S.S. Dragomir

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OSTROWSKI'S TYPE INEQUALITIES FOR COMPLEX FUNCTIONS DEFINED ON UNIT CIRCLE WITH APPLICATIONS FOR UNITARY OPERATORS IN HILBERT SPACES

S.S. Dragomir

Abstract. Some Ostrowski's type inequalities for the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right) du\,(t)$ of continuous complex valued integrands $f\colon \mathcal{C}\,(0,1) \to \mathbb{C}$ defined on the complex unit circle $\mathcal{C}\,(0,1)$ and various subclasses of integrators $u\colon\ [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ of bounded variation are given. Natural applications for functions of unitary operators in Hilbert spaces are provided as well.

1. Introduction

The problem of approximating the Stieltjes integral $\int_a^b f\,(t)\,du\,(t)$ by the quantity $f\,(x)\,[u\,(b) - u\,(a)]$, which is a natural generalization of the Ostrowski problem analyzed in 1937 (see [6]), was apparently first considered in the literature by the author in 2000 (see [1]) where we obtained the following result:

$$\left|\,[u\,(b) - u\,(a)]\,f\,(x) - \int_a^b f\,(t)\;du\,(t)\right| \le H\Big[\,(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\,\Big]$$

$$\le H \times \begin{cases} \left[(x-a)^r + (b-x)^r\right]\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]; \\ \left[(x-a)^{qr} + (b-x)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x (f)\right)^p + \left(\bigvee_x^b (f)\right)^p\right]^{\frac{1}{p}} \\ \qquad \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f)\ . \end{cases} \tag{1.1}$$

for each $x \in [a,b]$, provided f is of bounded variation on $[a,b]$, $\bigvee_a^b (f)$ is its total variation on $[a,b]$, while $u\colon\ [a,b] \to \mathbb{R}$ is $r-H$-Hölder continuous, i.e., we recall that:

$$|u\,(x) - u\,(y)| \le H\,|x-y|^r \quad \text{for each} \quad x,y \in [a,b]\ , \tag{1.2}$$

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where $H>0$ and $r\in(0,1]$.

The dual case, i.e., when the *integrand* f is $q-K$-Hölder continuous and the *integrator* u is of bounded variation was obtained by the author in 2001 and can be stated as [2]

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right|\le K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q\bigvee_a^b(u) \tag{1.3}$$

for each $x\in[a,b]$.

The above inequalities provide, as important consequences, the following midpoint inequalities:

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right|\le\begin{cases}\frac{1}{2^r}(b-a)^r H\bigvee_a^b(f)\\ \frac{1}{2^q}(b-a)^q K\bigvee_a^b(u)\end{cases} \tag{1.4}$$

which can be numerically implemented and provide a quadrature rule for approximating the Stieltjes integral $\int_a^b f(t)\,du(t)$.

Let U be a selfadjoint operator on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ with the spectrum $\operatorname{Sp}(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f\colon[m,M]\to\mathbb{R}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieltjes integral:

$$\langle f(U)x,y\rangle=\int_{m-0}^M f(\lambda)\,d(\langle E_\lambda x,y\rangle)\,, \tag{1.5}$$

for any x, $y\in H$. The function $g_{x,y}(\lambda):=\langle E_\lambda x,y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0)=0\quad\text{and}\quad g_{x,y}(M)=\langle x,y\rangle$$

for any $x,y\in H$. It is also well known that $g_x(\lambda):=\langle E_\lambda x,x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$.

On utilizing the spectral representation (1.5) and the Ostrowski's type inequality (1.3) we obtained the following result for continuous functions of selfadjoint operators:

Theorem 1 ([4]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $\operatorname{Sp}(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f\colon[m,M]\to\mathbb{R}$ is $r-H$-Hölder continuous on $[m,M]$, then we have the inequality*

$$\begin{aligned}|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle|&\le H\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r\\ &\le H\|x\|\|y\|\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r \end{aligned}\tag{1.6}$$

for any x, $y\in H$ and $s\in[m,M]$.

The following dual result also holds:

Theorem 2 ([3]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f\colon [m, M] \to \mathbb{R}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\begin{aligned} |f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| &\le \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\ &\quad + \langle (1_H - E_s)x, x\rangle^{1/2} \langle (1_H - E_s)y, y\rangle^{1/2} \bigvee_s^M (f) \\ &\le \|x\|\,\|y\| \left(\frac{1}{2} \bigvee_m^M (f) + \frac{1}{2} \left| \bigvee_m^s (f) - \bigvee_s^M (f) \right| \right) \le \|x\|\,\|y\| \bigvee_m^M (f) \end{aligned} \tag{1.7}$$

for any x, $y \in H$ and for any $s \in [m, M]$, where 1_H is the identity operator on H.

Motivated by the above results, we investigate in the current paper the magnitude of the difference

$$f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) \quad \text{with} \quad s \in [a, b] \subseteq [0, 2\pi]$$

for continuous complex valued function $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of functions $u\colon [a, b] \subseteq [0, 2\pi] \to \mathbb{C}$ of bounded variation. Natural applications for functions of unitary operators in Hilbert spaces are provided as well.

2. Scalar Ostrowski's type inequalities

Theorem 3. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z) - f(w)| \le H|z - w|^r \tag{2.1}$$

for any $w, z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0, 1]$ are given.

If $[a, b] \subseteq [0, 2\pi]$ and the function $u\colon [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then

$$\begin{aligned} &\left| f(e^{is})[u(b) - u(a)] - \int_a^b f(e^{it})\, du(t) \right| \\ &\le 2^r H \max_{t \in [a,b]} \left| \sin\left(\frac{s-t}{2}\right) \right|^r \bigvee_a^b (u) \end{aligned} \tag{2.2}$$

for any $s \in [a, b]$, where $\bigvee_a^b (u)$ denotes the total variation of u on the interval $[a, b]$.

Proof. Observe that

$$f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) = \int_a^b \left[f\left(e^{is}\right) - f\left(e^{it}\right)\right] du(t) \tag{2.3}$$

for any $s \in [a, b]$.

It is known that if $p\colon [c,d] \to \mathbb{C}$ is a continuous function and $v\colon [c,d] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and the following inequality holds

$$\left| \int_c^d p(t)\,dv(t) \right| \le \max_{t\in[c,d]} |p(t)| \bigvee_c^d (v) \tag{2.4}$$

where $\bigvee_c^d (v)$ denotes the total variation of v on $[c,d]$.

Applying the property (2.4) to the identity (2.3) and utilizing the Hölder's type condition (2.1) we have successively

$$\begin{aligned} \left| f(e^{is})[u(b)-u(a)] - \int_a^b f(e^{it})\,du(t) \right| &= \max_{t\in[a,b]} \left| f\left(e^{is}\right) - f\left(e^{it}\right) \right| \bigvee_a^b (u) \\ &\le H \max_{t\in[a,b]} \left| e^{is} - e^{it} \right|^r \bigvee_a^b (u) \; . \end{aligned} \tag{2.5}$$

Since

$$\left| e^{is} - e^{it} \right|^2 = \left| e^{is} \right|^2 - 2\operatorname{Re}\left(e^{i(s-t)} \right) + \left| e^{it} \right|^2 = 2 - 2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2} \right)$$

for any $t,\ s \in \mathbb{R}$, then

$$\left| e^{is} - e^{it} \right|^r = 2^r \left| \sin\left(\frac{s-t}{2} \right) \right|^r \tag{2.6}$$

for any $t,\ s \in \mathbb{R}$.

Now, by (2.5) and (2.6) we deduce the desired result (2.2). □

Remark 1. If $a = 0$ and $b = 2\pi$, then for any $s \in [0,2\pi]$ there exists a unique $t \in [0,2\pi]$ such that $\frac{1}{2}|t-s| = \frac{\pi}{2}$, therefore $\max_{t\in[0,2\pi]} \left| \sin\left(\frac{s-t}{2} \right) \right| = 1$ for all $s \in [0,2\pi]$ and we deduce from (2.2) the following inequality of interest

$$\left| f\left(e^{is}\right)[u(2\pi) - u(0)] - \int_0^{2\pi} f\left(e^{it}\right)\,du(t) \right| \le 2^r H \bigvee_0^{2\pi} (u) \tag{2.7}$$

that holds for each $s \in [0,2\pi]$.

Remark 2. If $[a,b] \subset [0,2\pi]$ and $0 < b-a \le \pi$ then for all $t,s \in [a,b]$ we have $\frac{1}{2}|t-s| \le \frac{1}{2}(b-a) \le \frac{\pi}{2}$. Since the function sin is increasing on $\left[0,\frac{\pi}{2}\right]$, then we have successively that

$$\begin{aligned} \max_{t\in[a,b]} \left| \sin\left(\frac{s-t}{2} \right) \right| &= \sin\left(\max_{t\in[a,b]} \frac{1}{2}|t-s| \right) = \sin\left(\frac{1}{2}\max\{b-s, s-a\} \right) \\ &= \sin\left(\frac{1}{4}(b-a) + \frac{1}{2}\left| s - \frac{a+b}{2} \right| \right) \end{aligned} \tag{2.8}$$

for any $s \in [a,b]$.

Therefore, under the assumptions of Theorem 3 and if $[a,b] \subset [0,2\pi]$ with $0 < b-a \leq \pi$, then

$$\left|f(e^{is})[u(b)-u(a)] - \int_a^b f\left(e^{it}\right) du\left(t\right)\right|$$

$$\leq 2^r H \sin^r \left[\frac{1}{4}(b-a) + \frac{1}{2}\left|s - \frac{a+b}{2}\right|\right] \bigvee_a^b (u) \leq 2^r H \sin^r \left[\frac{1}{2}(b-a)\right] \bigvee_a^b (u) \tag{2.9}$$

for all $s \in [a,b]$.

In particular, the best inequality we can get from (2.9) is incorporated in (2.10)

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\left[u\left(b\right)-u\left(a\right)\right] - \int_a^b f\left(e^{it}\right)\, du\left(t\right)\right| \leq 2^r H \sin^r \left[\frac{1}{4}\left(b-a\right)\right] \bigvee_a^b (u)\ .$$

The case when $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the Lipschitz condition $|f(z) - f(w)| \leq L|z-w|$ for any $w,\ z \in \mathcal{C}(0,1)$, where $L > 0$ is given, is of interest due to various examples one can consider. Also in this case we can show that the corresponding version of the inequality (2.11) is sharp.

Corollary 1. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}(0,1)$. If $[a,b] \subset [0,2\pi]$ with $0 < b-a \leq \pi$ and the function $u\colon [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then we have*

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\left[u\left(b\right)-u\left(a\right)\right] - \int_a^b f\left(e^{it}\right) du\left(t\right)\right| \leq 2L \sin \left[\frac{1}{4}\left(b-a\right)\right] \bigvee_a^b (u)\ . \tag{2.11}$$

The constant 2 cannot be replaced by a smaller quantity.

Proof. We need to prove only the sharpness of the constant.

If we consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z$, then obviously f is Lipschitzian with the constant $L = 1$. Also, consider in (2.11) $a = 0$ and $b = \pi$ to get

$$\left|i\left[u\left(\pi\right)-u\left(0\right)\right] - \int_0^\pi e^{it} du\left(t\right)\right| \leq \sqrt{2} \bigvee_0^\pi (u)\ . \tag{2.12}$$

Utilising the integration by parts formula for the Riemann-Stieltjes integral we have

$$\int_0^\pi e^{it} du\left(t\right) = e^{it} u\left(t\right)\Big|_0^\pi - i\int_0^\pi e^{it} u\left(t\right)\, dt = -u\left(\pi\right) - u\left(0\right) - i\int_0^\pi e^{it} u\left(t\right)\, dt$$

and replacing into the inequality (2.12) we deduce

$$\left|i\left[u\left(\pi\right)-u\left(0\right)\right] + u\left(\pi\right) + u\left(0\right) + i\int_0^\pi e^{it} u\left(t\right)\, dt\right| \leq \sqrt{2} \bigvee_0^\pi (u)$$

which is equivalent with

$$\left|\left(i-1\right)u\left(\pi\right) + \left(i+1\right)u\left(0\right) - \int_0^\pi e^{it} u\left(t\right) dt\right| \leq \sqrt{2} \bigvee_0^\pi (u) \tag{2.13}$$

that holds for any functions of bounded variation $u\colon [0,\pi]\to\mathbb{C}$ and is of interest in itself.

Now, assume that there exists a constant $C>0$ such that

$$\left|(i-1)\,u(\pi)+(i+1)\,u(0)-\int_0^\pi e^{it}u(t)\,dt\right|\le C\bigvee_0^\pi(u) \tag{2.14}$$

for any functions of bounded variation $u\colon [0,\pi]\to\mathbb{C}$.

Consider the function $u\colon [0,\pi]\to\mathbb{R}$ with

$$u(t)=\begin{cases}0 & \text{if} \quad 0\le t<\pi\\ 1 & \text{if} \quad t=\pi\,.\end{cases}$$

Then u is of bounded variation, $\int_0^\pi e^{it}u(t)\,dt=0$, $\bigvee_0^\pi(u)=1$ and from (2.14) we get $C\ge\sqrt{2}$ showing that (2.14) is sharp and therefore (2.11) is sharp. □

Remark 3. The case of Riemann integral, namely when $u(t)=t$, $t\in[a,b]\subseteq[0,2\pi]$, is as follows

$$\left|f\left(e^{is}\right)-\frac{1}{b-a}\int_a^b f\left(e^{it}\right)\,du(t)\right|\le 2^rH\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.15}$$

for any $s\in[a,b]$ provided that $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1).

When u is an integral, then the following weighted integral inequality also holds.

Remark 4. If $w\colon [a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is Lebesgue integrable on $[a,b]$ and $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1), then

$$\begin{aligned}&\left|f\left(e^{is}\right)\int_a^b w(t)\,dt-\int_a^b f\left(e^{it}\right)w(t)\,dt\right|\\ &\qquad\le 2^rH\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r\int_a^b|w(t)|\,dt\end{aligned} \tag{2.16}$$

for any $s\in[a,b]$.

In particular, if $w(t)\ge 0$ for $t\in[a,b]$ and $\int_a^b w(t)\,dt>0$ then

$$\left|f\left(e^{is}\right)-\frac{1}{\int_a^b w(t)\,dt}\int_a^b f\left(e^{it}\right)w(t)\,dt\right|\le 2^rH\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.17}$$

for any $s\in[a,b]$.

Theorem 4. *Assume that $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$. If $[a,b]\subseteq[0,2\pi]$ and the function $u\colon [a,b]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ on $[a,b]$, then*

$$\begin{aligned}&\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)\,du(t)\right|\le 4LK\left[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\right]\\ &\qquad\le 8LK\sin^2\left(\frac{b-a}{4}\right)\end{aligned} \tag{2.18}$$

for any $s \in [a,b]$.

Proof. It is well known that if $p\colon [a,b] \to \mathbb{C}$ is a Riemann integrable function and $v\colon [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, i.e.,

$$|f(s) - f(t)| \le M|s-t| \text{ for any } t, s \in [a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le M\int_a^b |p(t)|\,dt. \tag{2.19}$$

Utilising the property (2.19), we have from (2.3) that

$$\left|f\left(e^{is}\right)[u(b)-u(a)] - \int_a^b f\left(e^{it}\right)du(t)\right| = \left|\int_a^b \left[f\left(e^{is}\right) - f\left(e^{it}\right)\right]du(t)\right|$$

$$\le K\int_a^b \left|f\left(e^{is}\right) - f\left(e^{it}\right)\right| dt \le KL\int_a^b \left|e^{is} - e^{it}\right| dt \tag{2.20}$$

for any $s \in [a,b]$.

Since, by (2.6), $\left|e^{is} - e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$ for any t, $s \in \mathbb{R}$, then

$$\begin{aligned}\int_a^b \left|e^{is}-e^{it}\right| dt &= 2\int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right| dt \\ &= 2\Big[\int_a^s \sin\left(\frac{s-t}{2}\right)dt + \int_s^b \sin\left(\frac{t-s}{2}\right)dt\Big] \\ &= 2\Big[1-\cos\left(\frac{s-a}{2}\right)\Big] + 2\Big[1-\cos\left(\frac{b-s}{2}\right)\Big] \\ &= 4\Big[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\Big] \le 8\sin^2\left(\frac{b-a}{4}\right) \end{aligned} \tag{2.21}$$

for any $s \in [a,b] \subseteq [0,2\pi]$, and the inequality (2.18) is proved. □

The best inequality we can get from (2.18) is incorporated in

Corollary 2. *With the assumptions in Theorem 4 we have the inequality*

$$\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)] - \int_a^b f\left(e^{it}\right)du(t)\right| \le 8LK\sin^2\left(\frac{b-a}{8}\right). \tag{2.22}$$

The multiplicative constant 8 *cannot be replaced by a smaller quantity.*

Proof. We need to prove only the sharpness of the constant.

If we consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z$, then obviously f is Lipschitzian with the constant $L = 1$. Also, consider in (2.22) $a = 0$ and $b = 2\pi$ to get

$$\left|-[u(2\pi)-u(0)] - \int_0^{2\pi} e^{it}du(t)\right| \le 4K. \tag{2.23}$$

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_0^{2\pi} e^{it}du(t) = e^{it}u(t)\Big|_0^{2\pi} - i\int_0^{2\pi} e^{it}u(t)\,dt = u(2\pi)-u(0) - i\int_0^{2\pi} e^{it}u(t)\,dt,$$

which inserted in (2.23) produces the inequality

$$\left| -2\left[u\left(2\pi\right)-u\left(0\right)\right]+i\int_0^{2\pi} e^{it}u\left(t\right)\,dt\right| \leq 4K$$

which is equivalent with

$$\left| \int_0^{2\pi} e^{it}u\left(t\right)\,dt-\frac{2}{i}\left[u\left(2\pi\right)-u\left(0\right)\right]\right| \leq 4K \tag{2.24}$$

that holds for any K-Lipschitzian function $u\colon\ [0,2\pi]\to\mathbb{C}$ and is of interest in itself.

Now, assume that the inequality (2.24) holds with a constant $D>0$, namely

$$\left| \int_0^{2\pi} e^{it}u\left(t\right)\,dt-\frac{2}{i}\left[u\left(2\pi\right)-u\left(0\right)\right]\right| \leq DK \tag{2.25}$$

for any K-Lipschitzian function $u\colon\ [0,2\pi]\to\mathbb{C}$.

Consider $u\colon\ [0,2\pi]\to\mathbb{R}$, $u\left(t\right)=\left|t-\pi\right|$. Then, by the continuity property of the modulus we have that u is Lipschitzian with the constant $K=1$.

We also have that

$$\begin{aligned}\int_0^{2\pi} e^{it}u\left(t\right)\,dt &= \int_0^{2\pi} e^{it}\left|t-\pi\right|\,dt=\int_0^{2\pi}\left|t-\pi\right|\left(\cos t+i\sin t\right)\,dt\\ &=\int_0^{2\pi}\left|t-\pi\right|\cos t\,dt+i\int_0^{2\pi}\left|t-\pi\right|\sin t\,dt\,.\end{aligned}$$

Observe that by symmetry reasons $\int_0^{2\pi}\left|t-\pi\right|\sin t\,dt=0$ and

$$\int_0^{2\pi}\left|t-\pi\right|\cos t\,dt=2\int_0^{\pi}\left(\pi-t\right)\cos t\,dt=2\Big[\left(\pi-t\right)\sin t\big|_0^{\pi}+\int_0^{\pi}\sin t\,dt\Big]=4$$

and by (2.25) we get $D\geq 4$ which proves the desired sharpness of the constant 8 in (2.22). □

Remark 5. If $u\left(t\right)=t$, $t\in[a,b]$, then we get from (2.18) and (2.22) the following inequalities for the Riemann integral

$$\begin{aligned}\left|f\left(e^{is}\right)\left(b-a\right)-\int_a^b f\left(e^{it}\right)\,dt\right| &\leq 4L\Big[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\Big]\\ &\leq 8L\sin^2\left(\frac{b-a}{4}\right)\end{aligned} \tag{2.26}$$

for any $s\in[a,b]$ and

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\left(b-a\right)-\int_a^b f\left(e^{it}\right)\,dt\right| \leq 8L\sin^2\left(\frac{b-a}{8}\right) \tag{2.27}$$

provided that $f\colon\mathcal{C}\left(0,1\right)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}\left(0,1\right)$.

Remark 6. If $w\colon [a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is essentially bounded on $[a,b]$ and $f\colon \mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$, then we have the following weighted integral inequality

$$\left|f\left(e^{is}\right)\int_a^b w(t)\,dt-\int_a^b f\left(e^{it}\right)w(t)\,dt\right|\le 4L\|w\|_\infty\left[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\right]$$
$$\le 8L\|w\|_\infty\sin^2\left(\frac{b-a}{4}\right) \tag{2.28}$$

for any $s\in[a,b]$ where $\|w\|_\infty:=\operatorname{ess\,sup}_{t\in[a,b]}|w(t)|$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)\int_a^b w(t)\,dt-\int_a^b f\left(e^{it}\right)w(t)\,dt\right|\le 8L\|w\|_\infty\sin^2\left(\frac{b-a}{8}\right). \tag{2.29}$$

The case of monotonic integrators is as follows:

Theorem 5. *Assume that $f\colon\mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on the circle $\mathcal{C}(0,1)$. If $[a,b]\subseteq[0,2\pi]$ and the function $u\colon[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right|\le 2L\left[\sin\left(\frac{b-s}{2}\right)u(b)-\sin\left(\frac{s-a}{2}\right)u(a)\right]$$
$$+L\int_a^b\operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt \tag{2.30}$$

for any $s\in[a,b]$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right|\le 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)]$$
$$+L\int_a^b\operatorname{sgn}\left(\frac{a+b}{2}-t\right)\cos\left(\frac{\frac{a+b}{2}-t}{2}\right)u(t)\,dt. \tag{2.31}$$

Proof. It is well known that if $p\colon[a,b]\to\mathbb{C}$ is a continuous function and $v\colon[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t). \tag{2.32}$$

Utilising the property (2.32), we have from (2.3) that

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right|=\left|\int_a^b\left[f\left(e^{is}\right)-f\left(e^{it}\right)\right]du(t)\right|$$
$$\le\int_a^b\left|f\left(e^{is}\right)-f\left(e^{it}\right)\right|du(t)\le L\int_a^b\left|e^{is}-e^{it}\right|du(t) \tag{2.33}$$

for any $s\in[a,b]$.

Since, by (2.6), $\left|e^{is}-e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$ for any $t,s\in\mathbb{R}$, then

$$\int_a^b \left|e^{is}-e^{it}\right| du(t) = 2\int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right| du(t)$$
$$= 2\left[\int_a^s \sin\left(\frac{s-t}{2}\right) du(t) + \int_s^b \sin\left(\frac{t-s}{2}\right) du(t)\right] \tag{2.34}$$

for any $s\in[a,b]\subseteq[0,2\pi]$.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_a^s \sin\left(\frac{s-t}{2}\right) du(t) = \sin\left(\frac{s-t}{2}\right)u(t)\Big|_a^s + \frac{1}{2}\int_a^s \cos\left(\frac{s-t}{2}\right)u(t)\,dt$$
$$= -\sin\left(\frac{s-a}{2}\right)u(a) + \frac{1}{2}\int_a^s \cos\left(\frac{s-t}{2}\right)u(t)\,dt$$

and

$$\int_s^b \sin\left(\frac{t-s}{2}\right) du(t) = \sin\left(\frac{t-s}{2}\right)u(t)\Big|_s^b - \frac{1}{2}\int_s^b \cos\left(\frac{t-s}{2}\right)u(t)\,dt$$
$$= \sin\left(\frac{b-s}{2}\right)u(b) - \frac{1}{2}\int_s^b \cos\left(\frac{t-s}{2}\right)u(t)\,dt\,,$$

which, by (2.34), produce the equality

$$\int_a^b \left|e^{is}-e^{it}\right| du(t) = 2\left[\sin\left(\frac{b-s}{2}\right)u(b) - \sin\left(\frac{s-a}{2}\right)u(a)\right]$$
$$+\int_a^s \cos\left(\frac{s-t}{2}\right)u(t)\,dt - \int_s^b \cos\left(\frac{t-s}{2}\right)u(t)\,dt$$
$$= 2\left[\sin\left(\frac{b-s}{2}\right)u(b) - \sin\left(\frac{s-a}{2}\right)u(a)\right]$$
$$+\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt\,. \tag{2.35}$$

Utilising (2.33) we deduce the desired result (2.30). □

Remark 7. We remark that if $a=0$ and $b=2\pi$, then we get from (2.30) and (2.31) that

$$\left|f\left(e^{is}\right)\left[u(2\pi)-u(0)\right] - \int_0^{2\pi} f\left(e^{it}\right) du(t)\right| \le 2L\sin\left(\frac{s}{2}\right)\left[u(2\pi)-u(0)\right]$$
$$+L\int_0^{2\pi} \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt \tag{2.36}$$

for any $s\in[a,b]$.

In particular, we have

$$\left|f(-1)\left[u(2\pi)-u(0)\right]-\int_0^{2\pi} f\left(e^{it}\right)\,du(t)\right| \le \sqrt{2}L\left[u(2\pi)-u(0)\right]$$
$$+L\int_0^{2\pi}\operatorname{sgn}(\pi-t)\sin\left(\frac{t}{2}\right)u(t)\,dt\,. \tag{2.37}$$

Corollary 3. *Assume that f and u are as in Theorem 5 then for any $[a,b]\subset[0,2\pi]$ with $0<b-a\le\pi$ we have the sequence of inequalities*

$$\left|f(e^{is})[u(b)-u(a)]-\int_a^b f(e^{it})\,du(t)\right| \le 2L\Big[\sin\left(\frac{b-s}{2}\right)u(b)-\sin\left(\frac{s-a}{2}\right)u(a)\Big]$$
$$+L\int_a^b\operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt$$
$$\le 2L\Big[\sin\left(\frac{b-s}{2}\right)[u(b)-u(s)]+\sin\left(\frac{s-a}{2}\right)[u(s)-u(a)]\Big]=:B(s) \tag{2.38}$$

where

$$B(s)\le 2L\times\begin{cases}\sin\left[\frac{1}{4}(b-a)+\frac{1}{2}\left|s-\frac{a+b}{2}\right|\right][u(b)-u(a)]\\ 2\sin\left(\frac{b-a}{4}\right)\cos\left(\frac{s-\frac{a+b}{2}}{2}\right)\left[\frac{u(b)-u(a)}{2}+\left|u(s)-\frac{u(b)+u(a)}{2}\right|\right]\end{cases}$$

for any $s\in[a,b]$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)\,du(t)\right| \le 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)]$$
$$+L\int_a^b\operatorname{sgn}\left(\frac{a+b}{2}-t\right)\cos\left(\frac{\frac{a+b}{2}-t}{2}\right)u(t)\,dt=:M \tag{2.39}$$

where

$$M\le 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)]\,.$$

Proof. Since $0<b-a\le\pi$, then $\frac{|s-t|}{2}\le\frac{\pi}{2}$ for $s,t\in[a,b]$. Utilising the fact that u is monotonic nondecreasing on $[a,b]$ and $\cos\left(\frac{|s-t|}{2}\right)\ge 0$ for $s,t\in[a,b]$, then

$$\int_a^s\cos\left(\frac{s-t}{2}\right)u(t)\,dt\le u(s)\int_a^s\cos\left(\frac{s-t}{2}\right)dt=2u(s)\sin\left(\frac{s-a}{2}\right) \tag{2.40}$$

and

$$\int_s^b\cos\left(\frac{s-t}{2}\right)u(t)\,dt\ge u(s)\int_s^b\cos\left(\frac{s-t}{2}\right)dt=2u(s)\sin\left(\frac{b-s}{2}\right)$$

i.e.,

$$-\int_s^b\cos\left(\frac{s-t}{2}\right)u(t)\,dt\le -2u(s)\sin\left(\frac{b-s}{2}\right). \tag{2.41}$$

Summing (2.40) with (2.41) we deduce that

$$\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt \le 2u(s)\sin\left(\frac{s-a}{2}\right) - 2u(s)\sin\left(\frac{b-s}{2}\right)$$

giving that

$$2L\left[\sin\left(\frac{b-s}{2}\right)u(b) - \sin\left(\frac{s-a}{2}\right)u(a)\right] + L\int_a^b \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt$$
$$\le 2L\left[\sin\left(\frac{b-s}{2}\right)[u(b)-u(s)] + \sin\left(\frac{s-a}{2}\right)[u(s)-u(a)]\right],$$

which proves the second inequality in (2.38).

The bounds for $B(s)$ follows from the elementary property stating that

$$\alpha x + \beta y \le \max\{\alpha, \beta\}(x+y)$$

where α, β, x, $y \ge 0$. □

3. A quadrature rule

We consider the following partition of the interval $[a,b]$

$$\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \le k \le n-1$. Define $h_k := x_{k+1} - x_k$, $0 \le k \le n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \le k \le n-1\}$ the norm of the partition Δ_n.

For the continuous function $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ and the function $u\colon\ [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ of bounded variation on $[a,b]$, define the quadrature rule

$$O_n(f,u,\Delta_n,\xi) := \sum_{k=0}^{n-1} f\left(e^{i\xi_k}\right)[u(x_{k+1}) - u(x_k)] \tag{3.1}$$

and the remainder $R_n(f,u,\Delta_n,\xi)$ in approximating the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right)du(t)$ by $O_n(f,u,\Delta_n,\xi)$. Then we have

$$\int_a^b f\left(e^{it}\right)\,du(t) = O_n(f,u,\Delta_n,\xi) + R_n(f,u,\Delta_n,\xi)\ . \tag{3.2}$$

The following result provides a priory bounds for $R_n(f,u,\Delta_n,\xi)$ in several instances of f and u as above.

Proposition 1. *Assume that $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z) - f(w)| \le H|z-w|^r$$

for any w, $z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given.

If $[a,b] \subseteq [0,2\pi]$ and the function $u\colon\ [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then for any partition $\Delta_n\colon a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm

$\nu(\Delta_n) \leq \pi$ we have the error bound

$$|R_n(f,u,\Delta_n,\xi)| \leq 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k+x_{k+1}}{2}\right|\right] \bigvee_{x_k}^{x_{k+1}} (u)$$

$$\leq 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{2}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}} (u)$$

$$\leq H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}} (u) \leq H\nu^r(\Delta_n) \bigvee_a^b (u) \tag{3.3}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \leq k \leq n-1$.

Proof. Since $\nu(\Delta_n) \leq \pi$, then on writing inequality (2.9) on each interval $[x_k, x_{k+1}]$ and for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \leq k \leq n-1$, we have

$$\left|f(e^{i\xi_k})[u(x_{k+1}) - u(x_k)] - \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) du(t)\right|$$

$$\leq 2^r H \sin^r \left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k+x_{k+1}}{2}\right|\right] \bigvee_{x_k}^{x_{k+1}} (u)$$

$$\leq 2^r H \sin^r \left[\frac{1}{2}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}} (u) \leq H(x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}} (u) \tag{3.4}$$

where for the last inequality we have used the fact that $\sin x \leq x$ for $x \in \left[0, \frac{\pi}{2}\right]$.

Summing over k from 0 to $n-1$ in (3.4) and utilizing the generalized triangle inequality, we deduce the first part of (3.3). The second part is obvious. □

Corollary 4. *Assume that f, u and Δ_n are as in Proposition 1. Define the midpoint trapezoid type quadrature rule by*

$$T_n(f,u,\Delta_n) := \sum_{k=0}^{n-1} f\left(e^{\frac{x_{k+1}+x_k}{2}i}\right)[u(x_{k+1}) - u(x_k)] \tag{3.5}$$

and the error $E_n(f,u,\Delta_n)$ by

$$\int_a^b f\left(e^{it}\right) du(t) = T_n(f,u,\Delta_n) + E_n(f,u,\Delta_n). \tag{3.6}$$

Then we have the error bounds

$$|E_n(f,u,\Delta_n)| \leq 2^r H \sum_{k=0}^{n-1} \sin^r \left[\frac{1}{4}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}} (u)$$

$$\leq \frac{1}{2^r} H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}} (u) \leq \frac{1}{2^r} H\nu^r(\Delta_n) \bigvee_a^b (u). \tag{3.7}$$

The case of both integrator and integrand being Lipschitzian is incorporated in the following result:

Proposition 2. *Assume that* $f\colon \mathcal{C}(0,1)\to\mathbb{C}$ *is Lipschitzian with the constant* $L>0$ *on the circle* $\mathcal{C}(0,1)$*. If* $[a,b]\subseteq[0,2\pi]$ *and the function* $u\colon [a,b]\to\mathbb{C}$ *is Lipschitzian with the constant* $K>0$ *on* $[a,b]$*, then for any partition* $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ *we have the error bound*

$$\begin{aligned} |R_n(f,u,\Delta_n,\xi)| &\le 4LK\sum_{k=0}^{n-1}\left[\sin^2\left(\frac{\xi_k-x_k}{4}\right)+\sin^2\left(\frac{x_{k+1}-\xi_k}{4}\right)\right] \\ &\le 8LK\sum_{k=0}^{n-1}\sin^2\left(\frac{x_{k+1}-x_k}{4}\right)\le\frac{1}{2}LK\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2 \\ &\le\frac{1}{2}LK(b-a)\,\nu(\Delta_n) \end{aligned}\tag{3.8}$$

for any intermediate points $\xi_k\in[x_k,x_{k+1}]$ *where* $0\le k\le n-1$.

In particular, we have

$$\begin{aligned} |E_n(f,u,\Delta_n)| &\le 8LK\sum_{k=0}^{n-1}\sin^2\left(\frac{x_{k+1}-x_k}{8}\right) \\ &\le\frac{1}{8}LK\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2\le\frac{1}{8}LK(b-a)\,\nu(\Delta_n)\,. \end{aligned}\tag{3.9}$$

The proof follows by Theorem 4 and the details are omitted.

Proposition 3. *Assume that* $f\colon \mathcal{C}(0,1)\to\mathbb{C}$ *is Lipschitzian with the constant* $L>0$ *on the circle* $\mathcal{C}(0,1)$*. If* $[a,b]\subseteq[0,2\pi]$ *and the function* $u\colon [a,b]\to\mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$*, then for any partition* $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ *with the norm* $\nu(\Delta_n)\le\pi$ *we have the error bound*

$$\begin{aligned} |R_n(f,u,\Delta_n,\xi)| &\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{x_{k+1}-\xi_k}{2}\right)u(x_{k+1})-\sin\left(\frac{\xi_k-x_k}{2}\right)u(x_k)\right] \\ &\quad + L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\operatorname{sgn}(\xi_k-t)\cos\left(\frac{\xi_k-t}{2}\right)u(t)\,dt \\ &\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{x_{k+1}-\xi_k}{2}\right)[u(x_{k+1})-u(\xi_k)]\right. \\ &\quad \left.+\sin\left(\frac{\xi_k-x_k}{2}\right)[u(\xi_k)-u(x_k)]\right] \\ &\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{4}(x_{k+1}-x_k)+\frac{1}{2}\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right][u(x_{k+1})-u(x_k)] \\ &\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{2}(x_{k+1}-x_k)\right][u(x_{k+1})-u(x_k)] \end{aligned}$$

$$\leq L \sum_{k=0}^{n-1} (x_{k+1} - x_k) [u (x_{k+1}) - u (x_k)] \leq \nu (\Delta_n) L [u (b) - u (a)] \tag{3.10}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ *where* $0 \leq k \leq n-1$.

In particular, we have

$$\begin{aligned} |E_n (f, u, \Delta_n)| &\leq 2L \sum_{k=0}^{n-1} \sin \left(\frac{x_{k+1} - x_k}{4} \right) [u (x_{k+1}) - u (x_k)] \\ &\quad + L \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \operatorname{sgn} \left(\frac{x_k + x_{k+1}}{2} - t \right) \cos \left(\frac{\frac{x_k + x_{k+1}}{2} - t}{2} \right) u (t) \, dt \\ &\leq 2L \sum_{k=0}^{n-1} \sin \left(\frac{x_{k+1} - x_k}{4} \right) [u (x_{k+1}) - u (x_k)] \\ &\leq \frac{1}{2} L \sum_{k=0}^{n-1} (x_{k+1} - x_k) [u (x_{k+1}) - u (x_k)] \leq \frac{1}{2} L \nu (\Delta_n) [u (b) - u (a)] . \end{aligned} \tag{3.11}$$

The proof follows by Corollary 3 and the details are omitted.

4. Applications for functions of unitary operators

We recall that the bounded linear operator U on the Hilbert space H is unitary iff $U^* = U^{-1}$.

It is well known that (see for instance [5, p. 275-p. 276]), if U is a unitary operator, then there exists a family of projections $\{E_\lambda\}_{\lambda \in [0,2\pi]}$, called the spectral family of U with the following properties

a) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;

b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the identity operator on H);

c) $E_{\lambda+0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;

d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda \in [0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C} (0, 1) \to \mathbb{C}$ on the complex unit circle, we have

$$f (U) = \int_0^{2\pi} f \left(e^{i\lambda} \right) dE_\lambda \tag{4.1}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$f (U) x = \int_0^{2\pi} f \left(e^{i\lambda} \right) dE_\lambda x , \tag{4.2}$$

$$\langle f (U) x, y \rangle = \int_0^{2\pi} f \left(e^{i\lambda} \right) d \langle E_\lambda x, y \rangle \tag{4.3}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2 , \tag{4.4}$$

for any x, $y \in H$.

We consider the following partition of the interval $[a,b]$

$$\Delta_n : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 2\pi$$

and the intermediate points $\xi_k \in [\lambda_k, \lambda_{k+1}]$ where $0 \le k \le n-1$. Define $h_k := \lambda_{k+1} - \lambda_k$, $0 \le k \le n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \le k \le n-1\}$ the norm of the partition Δ_n.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums

$$O_n(f,U,\Delta_n,\xi;x,y) := \sum_{k=0}^{n-1} f\left(e^{i\xi_k}\right)\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x,y\right\rangle \tag{4.5}$$

and

$$T_n(f,U,\Delta_n;x,y) := \sum_{k=0}^{n-1} f\left(e^{\frac{\lambda_{k+1}+\lambda_k}{2}i}\right)\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x,y\right\rangle \tag{4.6}$$

where x, $y \in H$.

Theorem 6. *With the above assumptions for $U, \{E_\lambda\}_{\lambda\in[0,2\pi]}, \Delta_n$ with $\nu(\Delta_n) \le \pi$ and if $f\colon \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the Hölder's type condition $|f(z) - f(w)| \le H|z-w|^r$ for any w, $z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given, then we have the representation*

$$\langle f(U)x,y\rangle = O_n(f,U,\Delta_n,\xi;x,y) + R_n(f,U,\Delta_n,\xi;x,y) \tag{4.7}$$

with the error $R_n(f,U,\Delta_n,\xi;x,y)$ satisfying the bounds

$$\begin{aligned}
&\left|R_n(f,U,\Delta_n,\xi;x,y)\right| \\
&\quad\le 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{4}(\lambda_{k+1}-\lambda_k) + \frac{1}{2}\left|\xi_k - \frac{\lambda_k+\lambda_{k+1}}{2}\right|\right] \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\quad\le 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{2}(\lambda_{k+1}-\lambda_k)\right] \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\quad\le H \sum_{k=0}^{n-1} (\lambda_{k+1}-\lambda_k)^r \bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \le H\nu^r(\Delta_n) \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\quad\le H\nu^r(\Delta_n)\|x\|\|y\| \qquad (4.8)
\end{aligned}$$

for any x, $y \in H$ and the intermediate points $\xi_k \in [\lambda_k, \lambda_{k+1}]$ where $0 \le k \le n-1$.

In particular we have

$$\langle f(U)x,y\rangle = T_n(f,U,\Delta_n;x,y) + E_n(f,U,\Delta_n;x,y) \tag{4.9}$$

with the error

$$
\begin{aligned}
&\left|E_n\left(f,U,\Delta_n;x,y\right)\right| \\
&\leq 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{4}\left(\lambda_{k+1}-\lambda_k\right)\right] \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\leq \frac{1}{2^r} H \sum_{k=0}^{n-1} \left(\lambda_{k+1}-\lambda_k\right)^r \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\leq \frac{1}{2^r} H\nu^r\left(\Delta_n\right) \bigvee_0^{2\pi} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2^r} H\nu^r\left(\Delta_n\right)\|x\|\,\|y\| \qquad (4.10)
\end{aligned}
$$

for any x, $y \in H$.

Proof. For given $x, y \in H$, define the function $u(\lambda) := \langle E_\lambda x, y\rangle$, $\lambda \in [0, 2\pi]$. We will show that u is of bounded variation and

$$\bigvee_0^{2\pi}(u) =: \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \|x\|\,\|y\| \,. \qquad (4.11)$$

It is well known that, if P is a nonnegative selfadjoint operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a *generalization of the Schwarz inequality* in H

$$|\langle Px,y\rangle|^2 \leq \langle Px,x\rangle\,\langle Py,y\rangle \,, \qquad (4.12)$$

for any x, $y \in H$.

Now, if $d\colon 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 2\pi$ is an arbitrary partition of the interval $[0, 2\pi]$, then we have by Schwarz's inequality for nonnegative operators (4.12) that

$$
\begin{aligned}
&\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) = \sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\} \\
&\leq \sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\} := I\,. \qquad (4.13)
\end{aligned}
$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$
\begin{aligned}
I &\leq \sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2} \\
&\leq \sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2} \\
&= \left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,x\right\rangle\right)\right]^{1/2}\left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}y,y\right\rangle\right)\right]^{1/2} = \|x\|\,\|y\| \qquad (4.14)
\end{aligned}
$$

for any x, $y \in H$.

On making use of (4.13) and (4.14) we deduce the desired result (4.11).

Now, applying Proposition 1 to the spectral representation (4.3) we deduce the desired result (4.7) with the error bound (4.8). The details are omitted. □

Remark 8. In the case when the partition reduces to the whole interval $[0, 2\pi]$, then utilizing the inequality (2.7) we deduce the bound

$$\left|f\left(e^{is}\right)\langle x,y\rangle-\langle f(U)x,y\rangle\right|\leq 2^{r}H\bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq 2^{r}H\|x\|\|y\| \tag{4.15}$$

for any $s\in[0,2\pi]$ and any vectors $x,\,y\in H$.

In the case when the division is

$$\Delta_2 : 0=\lambda_0<\lambda_1=\pi<\lambda_2=2\pi$$

and we take the intermediate points $u\in[0,\pi]$ and $v\in[\pi,2\pi]$, then we get from Theorem 6 that

$$\begin{aligned}&\left|f(e^{iu})\langle E_\pi x,y\rangle+f(e^{iv})\langle(I-E_\pi)x,y\rangle-\langle f(U)x,y\rangle\right|\\&\leq 2^{r}H\Big[\sin^{r}\left[\frac{1}{4}\pi+\frac{1}{2}\Big|u-\frac{\pi}{2}\Big|\right]\bigvee_{0}^{\pi}(\langle E_{(\cdot)}x,y\rangle)\\&\qquad+\sin^{r}\left[\frac{1}{4}\pi+\frac{1}{2}\Big|v-\frac{3\pi}{2}\Big|\right]\bigvee_{\pi}^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right)\Big]\end{aligned} \tag{4.16}$$

for any vectors $x,\,y\in H$.

The best inequality we can get from (4.17) is obtained for $u=\frac{\pi}{2}$ and $v=\frac{3\pi}{2}$, namely

$$\begin{aligned}&\left|f(i)\langle E_\pi x,y\rangle+f(-i)\langle(1_H-E_\pi)x,y\rangle-\langle f(U)x,y\rangle\right|\\&\leq 2^{\frac{r}{2}}H\bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq 2^{\frac{r}{2}}H\|x\|\|y\|\end{aligned} \tag{4.17}$$

for any vectors $x,\,y\in H$.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums depending only of one vector $x\in H$

$$\tilde{O}_n(f,U,\Delta_n,\xi;x):=\sum_{k=0}^{n-1}f\left(e^{i\xi_k}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \tag{4.18}$$

and

$$\tilde{T}_n(f,U,\Delta_n;x,y):=\sum_{k=0}^{n-1}f\left(e^{\frac{\lambda_{k+1}+\lambda_k}{2}i}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle . \tag{4.19}$$

Theorem 7. *With the above assumptions for* $U,\{E_\lambda\}_{\lambda\in[0,2\pi]},\Delta_n$ *with* $\nu(\Delta_n)\leq\pi$ *and, if* $f:\mathcal{C}(0,1)\to\mathbb{C}$ *is Lipschitzian with the constant* $L>0$ *on the circle* $\mathcal{C}(0,1)$, *then we have the representation*

$$\langle f(U)x,x\rangle = \tilde{O}_n(f,U,\Delta_n,\xi;x) + \tilde{R}_n(f,U,\Delta_n,\xi;x) \tag{4.20}$$

with the error $\tilde{R}_n(f,U,\Delta_n,\xi;x)$ *satisfying the bounds*

$$\begin{aligned}
&\left|\tilde{R}_n(f,U,\Delta_n,\xi;x)\right| \\
&\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{\lambda_{k+1}-\xi_k}{2}\right)\langle E_{\lambda_{k+1}}x,x\rangle - \sin\left(\frac{\xi_k-\lambda_k}{2}\right)\langle E_{\lambda_k}x,x\rangle\right] \\
&\quad + L\sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}}\operatorname{sgn}(\xi_k-t)\cos\left(\frac{\xi_k-t}{2}\right)\langle E_t x,x\rangle\,dt \\
&\le 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{\lambda_{k+1}-\xi_k}{2}\right)\left[\left\langle\left(E_{\lambda_{k+1}}-E_{\xi_k}\right)x,x\right\rangle\right]\right. \\
&\quad \left.+\sin\left(\frac{\xi_k-\lambda_k}{2}\right)\langle(E_{\xi_k}-E_{\lambda_k})x,x\rangle\right] \\
&\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{4}(\lambda_{k+1}-\lambda_k)+\frac{1}{2}\left|\xi_k-\frac{\lambda_k+\lambda_{k+1}}{2}\right|\right]\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&\le 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{2}(\lambda_{k+1}-\lambda_k)\right]\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle
\end{aligned}$$

$$\le L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \le \nu(\Delta_n)L\|x\|^2 \tag{4.21}$$

for any $x\in H$ *and the intermediate points* $\xi_k\in[\lambda_k,\lambda_{k+1}]$ *where* $0\le k\le n-1$.

In particular we have

$$\langle f(U)x,x\rangle = \tilde{T}_n(f,U,\Delta_n;x) + \tilde{E}_n(f,U,\Delta_n;x) \tag{4.22}$$

with the error

$$\begin{aligned}
|\tilde{E}_n(f,U,\Delta_n;x)| &\le 2L\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&\quad + L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\operatorname{sgn}\left(\frac{\lambda_k+\lambda_{k+1}}{2}-t\right)\cos\left(\frac{\frac{\lambda_k+\lambda_{k+1}}{2}-t}{2}\right)\langle E_t x,x\rangle\,dt \\
&\le 2L\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle
\end{aligned}$$

$$\le \frac{1}{2}L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \le \frac{1}{2}L\nu(\Delta_n)\|x\|^2 \tag{4.23}$$

for any $x\in H$.

The proof follows by Proposition 3 applied for the monotonic nondecreasing function $u(t) := \langle E_t x, x\rangle$, $t \in [0, 2\pi]$.

Remark 9. We remark that if the partition reduces to the whole interval $[0, 2\pi]$ then we get from (2.36) that

$$\begin{aligned} \left|f(e^{is})\|x\|^2 - \langle f(U)x, x\rangle\right| &\leq 2L\sin\left(\frac{s}{2}\right)\|x\|^2 \\ &+ L\int_0^{2\pi} \operatorname{sgn}(s-t)\cos\left(\frac{s-t}{2}\right)\langle E_t x, x\rangle\, dt \end{aligned} \tag{4.24}$$

for any $s \in [a, b]$ and $x \in H$.

In particular, we have

$$\begin{aligned} \left|f(-1)\|x\|^2 - \langle f(U)x, x\rangle\right| &\leq \sqrt{2}L\|x\|^2 \\ &+ L\int_0^{2\pi} \operatorname{sgn}(\pi-t)\sin\left(\frac{t}{2}\right)\langle E_t x, x\rangle\, dt \end{aligned} \tag{4.25}$$

for any $x \in H$.

Example 1. In order to provide some simple examples for the inequalities above we choose two complex functions as follows.

a) Consider the power function $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $f(z) = z^m$ where m is a nonzero integer. Then, obviously, for any z, w belonging to the unit circle $\mathcal{C}(0,1)$ we have the inequality

$$|f(z) - f(w)| \leq |m|\,|z-w|$$

which shows that f is Lipschitzian with the constant $L = |m|$ on the circle $\mathcal{C}(0,1)$.

Then from (4.15), we get for any unitary operator U that

$$\left|e^{ims}\langle x, y\rangle - \langle U^m x, y\rangle\right| \leq 2|m| \bigvee_0^{2\pi}\left(\langle E_{(\cdot)}x, y\rangle\right) \leq 2|m|\,\|x\|\,\|y\| \tag{4.26}$$

for any $s \in [0, 2\pi]$ and x, $y \in H$.

Also, from (4.16) and the intermediate points $u \in [0, \pi]$ and $v \in [\pi, 2\pi]$, we have for any unitary operator U

$$\begin{aligned} &\left|e^{imu}\langle E_\pi x, y\rangle + e^{imv}\langle(1_H - E_\pi)x, y\rangle - \langle U^m x, y\rangle\right| \\ &\leq 2|m|\Big[\sin\left[\frac{1}{4}\pi + \frac{1}{2}\left|u - \frac{\pi}{2}\right|\right]\bigvee_0^{\pi}\left(\langle E_{(\cdot)}x, y\rangle\right) \\ &+ \sin\left[\frac{1}{4}\pi + \frac{1}{2}\left|v - \frac{3\pi}{2}\right|\right]\bigvee_\pi^{2\pi}\left(\langle E_{(\cdot)}x, y\rangle\right)\Big] \end{aligned} \tag{4.27}$$

for any vectors x, $y \in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

The best inequality we can get from (4.27) is obtained for $u = \frac{\pi}{2}$ and $v = \frac{3\pi}{2}$, namely

$$\left| i^m \langle E_\pi x, y\rangle + (-i)^m \langle (1_H - E_\pi) x, y\rangle - \langle U^m x, y\rangle \right| \leq \sqrt{2}\,|m| \bigvee_0^{2\pi} \left(\langle E_{(\cdot)} x, y\rangle\right) \leq \sqrt{2}\,|m| \,\|x\|\,\|y\| \,, \tag{4.28}$$

for any vectors x, $y \in H$.

b) For $a \neq \pm 1, 0$ consider the function $f \colon \mathcal{C}(0,1) \to \mathbb{C}$, $f_a(z) = \frac{1}{1-az}$. Observe that

$$|f_a(z) - f_a(w)| = \frac{|a|\,|z-w|}{|1-az|\,|1-aw|} \tag{4.29}$$

for any z, $w \in \mathcal{C}(0,1)$.

If $z = e^{it}$ with $t \in [0, 2\pi]$, then we have

$$\begin{aligned} |1-az|^2 &= 1 - 2a \operatorname{Re}(\bar{z}) + a^2 |z|^2 = 1 - 2a\cos t + a^2 \\ &\geq 1 - 2|a| + a^2 = (1-|a|)^2 \end{aligned}$$

therefore

$$\frac{1}{|1-az|} \leq \frac{1}{|1-|a||} \quad \text{and} \quad \frac{1}{|1-aw|} \leq \frac{1}{|1-|a||} \tag{4.30}$$

for any z, $w \in \mathcal{C}(0,1)$.

Utilising (4.29) and (4.30) we deduce

$$|f_a(z) - f_a(w)| \leq \frac{|a|}{(1-|a|)^2} |z-w| \tag{4.31}$$

for any $z, w \in \mathcal{C}(0,1)$, showing that the function f_a is Lipschitzian with the constant $L_a = \frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

Applying the inequality (4.15), we get for any unitary operator U that

$$\left| (1 - ae^{is})^{-1} \langle x, y\rangle - \left\langle (1_H - aU)^{-1} x, y\right\rangle \right| \leq \frac{2|a|}{(1-|a|)^2} \bigvee_0^{2\pi} \left(\langle E_{(\cdot)} x, y\rangle\right) \leq \frac{2|a|}{(1-|a|)^2} \|x\|\,\|y\| \tag{4.32}$$

for any $s \in [0, 2\pi]$ and x, $y \in H$.

Also, from (4.16) and the intermediate points $u \in [0, \pi]$ and $v \in [\pi, 2\pi]$, we have for any unitary operator U

$$\begin{aligned} &\left| (1 - ae^{iu})^{-1} \langle E_\pi x, y\rangle + (1 - ae^{iv})^{-1} \langle (1_H - E_\pi) x, y\rangle - \left\langle (1_H - aU)^{-1} x, y\right\rangle \right| \\ &\leq \frac{2|a|}{(1-|a|)^2} \Big[\sin\left[\frac{1}{4}\pi + \frac{1}{2}\left|u - \frac{\pi}{2}\right|\right] \bigvee_0^{\pi} \left(\langle E_{(\cdot)} x, y\rangle\right) \\ &\quad + \sin\left[\frac{1}{4}\pi + \frac{1}{2}\left|v - \frac{3\pi}{2}\right|\right] \bigvee_\pi^{2\pi} (\langle E_{(\cdot)} x, y\rangle) \Big] \end{aligned} \tag{4.33}$$

for any vectors $x, y \in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

The best inequality we can get from (4.27) is obtained for $u = \frac{\pi}{2}$ and $v = \frac{3\pi}{2}$, namely

$$\left|(1-ai)^{-1}\langle E_\pi x, y\rangle + (1+ai)^{-1}\langle (1_H - E_\pi)x, y\rangle - \left\langle (1_H - aU)^{-1}x, y\right\rangle\right|$$

$$\leq \frac{\sqrt{2}\,|a|}{(1-|a|)^2}\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{\sqrt{2}\,|a|}{(1-|a|)^2}\|x\|\,\|y\| \tag{4.34}$$

for any vectors $x, y \in H$.

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Mathematics, College of Engineering and Science,
Victoria University, PO Box 14428,
Melbourne City, MC 8001, Australia

School of Computer Science and Applied Mathematics,
University of the Witwatersrand,
Private Bag 3, Johannesburg 2050, South Africa
E-mail: sever.dragomir@vu.edu.au