-p)(x-a(t))^{p-2}} \\ &= \Big(1+\frac{c\hat{a}(t)}{x}\Big)\frac{\mu-r}{(1-p)\tilde{\sigma}_\theta^2}\Big(1+\frac{\tilde{\sigma}_\theta^2 s g_s+\varrho\beta\tilde{\sigma}_\theta g_y}{(\mu-r)g}\Big),\end{aligned}$$

where

$$\hat{a}(t) := -\frac{a(t)}{c} = \frac{1}{r}(1-\mathrm{e}^{-r(T-t)}), \tag{4.20}$$

$$\tilde{\sigma}_\theta(s,y) := \sigma(y)s^{\theta}. \tag{4.21}$$

In terms of notation

$$\begin{aligned}M_{\tilde{\sigma}_\theta} &:= \frac{\mu-r}{(1-p)\tilde{\sigma}_\theta^2(s,y)}, \\ C_{\tilde{\sigma}_\theta} &:= \frac{\tilde{\sigma}_\theta^2(s,y)sg_s+\varrho\beta\tilde{\sigma}_\theta(s,y)g_y}{(\mu-r)g},\end{aligned}$$

the optimal strategy π^* given by (4.19) has the simple expression

$$\pi^* = \Big(1+\frac{c\hat{a}(t)}{x}\Big)M_{\tilde{\sigma}_\theta}(1+C_{\tilde{\sigma}_\theta}). \tag{4.22}$$

Each of the above $M_{\tilde{\sigma}_\theta}$ (Merton strategy) and $C_{\tilde{\sigma}_\theta}$ (correction) is an extension of the one corresponding to the CEV model, in which the optimal strategy π^* is given by (cf. Gao [17])

$$\begin{aligned}\pi^* &= \Big(1+\frac{c\hat{a}(t)}{x}\Big)M_{\sigma_\theta}(1+C_{\sigma_\theta}), \\ M_{\sigma_\theta} &:= \frac{\mu-r}{(1-p)\sigma_\theta^2(s)}, \\ C_{\sigma_\theta} &:= -\frac{2\theta(1-p)}{\mu-r}I(t),\end{aligned} \tag{4.23}$$

where

$$\sigma_\theta(s) := \sigma_0 s^{\theta}, \tag{4.24}$$

$$I(t) := \frac{1-\mathrm{e}^{2\theta^2(\lambda_1-\lambda_2)(T-t)}}{1/\lambda_1-(1/\lambda_2)\mathrm{e}^{2\theta^2(\lambda_1-\lambda_2)(T-t)}} \tag{4.25}$$

for some constant σ_0. Here, we note that if the elasticity parameter θ is zero, then σ_θ becomes the constant σ_0 and (4.23) reduces to the optimal strategy corresponding to the Black-Scholes model, where risky asset price evolves with a geometric Brownian motion. Indeed, this is given by the classical (original) Merton coefficient (see Devolder et al. [12]) as follows:

$$\pi^* = \Big(1 + \frac{c\hat{a}(t)}{x}\Big) M_{\sigma_0}(1 + C_{\sigma_0}),$$
$$M_{\sigma_0} := \frac{\mu - r}{(1-p){\sigma_0}^2},$$
$$C_{\sigma_0} := 0.$$

At the moment it is not clear, due to the g-dependence of $C_{\tilde{\sigma}_\theta}$, whether (4.22) reduces to (4.23) if $\sigma(y)$ becomes the constant σ_0. After Theorem 3 is obtained below, however, one can notice that this is the case. Table 1 summarizes the comparison of the optimal strategy among the Black-Scholes, CEV and SVCEV models.

	Black-Scholes	CEV	SVCEV
Merton strategy	$\frac{\mu - r}{(1-p)\sigma_0^2}$	$\frac{\mu - r}{(1-p)\sigma_\theta^2(s)}$	$\frac{\mu - r}{(1-p)\tilde{\sigma}_\theta^2(s,y)}$
correction	0	$-\frac{2\theta(1-p)I(t)}{\mu - r}$	$\frac{\tilde{\sigma}_\theta^2(s,y) s g_s + \varrho\beta\tilde{\sigma}_\theta(s,y) g_y}{(\mu - r)g}$

Table 1: Comparison among the three models.

In the next theorem we obtain an asymptotic form of the optimal strategy π^* based upon the results of Theorem 1 and 2.

Theorem 3. *If π^* is expanded as $\pi^* = \pi_0^* + \sqrt{\varepsilon}\pi_1^* + \varepsilon\pi_2^* + \ldots$, then the leading order term π_0^* and the first correction term π_1^* are given by*

$$\pi_0^* = \Big(1 + \frac{c\hat{a}(t)}{x}\Big)\frac{\mu - r}{(1-p)\tilde{\sigma}_\theta^2}\Big(1 - \Big(\frac{\sigma}{\overline{\sigma}}\Big)^2 \frac{2\theta(1-p)}{\mu - r} I(t)\Big), \tag{4.26}$$

$$\pi_1^* = \Big(1 + \frac{c\hat{a}(t)}{x}\Big)\frac{\mu - r}{(1-p)\tilde{\sigma}_\theta^2}\Big[1 + \frac{\tilde{\sigma}_\theta^2 s}{\mu - r}\Big(\frac{g_{1,s}}{g_0} - \frac{g_{0,s}}{{g_0}^2} g_1\Big) - \frac{\varrho\tilde{\sigma}_\theta\sqrt{2}\nu}{(\mu - r)g_0} \times\Big(\frac{p(\mu - r)^2\psi'(y) g_0}{2(1-p)s^{2\theta}} + \frac{1}{2}\varphi'(y)\Big(s^{2(\theta+1)} g_{0,ss} + \frac{p s^{2(\theta+1)} g_{0,s}^2}{(1-p)g_0}\Big)\Big)\Big], \tag{4.27}$$

where $I(t)$ is given by (4.25) and g_0 and g_1 are given by Theorem 1 and Theorem 2, respectively.

Proof. From the expansion $g = g_0 + \sqrt{\varepsilon} g_1 + \varepsilon g_2 + \ldots$, (4.19) becomes

(4.28)
$$\pi^* = \Big(1 + \frac{c\hat{a}(t)}{x}\Big)\frac{\mu - r}{(1-p)\tilde{\sigma}_\theta^2} \times \Big(1 + \frac{\tilde{\sigma}_\theta^2 s(g_{0,s} + \sqrt{\varepsilon} g_{1,s} + \varepsilon g_{2,s} + \ldots) + \sqrt{2}\nu\tilde{\sigma}_\theta\varrho(g_{0,y}/\sqrt{\varepsilon} + g_{1,y} + \sqrt{\varepsilon} g_{2,y} + \ldots)}{(\mu - r)(g_0 + \sqrt{\varepsilon} g_1 + \varepsilon g_2 + \ldots)}\Big).$$

By direct computation, the leading order term is then given by

$$\pi_0^* = \Big(1 + \frac{c\hat{a}(t)}{x}\Big)\frac{\mu - r}{(1-p)\tilde{\sigma}_\theta^2}\Big(1 + \frac{\tilde{\sigma}_\theta^2 s g_{0,s}}{(\mu - r)g_0}\Big). \tag{4.29}$$

Since from Theorem 1 one can deduce $g_{0,s}/g_0 = (1-p)B(t)(-2\theta)s^{-2\theta-1}$, where $B(t) = \overline{\sigma}^{-2}I(t)$ holds from (4.6) and (4.25), the above leading order control (4.29) becomes (4.26). Direct computation yields that the first correction term of (4.28) is given by (4.27). □

We note here that if the mean reversion of the volatility is extremely fast, then $\sigma(y)$ becomes effectively a constant and so the leading order optimal strategy (4.26) reduces to the one corresponding to the CEV model, which is given by (4.23).

5. Practical optimal strategy

In practice, the stochastic volatility level given by the hidden process Y_t is not directly observable, unlike the local volatility. So, in this section, we consider the HJB equation given by (3.2) and restrict ourselves to the trading strategy π that does not depend upon the unobserved level and obtain the leading order strategy and the first order correction term together with the corresponding value function (the maximum expected utilities) for the given utility function. For analytic simplicity, we take the contribution rate $c = 0$ so that $a(t) = 0$ in (3.5).

From the y-independence of g_0 and g_1, the order-1 terms of (3.2) give

$$g_{0,t} + \sup_{\pi_0} \mathcal{A}^{\sigma,\pi_0} g_0 + \mathcal{L}_Y g_2 = 0, \tag{5.1}$$

where the operator $\mathcal{A}^{\sigma,\pi_0}$ is defined by

$$\mathcal{A}^{\sigma,\pi_0} = \mu s\partial_s + \frac{1}{2}\sigma^2(y)s^{2(\theta+1)}\partial_{ss}^2 + rp + \frac{1}{2}p(p-1)\pi_0{}^2\sigma^2(y)s^{2\theta} + p\pi_0((\mu - r) + \sigma^2(y)s^{2\theta+1}\partial_s).$$

Since (5.1) is a Poisson equation for g_2, the centering condition is applied to obtain

$$g_{0,t} + \sup_{\pi_0} \langle \mathcal{A}^{\sigma,\pi_0} g_0 \rangle = 0. \tag{5.2}$$

Here, the supremum is attained at $\pi_0 = \pi_0^*$ given by

$$\pi_0^* = \frac{\mu - r}{(1-p)\overline{\sigma}_\theta^2} + \frac{s g_{0,s}}{(1-p) g_0},$$

where $\overline{\sigma}_\theta$ is given by $\overline{\sigma}_\theta(s) = \overline{\sigma} s^\theta$, and subsequently (5.2) becomes

$$g_{0,t} + \mathcal{A}^{\overline{\sigma},\pi_0^*} g_0 = 0, \tag{5.3}$$

which is exactly (4.11) with $\breve{\sigma}$ replaced by $\overline{\sigma}$. Consequently, by the same argument as in the proof of Theorem 1, we obtain

$$\begin{aligned}
{\pi_0}^* &= M_{\overline{\sigma}_\theta}(1 + C_{\overline{\sigma}_\theta}), \\
M_{\overline{\sigma}_\theta} &:= \frac{\mu - r}{(1-p){\overline{\sigma}_\theta}^2(s)}, \\
C_{\overline{\sigma}_\theta} &:= -\frac{2\theta(1-p)}{\mu - r} I(t),
\end{aligned} \tag{5.4}$$

where $I(t)$ is given by (4.25).

Next, we obtain the first correction π_1^* to the strategy. From the y-independence of g_0 and g_1, the order-$\sqrt{\varepsilon}$ terms of (3.2) give

$$g_{1,t} + \sup_{\pi_0,\pi_1} \mathcal{A}^{\sigma,\pi_0,\pi_1}(g_0, g_1, g_2) + \mathcal{L}_Y g_3 = 0, \tag{5.5}$$

where $\mathcal{A}^{\sigma,\pi_0,\pi_1}(g_0, g_1, g_2)$ is given by

$$\mathcal{A}^{\sigma,\pi_0,\pi_1}(g_0, g_1, g_2) = \mathcal{A}_0^{\sigma,\pi_0,\pi_1} g_0 + \mathcal{A}_1^{\sigma,\pi_0} g_1 + \mathcal{A}_2^{\sigma,\pi_0} g_2, \tag{5.6}$$

in terms of the operators $\mathcal{A}_0^{\sigma,\pi_0,\pi_1}$, $\mathcal{A}_1^{\sigma,\pi_0}$ and $\mathcal{A}_2^{\sigma,\pi_0}$ defined by

$$\begin{aligned}
\mathcal{A}_0^{\sigma,\pi_0,\pi_1} &= p(p-1)\sigma^2(y) s^{2\theta} \pi_0 \pi_1 + p\pi_1((\mu - r) + \sigma^2(y) s^{2\theta+1} \partial_s), \\
\mathcal{A}_1^{\sigma,\pi_0} &= \mu s \partial_s + \frac{1}{2}\sigma^2(y) s^{2(\theta+1)} \partial_{ss}^2 + rp \\
&\quad + \frac{1}{2} p(p-1) \pi_0^2 \sigma^2(y) s^{2\theta} + p\pi_0((\mu - r) + \sigma^2(y) s^{2\theta+1} \partial_s), \\
\mathcal{A}_2^{\sigma,\pi_0} &= \varrho\nu\sqrt{2} s^{\theta+1} \partial_s(\sigma(y)\partial_y) + p\varrho\nu\sqrt{2} s^\theta \pi_0 \sigma(y) \partial_y.
\end{aligned}$$

Now, the centering condition applied to (5.5) leads to

$$g_{1,t} + \sup_{\pi_0,\pi_1} \langle \mathcal{A}^{\sigma,\pi_0,\pi_1}(g_0, g_1, g_2)\rangle = 0. \tag{5.7}$$

Then, by direct computation, one can find that the supremum is attained at $\pi_0 = \pi_0^*$, which is the same as (5.4), and $\pi_1 = \pi_1^*$ given by

$$\pi_1^* = \frac{s}{1-p}\Big(\frac{g_{1,s}}{g_0} - \frac{g_{0,s}g_1}{g_0^2}\Big) + \frac{\sqrt{2}\varrho\nu}{(1-p)\overline{\sigma}_\theta^2(s)}\frac{\langle\sigma g_{2,y}\rangle}{g_0}, \tag{5.8}$$

where g_0 is given by the explicit solution in Theorem 1, g_1 is the solution of the PDE

$$\begin{aligned} g_{1,t} + \mathcal{A}_1^{\overline{\sigma},\pi_0^*} g_1 &= \mathcal{K}_1(t,s), \\ \mathcal{K}_1(t,s) &:= -\mathcal{A}_0^{\overline{\sigma},\pi_0^*,\pi_1^*} g_0 - \bar{\mathcal{A}}_2^{\pi_0^*} g_2, \\ \bar{\mathcal{A}}_2^{\pi_0^*} g_2 &:= \varrho\nu\sqrt{2}s^{\theta+1}\partial_s\langle\sigma g_{2,y}\rangle + p\varrho\nu\sqrt{2}s^\theta \pi_0^*\langle\sigma g_{2,y}\rangle, \end{aligned} \tag{5.9}$$

and, from (5.1), g_2 satisfies the ODE

$$\begin{aligned} \mathcal{L}_Y g_2 &= \mathcal{K}_2(t,s,y), \\ \mathcal{K}_2(t,s,y) &:= -g_{0,t} - \mathcal{A}^{\overline{\sigma},\pi_0^*} g_0. \end{aligned} \tag{5.10}$$

6. Numerical results

In this section, we illustrate graphically some properties of the formula derived for the corrected optimal control in Section 5. The parameters used in the following figures are $S_0 = 67$, $X_0 = 100$, $\overline{\sigma} = 0.41$, $\mu = 0.12$, $r = 0.05$, $\theta = 0.01$, $p = -1$, and $T = 20$, whenever they are required to be fixed.

Fig. 1 illustrates the behavior of the leading order optimal strategy π_0^* for the CRRA utility function. The leading optimal strategy corresponds to the optimal strategy corresponding to the CEV model. It is drawn against the excess return rate $\mu - r$ for two different values of the elasticity parameter θ, to compare with the classical Merton's strategy, which exactly corresponds to the case $\theta = 0$. The risk-free interest rate r is fixed and the choices of θ are -0.01, 0 and 0.01. Since the elasticity parameter is near 0 in many cases, the two values of θ can provide us with sufficient information about optimal strategies. First, the graphs show that the leading order optimal strategy π_0^* increases linearly with respect to the excess return rate, as it is expected. The investor increases the amount invested in the risky asset as its price increases. Second, the bigger the elasticity parameter θ is chosen, the relatively less investment on the risky asset is employed. This suggests that the sign of θ can be related to making a difference between risk-aversion and risk-taking of the investment strategy. In the case of a risky asset with a negative value of θ, more risk-taking strategy than the Merton's strategy can produce an optimal strategy, whereas, in the case of positive θ, more risk-aversive strategy than the Merton's strategy is desirable. Also, the graphs in Fig. 1 and Fig. 3 indicate that the leading optimal strategies converge to the Merton's strategy smoothly as the elasticity parameter θ goes to zero.

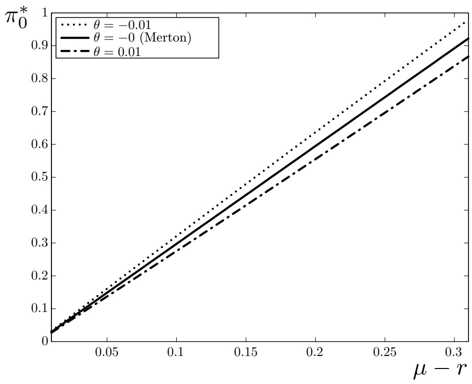

Figure 1. The leading order optimal strategy against the excess return rate for different values of θ.

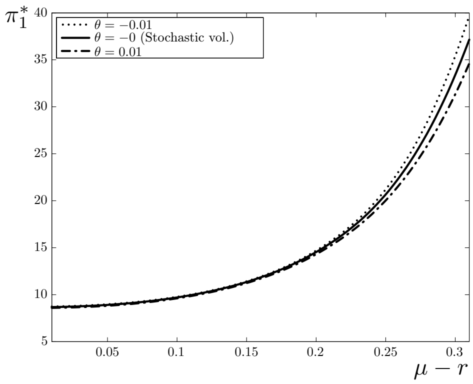

Figure 2. The first correction term against the excess return rate for different values of θ.

Fig. 2 displays the behavior of the first (stochastic volatility) correction term π_1^* (without the multiplication of $\sqrt{\varepsilon}$) against the excess return rate $\mu - r$ when $\theta = -0.01$, $\theta = 0$, and $\theta = 0.01$. The case of $\theta = 0$ corresponds to a stochastic volatility model, while the other cases represent the hybrid SVCEV models with different elasticities of variance. The graphs indicate that the stochastic volatility does not change the risk-taking or risk-aversion policy as shown in Fig. 1, but it tends to accelerate the optimal strategy more than linearly as the excess return rate $\mu - r$ increases.

Fig. 3 and 4 plot the sensitivity of the leading order optimal strategy and the first correction to the optimal strategy against the elasticity parameter θ. As indicated by Fig. 1 already, without the stochastic volatility, there is a smooth monotonically decreasing behavior in the leading order optimal strategy along the elasticity parameter θ. However, the stochastic volatility creates a nonmonotonic behavior of the corrected strategy (a hump type).

Fig. 5 shows the behavior of the first correction to the optimal strategy as a function of the risk-aversion coefficient p.

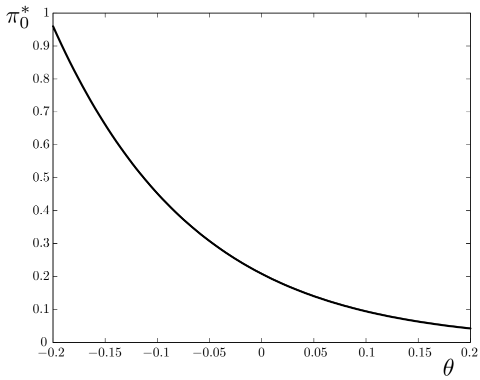

Figure 3. The leading order optimal strategy against the elasticity parameter θ.

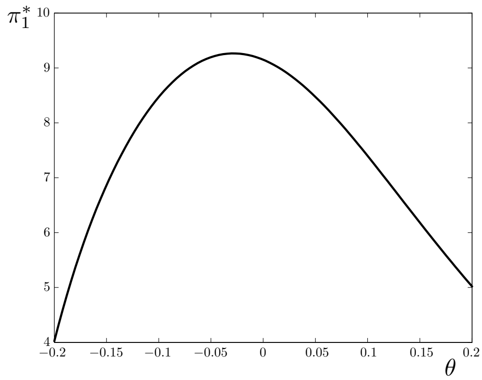

Figure 4. The first correction to the optimal strategy against the elasticity parameter θ.

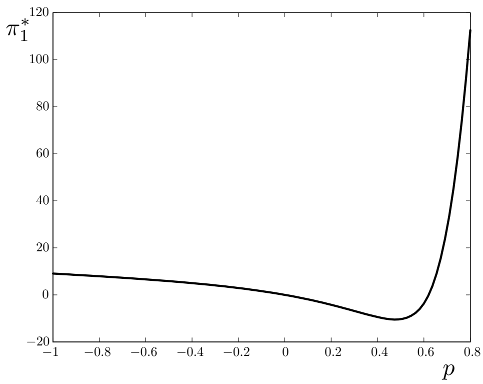

Figure 5. The first correction to the optimal strategy against the risk-aversion coefficient p.

7. Conclusion

As the risk of the financial market increases, a more sophisticated market model is required in portfolio selection problems. This paper has considered an optimal portfolio problem with a CRRA utility function under a hybrid market structure of stochastic and local volatility. The unobservable hidden variable driving the stochastic volatility has been saturated in the value functions by utilizing the ergodic property of the Ornstein-Uhlenbeck process. The fine structure of the influence of the stochastic volatility and the elasticity of variance on the optimal strategy is obtained with respect to the excess return rate, the elasticity parameter and the risk-aversion coefficient. The results may enable us to employ more delicate optimal strategy than the classical Merton's strategy for many portfolio selection problems.

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References

[1] *M. Asch, W. Kohler, G. Papanicolaou, M. Postel, B. White*: Frequency content of randomly scattered signals. SIAM Rev. *33* (1991), 519–625.
[2] *S. Beckers*: The constant elasticity of variance model and its implications for option pricing. Journal of Finance *35* (1980), 661–673.
[3] *J.-F. Boulier, S. Huang, G. Taillard*: Optimal management under stochastic interest rates: The case of a protected defined contribution pension fund. Insur. Math. Econ. *28* (2001), 173–189.
[4] *P. P. Boyle, Y. S. Tian*: Pricing lookback and barrier options under the CEV process. Journal of Financial and Quantitative Analysis *34* (1999), 241–264.
[5] *S. Cerrai*: A Khasminskii type averaging principle for stochastic reaction-diffusion equations. Ann. Appl. Probab. *19* (2009), 899–948.
[6] *S.-Y. Choi, J.-P. Fouque, J.-H. Kim*: Option pricing under hybrid stochastic and local volatility. Quant. Finance *13* (2013), 1157–1165.
[7] *J. C. Cox*: The constant elasticity of variance option pricing model. The Journal of Portfolio Management *23* (1996), 15–17.
[8] *J. C. Cox, C.-F. Huang*: Optimal consumption and portfolio policies when asset prices follow a diffusion process. J. Econ. Theory *49* (1989), 33–83.
[9] *J. C. Cox, S. Ross*: The valuation of options for alternative stochastic processes. Journal of Financial Economics *3* (1976), 145–166.
[10] *D. Davydov, V. Linetsky*: The valuation and hedging of barrier and lookback options under the CEV process. Manage. Sci. *47* (2001), 949–965.
[11] *G. Deelstra, M. Grasselli, P.-F. Koehl*: Optimal design of the guarantee for defined contribution funds. J. Econ. Dyn. Control *28* (2004), 2239–2260.
[12] *P. Devolder, M. Bosch Princep, I. Dominguez Fabian*: Stochastic optimal control of annuity contracts. Insur. Math. Econ. *33* (2003), 227–238.
[13] *W. H. Fleming, H. M. Soner*: Controlled Markov Processes and Viscosity Solutions. Stochastic Modelling and Applied Probability 25, Springer, New York, 2006.
[14] *J.-P Fouque, G. Papanicolaou, K. R. Sircar*: Derivatives in Financial Markets with Stochastic Volatility. Cambridge University Press, Cambridge, 2000.
[15] *J.-P. Fouque, G. Papanicolaou, R. Sircar, K. Sølna*: Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives. Cambridge University Press, Cambridge, 2011.
[16] *I. Fredholm*: Sur une classe d'équations fonctionnelles. Acta Math. *27* (1903), 365–390. (In French.)
[17] *J. Gao*: Optimal portfolios for DC pension plans under a CEV model. Insur. Math. Econ. *44* (2009), 479–490.
[18] *E. Ghysels, A. Harvey, E. Renault*: Stochastic volatility. Statistical Methods in Finance. Handbook of Statistics 14, North-Holland, Amsterdam, 1996.
[19] *S. Haberman, E. Vigna*: Optimal investment strategies and risk measures in defined contribution pension schemes. Insur. Math. Econ. *31* (2002), 35–69.
[20] *C. R. Harvey*: The specification of conditional expectations. Journal of Empirical Finance *8* (2001), 573–637.
[21] *J. C. Jackwerth, M. Rubinstein*: Recovering probability distributions from option prices. Journal of Finance *51* (1996), 1611–1631.

[22] *R. Z. Khas'minskii*: On stochastic processes defined by differential equations with a small parameter. Theory Probab. Appl. *11* (1966), 211–228; translation from Teor. Veroyatn. Primen. *11* (1966), 240–259. (In Russian.)
[23] *J.-H. Kim*: Asymptotic theory of noncentered mixing stochastic differential equations. Stochastic Processes Appl. *114* (2004), 161–174.
[24] *R. C. Merton*: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory *3* (1971), 373–413.
[25] *E.-J. Noh, J.-H. Kim*: An optimal portfolio model with stochastic volatility and stochastic interest rate. J. Math. Anal. Appl. *375* (2011), 510–522.
[26] *B. Øksendal*: Stochastic Differential Equations. An Introduction with Applications. Universitext, Springer, Berlin, 2003.
[27] *G. C. Papanicolaou, D. W. Stroock, S. R. S. Varadhan*: Martingale approach to some limit theorems. Proc. Conf. Durham, 1976. Duke Univ. Math. Ser., Vol. III, Duke Univ., Durham, 1977.
[28] *M. Rubinstein*: Nonparametric tests of alternative option pricing models using CBOE reported trades. Journal of Finance *40* (1985), 455–480.
[29] *J. Xiao, Z. Hong, C. Qin*: The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts. Insur. Math. Econ. *40* (2007), 302–310.
[30] *K. C. Yuen, H. Yang, K. L. Chu*: Estimation in the constant elasticity of variance model. British Actuarial Journal *7* (2001), 275–292.

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