Kamila Kliś-Garlicka
Hyperreflexivity of bilattices

Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 1, 119–125

Persistent URL: http://dml.cz/dmlcz/144884

Czechoslovak Mathematical Journal, 66 (141) (2016), 119–125

HYPERREFLEXIVITY OF BILATTICES

Kamila Kliś-Garlicka, Kraków

(Received January 19, 2015)

Abstract. The notion of a bilattice was introduced by Shulman. A bilattice is a subspace analogue for a lattice. In this work the definition of hyperreflexivity for bilattices is given and studied. We give some general results concerning this notion. To a given lattice $\mathcal{L}$ we can construct the bilattice $\Sigma_{\mathcal{L}}$. Similarly, having a bilattice Σ we may consider the lattice $\mathcal{L}_{\Sigma}$. In this paper we study the relationship between hyperreflexivity of subspace lattices and of their associated bilattices. Some examples of hyperreflexive or not hyperreflexive bilattices are given.

Keywords: reflexive bilattice; hyperreflexive bilattice; subspace lattice; bilattice

MSC 2010: 47A15, 47L99

1. Introduction

In [2] hyperreflexive subspace lattices were introduced and a number of results about these objects were obtained. Here we attempt to study the hyperreflexivity of bilattices. Bilattices were defined by Shulman in [6]. These structures were studied later in [5] in connection with operator synthesis and in [3] in the context of reflexivity. Let us first recall basic definitions. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $\mathcal{B}(\mathcal{H},\mathcal{K})$ the space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$, $\mathcal{B}(\mathcal{H})=\mathcal{B}(\mathcal{H},\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ the set of all orthogonal projections in $\mathcal{H}$. Given two projections $P,Q\in\mathcal{P}(\mathcal{H})$ we may consider their *meet* $P\wedge Q$ as the projection onto $P(\mathcal{H})\cap Q(\mathcal{H})$, and their *join* $P\vee Q$ as the projection onto the closure of $P(\mathcal{H})+Q(\mathcal{H})$. With those two operations $\mathcal{P}(\mathcal{H})$ is a complete lattice. A sublattice of $\mathcal{P}(\mathcal{H})$ containing the trivial projections 0 and I and SOT-closed is called a *subspace lattice*. For a set of operators $\mathcal{S}\subseteq\mathcal{B}(\mathcal{H})$, we denote $\operatorname{lat}\mathcal{S}=\{P\in\mathcal{P}(\mathcal{H})\colon\ SP=PSP,\ \forall\, S\in\mathcal{S}\}$ and for a family of projections

This research was financed by the Ministry of Science and Higher Education of the Republic of Poland.

$\mathcal{L} \subset \mathcal{P}(\mathcal{H})$ we denote by $\operatorname{alg} \mathcal{L}$ the algebra of all operators leaving invariant the ranges of all projections in $\mathcal{L}$, i.e. $\operatorname{alg} \mathcal{L} = \{A \in \mathcal{B}(\mathcal{H})\colon\ \mathcal{L} \subseteq \operatorname{lat}\{A\}\}$. An operator algebra $\mathcal{A}$ is called *reflexive* if $\mathcal{A} = \operatorname{alg} \operatorname{lat} \mathcal{A}$. On the other hand, a subspace lattice $\mathcal{L}$ is *reflexive* if $\mathcal{L} = \operatorname{lat} \operatorname{alg} \mathcal{L}$.

The *reflexive closure* of a subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ is the set

$$\operatorname{ref} \mathcal{S} = \{T \in \mathcal{B}(\mathcal{H}, \mathcal{K})\colon\ Tx \in \overline{\mathcal{S}x},\ \forall\, x \in \mathcal{H}\}.$$

A subspace $\mathcal{S}$ is called *reflexive* if $\mathcal{S} = \operatorname{ref} \mathcal{S}$.

The notion of hyperreflexivity was first introduced for operator algebras [1] and later extended to operator subspaces [4] and subspace lattices [2]. Hyperreflexivity is stronger than reflexivity. Denote by

$$\alpha(T, \mathcal{S}) = \sup\{\|QTP\|\colon \text{ for projections } P, Q \text{ such that } Q\mathcal{S}P = \{0\}\}.$$

A subspace $\mathcal{S}$ is called *hyperreflexive* if there exists a constant $\kappa > 0$ such that $d(T, \mathcal{S}) \leqslant \kappa\alpha(T, \mathcal{S})$, for all $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Here $d(\cdot, \cdot)$ denotes the distance in the norm metric. Every hyperreflexive subspace is reflexive, but not vice versa.

Let us now recall following [2] the analogues of these for the case of lattices:

$$\alpha(P, \mathcal{L}) = \sup\{\|P^{\perp}AP\|\colon\ A \in (\operatorname{alg} \mathcal{L})_1\},$$

where $(\operatorname{alg} \mathcal{L})_1$ denotes the set of all contractions in $\operatorname{alg} \mathcal{L}$. A subspace lattice $\mathcal{L}$ is called *hyperreflexive* if there exists a constant $\kappa > 0$ such that $d(P, \mathcal{L}) \leqslant \kappa\alpha(P, \mathcal{L})$, for all $P \in \mathcal{P}(\mathcal{H})$. The infimum of such constants κ will be denoted by $\kappa(\mathcal{L})$ and called the *constant of hyperreflexivity* for $\mathcal{L}$. Again every hyperreflexive subspace lattice is reflexive, but not vice versa.

A subspace analogue for a lattice is called a bilattice [6]. Namely, a *bilattice* is a set $\Sigma \subseteq \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{K})$ containing the pairs $(0, I)$, $(I, 0)$, $(0, 0)$ and satisfying $(P_1 \wedge P_2, Q_1 \vee Q_2), (P_1 \vee P_2, Q_1 \wedge Q_2) \in \Sigma$ whenever $(P_1, Q_1), (P_2, Q_2) \in \Sigma$. In this paper we will always regard only SOT-closed bilattices.

We also define analogues of the above notions for bilattices. Define following [5]

$$\operatorname{op} \Sigma = \{T \in \mathcal{B}(\mathcal{H}, \mathcal{K})\colon\ QTP = 0,\ \forall\, (P, Q) \in \Sigma\}.$$

Then $\operatorname{op} \Sigma$ is always a reflexive subspace and all reflexive subspaces are of this form. The bilattice $\operatorname{bil} \mathcal{S}$ of a subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ is defined to be the set

$$\operatorname{bil} \mathcal{S} = \{(P, Q)\colon\ Q\mathcal{S}P = \{0\}\}.$$

A bilattice Σ is called *reflexive* if $\operatorname{bil}\operatorname{op}\Sigma=\Sigma$. Given a bilattice $\Sigma\subseteq\mathcal{P}(\mathcal{H})\times\mathcal{P}(\mathcal{K})$ and a pair of projections $(P,Q)\in\mathcal{P}(\mathcal{H})\times\mathcal{P}(\mathcal{K})$, let

$$\alpha((P,Q),\Sigma)=\sup\{\|QTP\|\colon\ \|T\|\leqslant 1,\ T\in\operatorname{op}\Sigma\}$$

and

$$d((P,Q),\Sigma)=\inf\{\|P-L_1\|+\|Q-L_2\|\colon\ (L_1,L_2)\in\Sigma\}.$$

If $(L_1,L_2)\in\Sigma$ and $T\in\operatorname{op}\Sigma$, $\|T\|\leqslant 1$, then

$$\begin{aligned}\|QTP\|=\|QTP-L_2TL_1\|&\leqslant\|QTP-QTL_1\|+\|QTL_1-L_2TL_1\|\\&\leqslant\|P-L_1\|+\|Q-L_2\|.\end{aligned}$$

Hence $\alpha((P,Q),\Sigma)\leqslant d((P,Q),\Sigma)$.

Definition 1.1. A bilattice $\Sigma\subseteq\mathcal{P}(\mathcal{H})\times\mathcal{P}(\mathcal{K})$ is called *hyperreflexive* if there exists a constant $\kappa>0$ such that $d((P,Q),\Sigma)\leqslant\kappa\alpha((P,Q),\Sigma)$, for each pair $(P,Q)\in\mathcal{P}(\mathcal{H})\times\mathcal{P}(\mathcal{K})$. The infimum of such constants κ will be denoted by $\kappa(\Sigma)$ and called the *constant of hyperreflexivity* for Σ.

2. Results

Let us start with some basic facts.

Proposition 2.1. *For any bilattice $\Sigma\subseteq\mathcal{P}(\mathcal{H})\times\mathcal{P}(\mathcal{K})$ and a pair of projections $(P,Q)\in\mathcal{P}(\mathcal{H})\times\mathcal{P}(\mathcal{K})$ we have that $\alpha((P,Q),\Sigma)=0$ if and only if $(P,Q)\in\operatorname{bil}\operatorname{op}\Sigma$.*

Proof. If $\alpha((P,Q),\Sigma)=0$, then $QTP=0$ for each $T\in\operatorname{op}\Sigma$. Hence $(P,Q)\in\operatorname{bil}\operatorname{op}\Sigma$. The second implication is obvious. □

Proposition 2.2. *If Σ is hyperreflexive, then it is reflexive.*

Proof. Let $(P,Q)\in\operatorname{bil}\operatorname{op}\Sigma$. Then $\alpha((P,Q),\Sigma)=0$ and hyperreflexivity implies that $d((P,Q),\Sigma)=0$. Hence $(P,Q)\in\Sigma$. □

The converse of Proposition 2.2 is not true. The example of a reflexive but not hyperreflexive bilattice is given after the proof of Proposition 2.6.

Note that, given a lattice $\mathcal{L}$, one can form a billatice $\Sigma_{\mathcal{L}}$ by letting

$$\Sigma_{\mathcal{L}}=\{(P,Q)\colon\ \text{there exists } L\in\mathcal{L} \text{ with } P\leqslant L\leqslant Q^{\perp}\}.$$

There is a dual construction as well: given a bilattice Σ, let

$$\mathcal{L}_\Sigma = \{P \oplus Q^\perp \colon\ (P,Q) \in \Sigma\}.$$

To see what is the relationship between hyperreflexivity of lattices and of the bilattices connected with them we will need the following result:

Theorem 2.3. *Let $\mathcal{M}$ be a hyperreflexive subspace lattice with constant a, and let $\mathcal{L}$ be a sublattice of $\mathcal{M}$. If there is a constant $b > 0$ such that*

$$d(M,\mathcal{L}) \leqslant b\alpha(M,\mathcal{L})$$

for all $M \in \mathcal{M}$, then $\mathcal{L}$ is hyperreflexive with constant $\kappa(\mathcal{L}) \leqslant a + b + 2ab$.

P r o o f. Let $P \in \mathcal{P}(\mathcal{H})$. For any $\varepsilon > 0$ there is $M_0 \in \mathcal{M}$ such that

$$\|P - M_0\| \leqslant d(P,\mathcal{M}) + \varepsilon.$$

Since $\mathcal{L} \subset \mathcal{M}$, then $\alpha(P,\mathcal{M}) \leqslant \alpha(P,\mathcal{L})$. Note that for any $T \in (\operatorname{alg}\mathcal{L})_1$ we have

$$\begin{aligned}\|M_0^\perp T M_0\| &\leqslant \|M_0^\perp T M_0 - P^\perp T M_0\| + \|P^\perp T M_0 - P^\perp T P\| + \|P^\perp T P\| \\ &\leqslant \|P^\perp T P\| + 2\|P - M_0\|.\end{aligned}$$

Hence

$$\begin{aligned}\alpha(M_0,\mathcal{L}) &\leqslant \alpha(P,\mathcal{L}) + 2d(P,\mathcal{M}) + 2\varepsilon \leqslant \alpha(P,\mathcal{L}) + 2a\alpha(P,\mathcal{M}) + 2\varepsilon \\ &\leqslant (1+2a)\alpha(P,\mathcal{L}) + 2\varepsilon.\end{aligned}$$

Therefore

$$\begin{aligned}d(P,\mathcal{L}) &\leqslant \|P - M_0\| + d(M_0,\mathcal{L}) \leqslant d(P,\mathcal{M}) + d(M_0,\mathcal{L}) + \varepsilon \\ &\leqslant a\alpha(P,\mathcal{M}) + b\alpha(M_0,\mathcal{L}) + \varepsilon \leqslant a\alpha(P,\mathcal{L}) + b((1+2a)\alpha(P,\mathcal{L}) + 2\varepsilon).\end{aligned}$$

Thus $\mathcal{L}$ is hyperreflexive and $\kappa(\mathcal{L}) \leqslant a + b + 2ab$. □

Proposition 2.4. *If a bilattice Σ is hyperreflexive, then the lattice $\mathcal{L}_\Sigma$ is hyperreflexive.*

Proof. Since $\mathcal{L}_\Sigma \subset \mathcal{P}(\mathcal{H})\oplus\mathcal{P}(\mathcal{H})$ and the lattice $\mathcal{P}(\mathcal{H})\oplus\mathcal{P}(\mathcal{H})$ is hyperreflexive with constant at most 2 (see [2], Theorem 4.1), by Theorem 2.3 it is enough to show that there is $\kappa > 0$ such that for any $P, Q \in \mathcal{P}(\mathcal{H})$

$$d(P\oplus Q, \mathcal{L}_\Sigma) \leqslant \kappa\alpha(P\oplus Q, \mathcal{L}_\Sigma).$$

First, note that by the hyperreflexivity of Σ

$$\begin{aligned} (2.1)\qquad d(P\oplus Q, \mathcal{L}_\Sigma) &= \inf\{\max\{\|P-L_1\|, \|Q-L_2^\perp\|\}\colon\ (L_1,L_2)\in\Sigma\} \\ &\leqslant \inf\{\|P-L_1\| + \|Q^\perp - L_2\|\}\colon\ (L_1,L_2)\in\Sigma\} \\ &= d((P,Q^\perp),\Sigma) \leqslant \kappa(\Sigma)\alpha((P,Q^\perp),\Sigma). \end{aligned}$$

Recall ([3], Proof of Proposition 2.6) that $\operatorname{alg}\mathcal{L}_\Sigma = \begin{pmatrix} \operatorname{alg}\Sigma_l & 0 \\ \operatorname{op}\Sigma & \operatorname{alg}(\Sigma_r)^\perp \end{pmatrix}$, where $\Sigma_l = \{L_1\colon (L_1,0)\in\Sigma\}$ and $\Sigma_r = \{L_2\colon (0,L_2)\in\Sigma\}$. Since Σ is reflexive, by [3], Remark 2.2, $\Sigma_l = \Sigma_r = \mathcal{P}(\mathcal{H})$. Hence in our case, $\operatorname{alg}\mathcal{L}_\Sigma = \begin{pmatrix} \mathbb{C}I & 0 \\ \operatorname{op}\Sigma & \mathbb{C}I \end{pmatrix}$. So any contraction $T\in\operatorname{alg}\mathcal{L}_\Sigma$ has a matrix form $\begin{pmatrix} aI & 0 \\ B & bI \end{pmatrix}$ for some contraction $B\in\operatorname{op}\Sigma$. Hence we have

$$\|(P\oplus Q)^\perp T(P\oplus Q)\| = \|Q^\perp BP\|.$$

Therefore

$$\alpha((P,Q^\perp),\Sigma) = \alpha(P\oplus Q,\mathcal{L}_\Sigma),$$

which together with the inequality (2.1) proves the hyperreflexivity of $\mathcal{L}_\Sigma$. □

To see that hyperreflexivity of $\mathcal{L}_\Sigma$ does not imply hyperreflexivity of Σ we will consider the following example.

Example 2.5. Let $\dim\mathcal{H} > 1$ and take $\Sigma = \{(0,0),(I,0),(0,I)\}\subset\mathcal{P}(\mathcal{H})\times\mathcal{P}(\mathcal{H})$. As it was shown in [3], Example 2.7, the bilattice Σ is not reflexive, hence it cannot be hyperreflexive. We will prove that $\mathcal{L}_\Sigma = \{0\oplus I, I\oplus I, 0\oplus 0\}$ is hyperreflexive. Using Theorem 2.3 and repeating similar reasoning as in the proof of Proposition 2.4 it is enough to calculate the appropriate distances for any projection of the form $P\oplus Q\in(\mathcal{P}(\mathcal{H})\oplus\mathcal{P}(\mathcal{H}))\setminus\mathcal{L}_\Sigma$. Clearly, $d(P\oplus Q,\mathcal{L}_\Sigma) = 1$. Since $\operatorname{alg}\mathcal{L}_\Sigma = \begin{pmatrix} \mathcal{B}(\mathcal{H}) & 0 \\ \mathcal{B}(\mathcal{H}) & \mathcal{B}(\mathcal{H}) \end{pmatrix}$, we have that

$$\alpha(P\oplus Q,\mathcal{L}_\Sigma) = \sup\left\{\left\|\begin{pmatrix} P^\perp AP & 0 \\ Q^\perp BP & Q^\perp CQ \end{pmatrix}\right\|\colon\ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\in(\operatorname{alg}\mathcal{L}_\Sigma)_1\right\}.$$

If $P = I$ and $Q\neq I$, then

$$\alpha(I\oplus Q,\mathcal{L}_\Sigma) \geqslant \sup\{\|Q^\perp B\|\colon\ \|B\|\leqslant 1\} = 1.$$

If $P = 0$, then $Q \neq 0$, $Q \neq I$ and

$$\alpha(0 \oplus Q, \mathcal{L}_\Sigma) \geqslant \sup\{\|Q^\perp C Q\|\colon\ \|C\| \leqslant 1\}.$$

Choose $x, y \in \mathcal{H}$ such that $Qx = x$, $\|x\| = 1$ and $Q^\perp y = y$, $\|y\| = 1$. Define an operator C as the orthogonal projection onto the subspace $\mathbb{C}(x+y)$, then $\|Q^\perp C Q x\| = \|Q^\perp C x\| = \|Q^\perp (x+y)/2\| = \|y/2\| = 1/2$. Hence

$$\alpha(0 \oplus Q, \mathcal{L}_\Sigma) \geqslant \frac{1}{2}.$$

If P is a proper projection, then

$$\alpha(P \oplus Q, \mathcal{L}_\Sigma) \geqslant \sup\{\|P^\perp A P\|\colon\ \|A\| \leqslant 1\}$$

and repeating similar reasoning as before we may prove that the supremum on the right hand side is at least equal to $1/2$.

Hence we obtain that

$$d(P \oplus Q, \mathcal{L}_\Sigma) \leqslant 2\alpha(P \oplus Q, \mathcal{L}_\Sigma).$$

Applying Theorem 2.3 we have proved the hyperreflexivity of $\mathcal{L}_\Sigma$ with constant $\kappa(\mathcal{L}_\Sigma) \leqslant 12$.

Recall following [2] that two projections P, Q are *close* if $\|P - Q\| < 1$.

Proposition 2.6. *Let $\mathcal{L}$ be a subspace lattice. If $\Sigma_{\mathcal{L}}$ is hyperreflexive then $\mathcal{L}$ is hyperreflexive.*

P r o o f. Let $P \in \mathcal{P}(\mathcal{H})$. Then

$$\alpha(P, \mathcal{L}) = \sup\{\|P^\perp T P\|\colon\ T \in (\operatorname{alg} \mathcal{L})_1\}.$$

Since $\operatorname{alg} \mathcal{L} = \operatorname{op} \Sigma_{\mathcal{L}}$ (see [3], Proposition 2.3), we have that

$$\alpha((P, P^\perp), \Sigma_{\mathcal{L}}) = \sup\{\|P^\perp T P\|\colon\ T \in (\operatorname{op} \Sigma_{\mathcal{L}})_1\} = \alpha(P, \mathcal{L}),$$

where $(\operatorname{op} \Sigma_{\mathcal{L}})_1$ denotes the set of all contractions in $\operatorname{op} \Sigma_{\mathcal{L}}$. On the other hand,

$$\begin{aligned} d((P, P^\perp), \Sigma_{\mathcal{L}}) &= \inf\{\|P - E_1\| + \|P^\perp - E_2\|\colon\ (E_1, E_2) \in \Sigma_{\mathcal{L}}\} \\ &= \inf\{\|P - E_1\| + \|P - E_2^\perp\|\colon\ (E_1, E_2) \in \Sigma_{\mathcal{L}}\}. \end{aligned}$$

Note that $E_1 \leqslant E_2^\perp$. If P is close to E_1 and $E_2^\perp$, then by [2], Lemma 2.4, $E_1 = E_2^\perp$ which implies that there is $L \in \mathcal{L}$ such that $E_1 = E_2^\perp = L$. In that case

$$\|P - E_1\| + \|P - E_2^\perp\| = 2\|P - L\| \geqslant d(P, \mathcal{L}).$$

If P is not close to E_1 or $E_2^\perp$, then

$$\|P - E_1\| + \|P - E_2^\perp\| \geqslant 1 \geqslant d(P, \mathcal{L}).$$

Hence by the hyperreflexivity of $\Sigma_{\mathcal{L}}$ we have

$$d(P, \mathcal{L}) \leqslant d((P, P^\perp), \Sigma_{\mathcal{L}}) \leqslant \kappa(\Sigma_{\mathcal{L}})\alpha((P, P^\perp), \Sigma_{\mathcal{L}}) = \kappa(\Sigma_{\mathcal{L}})\alpha(P, \mathcal{L}).$$

□

Example 2.7. In [2], Example 7.2, the authors constructed an example of reflexive but not hyperreflexive subspace lattice $\mathcal{L}$. Namely, for $0 < \vartheta \leqslant \pi/4$ they consider $\mathcal{L}(\vartheta) = \{0, Q_1, Q_2, I\}$, with

$$Q_1 = \begin{pmatrix} (\cos\vartheta)^2 & \sin\vartheta\cos\vartheta \\ \sin\vartheta\cos\vartheta & (\sin\vartheta)^2 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} (\cos\vartheta)^2 & -\sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & (\sin\vartheta)^2 \end{pmatrix}.$$

It is shown that each $\mathcal{L}(\vartheta)$ is hyperreflexive but $\kappa(\mathcal{L}(\vartheta)) \to \infty$ as $\vartheta \to 0$. So the direct sum $\mathcal{L} = \mathcal{L}(\vartheta_1) \oplus \mathcal{L}(\vartheta_2) \oplus \ldots$ is reflexive but not hyperreflexive (by [2], Theorem 7.1), when $0 < \vartheta_n \leqslant \pi/4$ and $\vartheta_n \to 0$.

Consider now the bilattice $\Sigma_{\mathcal{L}}$. Due to [3], Corollary 2.5, we know that $\Sigma_{\mathcal{L}}$ is reflexive but it cannot be hyperreflexive by Proposition 2.6.

Problem 2.8. Is the converse of Proposition 2.6 true?

References

[1] *W. Arveson*: Interpolation problems in nest algebras. J. Funct. Anal. *20* (1975), 208–233.
[2] *K. R. Davidson, K. J. Harrison*: Distance formulae for subspace lattices. J. Lond. Math. Soc., (2) *39* (1989), 309–323.
[3] *K. Kliś-Garlicka*: Reflexivity of bilattices. Czech. Math. J. *63* (2013), 995–1000.
[4] *J. Kraus, D. R. Larson*: Reflexivity and distance formulae. Proc. Lond. Math. Soc. (3) *53* (1986), 340–356.
[5] *V. Shulman, L. Turowska*: Operator synthesis. I. Synthetic sets, bilattices and tensor algebras. J. Funct. Anal. *209* (2004), 293–331.
[6] *V. Shulman*: A review of "Nest Algebras by K. R. Davidson, Longman Sci. and Techn. Pitman Research Notes Math., 1988". Algebra and Analiz *2* (1990), 236–255. http://www.mathnet.ru/links/04e6653e78d90590f32a76de1b827b3b/aa194.pdf.

Author's address: Kamila Kliś-Garlicka, Department of Applied Mathematics, University of Agriculture, Balicka 253c, 30-198, Kraków, Poland, e-mail: rmklis@cyfronet.pl.