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Jacek Dębecki

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LINEAR NATURAL OPERATORS LIFTING p-VECTORS TO TENSORS OF TYPE $(q,0)$ ON WEIL BUNDLES

Jacek Dębecki, Kraków

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Abstract. We give a classification of all linear natural operators transforming p-vectors (i.e., skew-symmetric tensor fields of type $(p,0)$) on n-dimensional manifolds M to tensor fields of type $(q,0)$ on T^AM, where T^A is a Weil bundle, under the condition that $p\geqslant 1$, $n\geqslant p$ and $n\geqslant q$. The main result of the paper states that, roughly speaking, each linear natural operator lifting p-vectors to tensor fields of type $(q,0)$ on T^A is a sum of operators obtained by permuting the indices of the tensor products of linear natural operators lifting p-vectors to tensor fields of type $(p,0)$ on T^A and canonical tensor fields of type $(q-p,0)$ on T^A.

Keywords: natural operator; Weil bundle

MSC 2010: 58A32

1. Introduction

In this paper we give a classification of all linear natural operators transforming skew-symmetric tensor fields of type $(p,0)$ (which we call p-vectors) on n-dimensional manifolds M to tensor fields of type $(q,0)$ on T^AM, where T^A is a Weil bundle, under the condition that $p\geqslant 1$, $n\geqslant p$ and $n\geqslant q$. Similar problems in some special cases were studied earlier by Kolář [7], Grabowski and Urbański [4], and Mikulski [10]. The theorem we prove here generalizes the results of [2] and [1]. The former of the two papers was devoted to the case $q=p$, whereas in the latter canonical tensor fields of type $(p,0)$ on Weil bundles were studied. We now prove that in the general case each linear natural operator lifting p-vectors to tensors of type $(q,0)$ on T^A is a sum of operators obtained by permuting the indices of the tensor products of linear natural operators lifting p-vectors to tensors of type $(p,0)$ on T^A and canonical tensor fields of type $(q-p,0)$ on T^A. Therefore in the general case each natural operator under consideration can by constructed from those described in [2] and [1] by using well known operations on tensors. However, the proof of this fact is much more difficult then the proofs in both the special cases and needs some new ideas.

2. Background on the Weil bundles

For the convenience of the reader we first summarize without proofs some basic information on Weil bundles. As was proved by Eck [3], Kainz and Michor [5] and Luciano [9], every product preserving bundle functor is equivalent to a Weil bundle. A new approach to this matter was presented by Kolář in [6]. We give a brief sketch of this result following the last paper. For a general theory of natural bundles and natural operators we refer the reader to [8].

Let F be a functor which transforms each manifold M into a locally trivial bundle $\pi_M\colon FM \to M$ and each smooth map $f\colon M \to N$ into a smooth map $Ff\colon FM \to FN$ such that $\pi_N \circ Ff = f \circ \pi_M$. We call F a *bundle functor* if for every integer $n \geqslant 0$ and every embedding $f\colon M \to N$ between n-dimensional manifolds Ff is an embedding and $Ff(FM) = {\pi_N}^{-1}(f(M))$. Hence we can identify FU with $\pi_M^{-1}(U)$ for each open subset U of a manifold M. Such F is said to be *product preserving* if for all manifolds M and N the map $(Fp_M, Fp_N)\colon F(M \times N) \to FM \times FN$, where $p_M\colon M \times N \to M$ and $p_N\colon M \times N \to N$ are the projections, is a diffeomorphism. Hence we can identify $F(M \times N)$ with $FM \times FN$.

A *Weil algebra* is, by definition, a finite-dimensional associative and commutative $\mathbb{R}$-algebra A with unit which has an ideal N such that A/N is one-dimensional and $N^{r+1} = 0$ for an integer $r \geqslant 0$. The basic examples are the algebras $\mathbb{D}^r_k$ of r-jets at 0 of smooth functions $\mathbb{R}^k \to \mathbb{R}$. For an arbitrary Weil algebra A there is a surjective algebra homomorphism $\mathbb{D}^r_k \to A$ for some integers $r \geqslant 0$ and $k \geqslant 0$.

Let F be a product preserving bundle functor. Put $A = F\mathbb{R}$. Applying F to the addition and multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in the field $\mathbb{R}$ as well as to multiplying $\mathbb{R} \to \mathbb{R}$ by any real number in $\mathbb{R}$ we obtain an addition and multiplication $A \times A \to A$ in A as well as multiplying $A \to A$ by this real number in A, so A is an $\mathbb{R}$-algebra. In fact, it is a Weil algebra.

Conversely, let A be a Weil algebra and let $p\colon \mathbb{D}^r_k \to A$ be a surjective algebra homomorphism. We say that two smooth maps $\gamma, \delta\colon \mathbb{R}^k \to M$, where M is a manifold, determine the same *A-jet* if $p(j^r_0(\psi \circ \gamma)) = p(j^r_0(\psi \circ \delta))$ for every smooth function $\psi\colon M \to \mathbb{R}$. We will denote by $j^A\gamma$ the A-jet of a smooth map $\gamma\colon \mathbb{R}^k \to M$ and by T^AM the set of A-jets of all such maps. Since every chart $\varphi\colon U \to \mathbb{R}^n$ on M induces the chart $T^AU \ni j^A\gamma \mapsto (p(j^r_0(\varphi^1 \circ \gamma)), \dots, p(j^r_0(\varphi^n \circ \gamma))) \in A^n$ on T^AM, T^AM is a manifold, and so a bundle over M with the projection $T^AM \ni j^A\gamma \mapsto \gamma(0) \in M$. If $f\colon M \to N$ is a smooth map between manifolds then we define $T^Af\colon T^AM \to T^AN$

by $T^A f(j^A\gamma) = j^A(f \circ \gamma)$. The functor T^A is called the *Weil bundle* induced by A. It is a product preserving bundle functor. Though the construction of T^A depends on the choice of p, T^A is unique up to an equivalence.

Therefore we have a Weil algebra for every product preserving bundle functor and a product preserving bundle functor for every Weil algebra. These constructions are inverse to each other if isomorphic algebras and equivalent functors are identified. Thus we have a one-to-one correspondence between product preserving bundle functors and Weil algebras.

It is worth pointing out that the Weil bundle induced by the simplest nontrivial Weil algebra $\mathbb{D}_1^1$ is nothing but the usual tangent bundle functor T.

3. Construction of some natural operators

We now turn to the main subject of the paper.

Fix a Weil algebra A, as well as integers $n \geqslant 0$, $p \geqslant 0$ and $q \geqslant 0$.

Let us denote by $\mathrm{V}^r(M)$, where M is a smooth manifold and $r \geqslant 0$ is an integer, the vector space of all tensor fields of type $(r, 0)$ on M, and by $\mathrm{SV}^r(M)$ the subspace of $\mathrm{V}^r(M)$ consisting of all skew-symmetric tensor fields.

Definition 3.1. A *natural operator lifting p-vectors to tensors of type $(q, 0)$ on T^A* is a system of maps $L_M\colon \mathrm{SV}^p(M) \to \mathrm{V}^q(T^A M)$ indexed by n-dimensional manifolds and satisfying for all such manifolds M, N, every embedding $f\colon M \to N$ and all $t \in \mathrm{SV}^p(M)$ and $u \in \mathrm{SV}^p(N)$ the implication

$$\bigwedge^p Tf \circ t = u \circ f \Longrightarrow \bigotimes^q TT^A f \circ L_M(t) = L_N(u) \circ T^A f. \tag{3.1}$$

Of course, such a natural operator L is called *linear* if the map L_M is linear for each n-dimensional manifold M.

For every integer $r \geqslant 0$, every $k \in \{1, \ldots, r\}$ and every $a \in A$ we have the linear map $Z_a^k\colon \bigotimes^r A \to \bigotimes^r A$ such that

$$Z_a^k(b_1 \otimes \ldots \otimes b_r) = b_1 \otimes \ldots \otimes b_{k-1} \otimes ab_k \otimes b_{k+1} \otimes \ldots \otimes b_r$$

for all $b_1, \ldots, b_r \in A$.

Suppose that $q \geqslant p$. Let $\mathrm{ED}_p^q(A)$ denote the vector space of all $(q-p)$-linear maps $D\colon A \times \ldots \times A \to \bigotimes^q A$ such that

$$Z_a^i \circ D = Z_a^j \circ D \tag{3.2}$$

for all $i, j \in \{1, \dots, p\}$ and every $a \in A$, and

$$\text{(3.3)}\quad \begin{aligned} D(c_{p+1}, \dots, c_{k-1}, ab, c_{k+1}, \dots, c_q) = Z_a^k(D(c_{p+1}, \dots, c_{k-1}, b, c_{k+1}, \dots, c_q)) \\ + Z_b^k(D(c_{p+1}, \dots, c_{k-1}, a, c_{k+1}, \dots, c_q)) \end{aligned}$$

for every $k \in \{p+1, \dots, q\}$ and all $a, b, c_{p+1}, \dots, c_{k-1}, c_{k+1}, \dots, c_q \in A$.

If $p \geqslant 1$, then elements of the vector space $\mathrm{ED}_p^q(A)$ may be multiplied by elements of the algebra A. Indeed, it suffices to take any $k \in \{1, \dots p\}$ and put

$$aD = Z_a^k \circ D$$

for every $a \in A$ and every $D \in \mathrm{ED}_p^q(A)$. By (3.2), it is immaterial which $k \in \{1, \dots, p\}$ we choose. In addition, we see that $\mathrm{ED}_p^q(A)$ is an A-module.

Let $e_1, \dots, e_n$ denote the standard basis of the vector space $\mathbb{R}^n$.

Proposition 3.1. *If $p \geqslant 1$ and $q \geqslant p$, then for every $D \in \mathrm{ED}_p^q(A)$ there is a unique natural operator $\overline{D}$ lifting p-vectors to tensors of type $(q, 0)$ on T^A such that*

$$\text{(3.4)}\quad \overline{D}_U(t)(X) = \sum_{i_1=1}^{n} \dots \sum_{i_q=1}^{n} (T^A t^{i_1 \dots i_p}(X) \cdot D)(X^{i_{p+1}}, \dots, X^{i_q}) \otimes e_{i_1} \otimes \dots \otimes e_{i_q}$$

for every open subset U of $\mathbb{R}^n$, every $t \in \mathrm{SV}^p(U)$ and every $X \in T^A U$.

The right hand side of the above equality needs some explanation. Since $T^A\mathbb{R} = A$ and $t^{i_1 \dots i_p}\colon U \to \mathbb{R}$, we have $T^A t^{i_1 \dots i_p}(X) \in A$ for all $i_1, \dots, i_p \in \{1, \dots, n\}$. Moreover, since $T^A U$ is an open subset of A^n, the tangent bundle $TT^A U$ can be identified with $T^A U \times A^n$. But the isomorphism $A^n \ni X \mapsto \sum_{i=1}^{n} X^i \otimes e_i \in A \otimes \mathbb{R}^n$ enables us to identify A^n with $A \otimes \mathbb{R}^n$, and consequently $\bigotimes^q A^n$ with $\bigotimes^q A \otimes \bigotimes^q \mathbb{R}^n$.

In order to prove the proposition, we first show a lemma.

Suppose now that $q \geqslant p$. Let $\mathrm{E}_p(A)$ denote the subspace of the vector space $\bigotimes^p A$ consisting of the tensors V which for all $i, j \in \{1, \dots, p\}$ and every $a \in A$ satisfy the condition $Z_a^i(V) = Z_a^j(V)$, and let $D_{q-p}(A)$ denote the vector space of all $(q-p)$-linear maps $F\colon A \times \dots \times A \to \bigotimes^{q-p} A$ such that $F(c_{p+1}, \dots, c_{k-1}, ab, c_{k+1}, \dots, c_q) = Z_a^{k-p}(F(c_{p+1}, \dots, c_{k-1}, b, c_{k+1}, \dots, c_q)) + Z_b^{k-p}(F(c_{p+1}, \dots, c_{k-1}, a, c_{k+1}, \dots, c_q))$ for all $a, b, c_{p+1}, \dots, c_{k-1}, c_{k+1}, \dots, c_q \in A$ and every $k \in \{p+1, \dots, q\}$.

Lemma 3.1. *Let* $I\colon \mathrm{E}_p(A)\otimes \mathrm{D}_{q-p}(A)\to \mathrm{ED}^q_p(A)$ *be the unique linear map such that for every* $V\in \mathrm{E}_p(A)$, *every* $F\in \mathrm{D}_{q-p}(A)$ *and all* $a_{p+1},\dots,a_q\in A$

$$I(V\otimes F)(a_{p+1},\dots,a_q)=V\otimes F(a_{p+1},\dots,a_q).$$

Then I *is an isomorphism of vector spaces.*

Proof. Fix a $D\in \mathrm{ED}^q_p(A)$. Let $v_1,\dots,v_m$ be a basis of the vector space A. There are uniquely determined $F_{i_1\dots i_p}\colon A\times\dots\times A\to \bigotimes^{q-p}A$, where $i_1,\dots,i_p\in\{1,\dots,m\}$, such that for all $a_{p+1},\dots,a_q\in A$

$$D(a_{p+1},\dots,a_q)=\sum_{i_1=1}^{m}\dots\sum_{i_p=1}^{m}v_{i_1}\otimes\dots\otimes v_{i_p}\otimes F_{i_1\dots i_p}(a_{p+1},\dots,a_q).$$

From the uniqueness it follows that $F_{i_1\dots i_p}\in \mathrm{D}_{q-p}(A)$ for all $i_1,\dots,i_p\in\{1,\dots,m\}$. Let $F_1,\dots,F_d$ be a basis of the vector space $D_{q-p}(A)$. By the above, there are uniquely determined $V_1,\dots,V_d\in\bigotimes^p A$ such that for all $a_{p+1},\dots,a_q\in A$

$$D(a_{p+1},\dots,a_q)=\sum_{j=1}^{d}V_j\otimes F_j(a_{p+1},\dots,a_q).$$

From the uniqueness it follows that $V_1,\dots,V_d\in \mathrm{E}_p(A)$. Therefore

$$D=I\Big(\sum_{j=1}^{d}V_j\otimes F_j\Big)$$

and I is a surjection. It is also an injection, because of the uniqueness of $V_1,\dots,V_d$, and the lemma follows. □

Proof of Proposition 3.1. Fix a $D\in \mathrm{ED}^q_p(A)$. From what has already been proved, we have $D=I\Big(\sum\limits_{j=1}^{d}V_j\otimes F_j\Big)$, where $F_1,\dots,F_d$ is a basis of the vector space $D_{q-p}(A)$ and $V_1,\dots,V_d\in \mathrm{E}_p(A)$ are uniquely determined. For every n-dimensional manifold M and every $t\in \mathrm{SV}^p(M)$ we put

$$\overline{D}_M(t)=\sum_{j=1}^{d}\overline{V}_j(t)\otimes\overline{F}_j,$$

where $\overline{V}_j$ with $j\in\{1,\dots,d\}$ is the linear natural operator lifting p-vectors to tensors of type $(p,0)$ on T^A induced by V_j in the manner described in [2], and where $\overline{F}_j$ with $j\in\{1,\dots,d\}$ is the canonical tensor of type $(q-p,0)$ on T^A induced by F_j in the manner described in [1]. It is known that for every $j \in \{1, \dots, d\}$, every open subset U of $\mathbb{R}^n$, every $t \in \mathrm{SV}^p(U)$ and every $X \in T^A U$

$$\overline{V}_{j,U}(t)(X) = \sum_{i_1=1}^{n} \dots \sum_{i_p=1}^{n} (T^A t^{i_1 \dots i_p}(X) \cdot V_j) \otimes e_{i_1} \otimes \dots \otimes e_{i_p},$$

$$\overline{F}_{j,U}(X) = \sum_{i_{p+1}=1}^{n} \dots \sum_{i_q=1}^{n} F_j(X^{i_{p+1}}, \dots, X^{i_q}) \otimes e_{i_{p+1}} \otimes \dots \otimes e_{i_q}.$$

Using these formulas it is easily seen that $\overline{D}$ satisfies (3.4). Since we may take as f in (3.1) the inverse of any chart on an n-dimensional manifold, it is obvious that $\overline{D}$ satisfying (3.4) is unique. This proves the proposition. □

4. The main result

Let Ω denote the group of all permutations of the set $\{1, \dots, q\}$ and let $\omega \in \Omega$. For every manifold M we define ω_M to be the linear map $\mathrm{V}^q(M) \to \mathrm{V}^q(M)$ such that

$$\omega_M(V_1 \otimes \dots \otimes V_q) = V_{\omega(1)} \otimes \dots \otimes V_{\omega(q)}$$

for all $V_1, \dots, V_q \in V^1(M)$. Of course, if L is a linear natural operator lifting p-vectors to tensors of type $(q, 0)$ on T^A, then so is the system of maps $\omega_{T^A M} \circ L_M$ indexed by n-dimensional manifolds. We will denote it by $\omega \circ L$.

For all $k_1, \dots, k_p \in \{1, \dots, q\}$ such that $k_1 < \dots < k_p$ we define $\omega_{k_1 \dots k_p}$ to be the permutation of the set $\{1, \dots, q\}$ satisfying $\omega_{k_1 \dots k_p}(1) = k_1$, …, $\omega_{k_1 \dots k_p}(p) = k_p$ and $\omega_{k_1 \dots k_p}(i) < \omega_{k_1 \dots k_p}(j)$ for all $i, j \in \{p+1, \dots, q\}$ such that $i < j$.

We can now formulate our main result.

Theorem 4.1. *Suppose that $p \geqslant 1$, $n \geqslant p$ and $n \geqslant q$. Then for every linear natural operator L lifting p-vectors to tensors of type $(q, 0)$ on T^A there are uniquely determined $D_{k_1 \dots k_p} \in \mathrm{ED}_p^q(A)$, where $k_1, \dots, k_p \in \{1, \dots, q\}$ and $k_1 < \dots < k_p$, such that*

$$L = \sum_{1 \leqslant k_1 < \dots < k_p \leqslant q} \omega_{k_1 \dots k_p}^{-1} \circ \overline{D}_{k_1 \dots k_p}.$$

5. Proof of the main result

The remainder of the paper will be devoted to the proof of this theorem. Throughout the proof, L denotes a linear natural operator lifting p-vectors to tensors of type $(q,0)$ on T^A.

Our proof starts with several lemmas.

Let $n \geqslant p$ and let e be the p-vector on $\mathbb{R}^n$ given by the formula

$$e\colon\ \mathbb{R}^n \ni x \mapsto e_1 \wedge \ldots \wedge e_p \in \bigwedge^p \mathbb{R}^n,$$

where, as usual, $e_1, \ldots, e_n$ stands for the standard basis of the vector space $\mathbb{R}^n$.

Lemma 5.1. *Suppose that $p \geqslant 1$ and $n \geqslant p$. If $L_{\mathbb{R}^n}(e) = 0$, then $L = 0$.*

Proof. The proof of this lemma is similar to that of the analogous lemma in [2].

Let $\alpha_1 \geqslant 0, \ldots, \alpha_n \geqslant 0$ be integers. We first prove that for every $i \in \{0, \ldots, p-1\}$ we have $L_{\mathbb{R}^n}(e_{\alpha,i})|_{T_0^A\mathbb{R}^n} = 0$, where $e_{\alpha,i}\colon\ \mathbb{R}^n \ni x \mapsto (x^1)^{\alpha_1} \ldots (x^i)^{\alpha_i} e_1 \wedge \ldots \wedge e_p \in \bigwedge^p \mathbb{R}^n$. The proof is by induction on i. Let $i \geqslant 1$ and let $g\colon\ (-\varepsilon, \varepsilon) \to \mathbb{R}$, where $\varepsilon > 0$, be an embedding such that $g(0) = 0$ and $g' = 1 + g^{\alpha_i}$. If $L_{\mathbb{R}^n}(e_{\alpha,i-1})|_{T_0^A\mathbb{R}^n} = 0$, then (3.1) with

$$f\colon\ \mathbb{R}^{i-1} \times (-\varepsilon, \varepsilon) \times \mathbb{R}^{n-i} \ni x \mapsto (x^1, \ldots, x^{i-1}, g(x^i), x^{i+1}, \ldots, x^n) \in \mathbb{R}^n,$$

$t = e_{\alpha,i-1}$ and $u = e_{\alpha,i-1} + e_{\alpha,i}$ yields $L_{\mathbb{R}^n}(e_{\alpha,i})|_{T_0^A\mathbb{R}^n} = 0$, as desired. Next, consider $e_\alpha\colon\ \mathbb{R}^n \ni x \mapsto (x^1)^{\alpha_1} \ldots (x^n)^{\alpha_n} e_1 \wedge \ldots \wedge e_p \in \bigwedge^p \mathbb{R}^n$. Let $g\colon\ (-\varepsilon, \varepsilon)^{n-p+1} \to \mathbb{R}$ be such that $g(0) = 0$, $(\partial g/\partial x^p)(x^p, \ldots, x^n) = 1 + g(x^p, \ldots, x^n)^{\alpha_p}(x^{p+1})^{\alpha_{p+1}} \ldots (x^n)^{\alpha_n}$ for every $(x^p, \ldots, x^n) \in (-\varepsilon, \varepsilon)^{n-p+1}$ and that

$$f\colon\ \mathbb{R}^{p-1} \times (-\varepsilon, \varepsilon)^{n-p+1} \ni x \mapsto (x^1, \ldots, x^{p-1}, g(x^p, \ldots, x^n), x^{p+1}, \ldots, x^n) \in \mathbb{R}^n$$

is an embedding. Then (3.1) with the above f, $t = e_{\alpha,p-1}$ and $u = e_{\alpha,p-1} + e_\alpha$ leads to the equality $L_{\mathbb{R}^n}(e_\alpha)|_{T_0^A\mathbb{R}^n} = 0$. Finally, for all $i_1, \ldots, i_p \in \{1, \ldots, n\}$ such that $i_1 < \ldots < i_p$ we consider $e_{\alpha,i_1,\ldots,i_p}\colon\ \mathbb{R}^n \ni x \mapsto (x^1)^{\alpha_1} \ldots (x^n)^{\alpha_n} e_{i_1} \wedge \ldots \wedge e_{i_p} \in \bigwedge^p \mathbb{R}^n$. Let τ be the permutation of the set $\{1, \ldots, n\}$ such that $\tau(1) = i_1, \ldots, \tau(p) = i_p$ and let us denote $\beta_1 = \alpha_{\tau(1)}, \ldots, \beta_n = \alpha_{\tau(n)}$. Then (3.1) with

$$f\colon\ \mathbb{R}^n \ni x \mapsto (x^{\tau^{-1}(1)}, \ldots, x^{\tau^{-1}(n)}) \in \mathbb{R}^n,$$

$t = e_\beta$ and $u = e_{\alpha,i_1,\ldots,i_p}$ leads to the equality $L_{\mathbb{R}^n}(e_{\alpha,i_1,\ldots,i_p})|_{T_0^A\mathbb{R}^n} = 0$.

Obviously, for every smooth $t\colon \mathbb{R}^n \to \bigwedge^p \mathbb{R}^n$ and every integer $r \geqslant 0$ there are polynomials $u_{i_1\dots i_p} \in \mathbb{R}[x^1,\dots,x^n]$ for all $i_1,\dots,i_p \in \{1,\dots,n\}$ such that $i_1 < \dots < i_p$ with the property that $j_0^r t = j_0^r u$, where $u = \sum_{1\leqslant i_1<\dots<i_p\leqslant n} u_{i_1\dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$. But from what has already been proved, we have the equality $L_{\mathbb{R}^n}(u)|_{T_0^A\mathbb{R}^n} = 0$. Therefore the Peetre theorem applied to the operator which maps each smooth $t\colon \mathbb{R}^n \to \bigwedge^p \mathbb{R}^n$ to $\mathbb{R}^n \ni x \mapsto L_{\mathbb{R}^n}(t)(x,y) \in \bigotimes^q A^n$, where y is any point of the standard fibre of the bundle $T^A\mathbb{R}^n \to \mathbb{R}^n$, implies $L_{\mathbb{R}^n}(t)|_{T_0^A\mathbb{R}^n} = 0$ for every smooth $t\colon \mathbb{R}^n \to \bigwedge^p \mathbb{R}^n$.

Now (3.1) with $f\colon \mathbb{R}^n \ni x \mapsto x-c \in \mathbb{R}^n$, where $c \in \mathbb{R}^n$, any smooth $t\colon \mathbb{R}^n \to \bigwedge^p \mathbb{R}^n$ and $u = t \circ f^{-1}$ shows that $L_{\mathbb{R}^n}(t)|_{T_c^A\mathbb{R}^n} = 0$ for every $c \in \mathbb{R}^n$, which proves the lemma. $\square$

If L is a linear natural operator lifting p-vectors to tensors of type $(q,0)$ on T^A, then there are unique smooth functions $B^{i_1\dots i_q}\colon A^n \to \bigotimes^q A$, where $i_1,\dots,i_q \in \{1,\dots,n\}$, such that

$$L_{\mathbb{R}^n}(e)(X) = \sum_{i_1=1}^{n} \dots \sum_{i_q=1}^{n} B^{i_1\dots i_q}(X) \otimes e_{i_1} \otimes \dots \otimes e_{i_q}$$

for every $X \in A^n$. We will call them the *coordinates of* L. On account of Lemma 5.1, L is fully determined by its coordinates, provided $p \geqslant 1$ and $n \geqslant p$, which we assume from now on.

Note that using the coordinates of L we may rewrite the left hand side of the consequent of (3.1) in a more convenient form. Namely, if U is an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^n$ is an embedding, then

$$\bigotimes^q TT^A f(L_{\mathbb{R}^n}(e)(X)) = \sum_{i_1=1}^{n} \dots \sum_{i_q=1}^{n} \sum_{j_1=1}^{n} \dots \sum_{j_q=1}^{n} \Big(Z^1_{T^A \frac{\partial f^{i_1}}{\partial x^{j_1}}(X)} \circ \dots \circ Z^q_{T^A \frac{\partial f^{i_q}}{\partial x^{j_q}}(X)} \Big) (B^{j_1\dots j_q}(X)) \otimes e_{i_1} \otimes \dots \otimes e_{i_q}$$

for every $X \in T^A U$.

Lemma 5.2. *If $\{i_1,\dots,i_q\}$ does not contain $\{1,\dots,p\}$, then $B^{i_1\dots i_q} = 0$. Otherwise there is a unique $(q-p)$-linear map $C^{i_1\dots i_q}\colon A \times \dots \times A \to \bigotimes^q A$ such that for every $X \in A^n$ we have*

$$B^{i_1\dots i_q}(X) = C^{i_1\dots i_q}(X^{j_1},\dots,X^{j_{q-p}}), \tag{5.1}$$

where the sequence $(j_1,\dots,j_{q-p})$ is determined by the conditions $j_1 \leqslant \dots \leqslant j_{q-p}$ and $(1,\dots,p,j_1,\dots,j_{q-p}) = (i_{\sigma(1)},\dots,i_{\sigma(q)})$ for a permutation σ of the set $\{1,\dots,q\}$.

P r o o f. Since L is linear, from (3.1) with $f\colon \mathbb{R}^n \ni x \mapsto (\lambda_1 x^1, \ldots, \lambda_n x^n) \in \mathbb{R}^n$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus \{0\}$, $t = e$ and $u = \lambda_1 \ldots \lambda_p e$ we have

$$(5.2) \qquad \lambda_{i_1} \ldots \lambda_{i_q} B^{i_1 \ldots i_q}(X) = \lambda_1 \ldots \lambda_p B^{i_1 \ldots i_q}(\lambda_1 X^1, \ldots, \lambda_n X^n)$$

for all $i_1, \ldots, i_q \in \{1, \ldots, n\}$ and every $X \in A^n$. By continuity, (5.2) is also true for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. The homogeneous function theorem now gives the assertion of the lemma, and the proof is complete. □

Note that if $q < p$, then Lemmas 5.1 and 5.2 yield $L = 0$, which completes the proof of the theorem in this case. Hence from now on we make the assumption $q \geqslant p$. We will also need the assumption $n \geqslant q$ throughout the rest of the proof.

Let $\omega \in \Omega$. The coordinates of $\omega \circ L$ will be denoted by $B_\omega^{i_1 \ldots i_q}$, where $i_1, \ldots, i_q \in \{1, \ldots, n\}$. We also define ω_A to be the linear map $\bigotimes^q A \to \bigotimes^q A$ such that

$$\omega_A(a_1 \otimes \ldots \otimes a_q) = a_{\omega(1)} \otimes \ldots \otimes a_{\omega(q)}$$

for all $a_1, \ldots, a_q \in A$. It is a simple matter to observe that

$$(5.3) \qquad B_\omega^{i_1 \ldots i_q} = \omega_A \circ B^{i_{\omega^{-1}(1)} \ldots i_{\omega^{-1}(q)}}$$

for all $i_1, \ldots, i_q \in \{1, \ldots, n\}$.

Lemma 5.3. *Suppose $B^{i_1 \ldots i_q} = 0$ for all $i_1, \ldots, i_q \in \{1, \ldots, n\}$ such that for every $k \in \{1, \ldots, p\}$ there is one and only one $l \in \{1, \ldots, q\}$ for which $i_l = k$. Then $L = 0$.*

P r o o f. Let $g_1, \ldots, g_q \in \{1, \ldots, n\}$ be such that there exist integers $r_1, \ldots, r_p$ which satisfy the following conditions:

$$r_1, \ldots, r_p \geqslant 1, \quad r_1 + \ldots + r_p \leqslant q, \quad g_{r_1 + \ldots + r_{s-1} + k} = s$$

for all $s \in \{1, \ldots, p\}$, $k \in \{1, \ldots, r_s\}$, and $p < g_{r_1 + \ldots + r_p + 1} \leqslant \ldots \leqslant g_q$. Since $n \geqslant q$, we can choose $h_1, \ldots, h_q \in \{1, \ldots, n\}$ with the properties that $h_{r_1 + \ldots + r_s} = s$ for every $s \in \{1, \ldots, p\}$, $h_k \neq h_l$ if either $k, l \leqslant r_1 + \ldots + r_p$, $k \neq l$ or $k \leqslant r_1 + \ldots + r_p$, $l > r_1 + \ldots + r_p$, and $h_m = g_m$ if $m > r_1 + \ldots + r_p$. We define the embedding $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$f^s(x) = \begin{cases} \sum\limits_{k=1}^{r_s} x^{h_{r_1 + \ldots + r_{s-1} + k}} & \text{if } s \in \{1, \ldots, p\}, \\ x^s & \text{if } s \in \{p+1, \ldots, n\}. \end{cases}$$

Consider the consequent of (3.1) with the above f, $t = e$ and $u = e$. Comparing the parts of both sides which contain $e_{g_1} \otimes \ldots \otimes e_{g_q}$ and are linear with respect to each variable $X^{h_{r_1+\ldots+r_{s-1}+k}}$ with $s \in \{1, \ldots, p\}$, $k \in \{1, \ldots, r_s - 1\}$ we obtain for every $X \in A^n$

$$\begin{aligned}\sum_{\phi\in\Phi} B^{h_{\phi(1)}\cdots h_{\phi(q)}}(X) = \sum_{\psi_1\in\Psi_1} \cdots \sum_{\psi_p\in\Psi_p} & C^{g_1\cdots g_q}(X^{h_{\psi_1(1)}}, \ldots, X^{h_{\psi_1(r_1-1)}}, \ldots, \\ & X^{h_{\psi_p(r_1+\ldots+r_{p-1}+1)}}, \ldots, \\ & X^{h_{\psi_p(r_1+\ldots+r_p-1)}}, X^{h_{r_1+\ldots+r_p+1}}, \ldots, X^{h_q}),\end{aligned}$$

where Φ is the group of all permutations ϕ of the set $\{1, \ldots, q\}$ satisfying the conditions $\phi\{r_1 + \ldots + r_{s-1} + 1, \ldots, r_1 + \ldots + r_s\} \subset \{r_1 + \ldots + r_{s-1} + 1, \ldots, r_1 + \ldots + r_s\}$ for every $s \in \{1, \ldots, p\}$ and $\phi|\{r_1 + \ldots + r_p + 1, \ldots, q\} = \mathrm{id}_{\{r_1+\ldots+r_p+1,\ldots,q\}}$, whereas Ψ_s for every $s \in \{1, \ldots, p\}$ is the group of all permutations of $\{r_1 + \ldots + r_{s-1} + 1, \ldots, r_1 + \ldots + r_s - 1\}$. Combining this formula with (5.1) yields for every $Y \in A^n$

$$B^{g_1\cdots g_q}(Y) = \frac{1}{(r_1 - 1)! \ldots (r_p - 1)!} \sum_{\phi\in\Phi} B^{h_{\phi(1)}\cdots h_{\phi(q)}}(X) \tag{5.4}$$

where X is an element of the set A^n such that $X^{h_{r_1+\ldots+r_{s-1}+k}} = Y^s$ for all $s \in \{1, \ldots, p\}$, $k \in \{1, \ldots, r_s - 1\}$, and $X^{h_m} = Y^{h_m}$ for every $m \in \{r_1 + \ldots + r_p + 1, \ldots, q\}$.

Let now $i_1, \ldots, i_q \in \{1, \ldots, n\}$ be such that $\{1, \ldots, p\} \subset \{i_1, \ldots, i_q\}$. Then there are $g_1, \ldots, g_q \in \{1, \ldots, n\}$ such that there exist integers $r_1, \ldots, r_p$ which satisfy the following conditions: $r_1, \ldots, r_p \geqslant 1$, $r_1 + \ldots + r_p \leqslant q$, $g_{r_1+\ldots+r_{s-1}+k} = s$ for all $s \in \{1, \ldots, p\}$, $k \in \{1, \ldots, r_s\}$, and $p < g_{r_1+\ldots+r_p+1} \leqslant \ldots \leqslant g_q$, as well as an $\omega \in \Omega$ such that $g_k = i_{\omega(k)}$ for every $k \in \{1, \ldots, q\}$. Applying (5.4) to $\omega \circ L$ instead of L and using (5.3) we obtain for every $Y \in A^n$

$$\begin{aligned}B^{i_1\cdots i_q}(Y) &= \omega_A^{-1}(B_\omega^{g_1\cdots g_q}(Y)) \\ &= \frac{1}{(r_1 - 1)! \ldots (r_p - 1)!} \sum_{\phi\in\Phi} \omega_A^{-1}(B_\omega^{h_{\phi(1)}\cdots h_{\phi(q)}}(X)) \\ &= \frac{1}{(r_1 - 1)! \ldots (r_p - 1)!} \sum_{\phi\in\Phi} B^{h_{\phi(\omega^{-1}(1))}\cdots h_{\phi(\omega^{-1}(q))}}(X),\end{aligned}$$

where $h_1, \ldots, h_q$ and X are chosen for $g_1, \ldots, g_q$ and Y in the same manner as in (5.4). But for every $\phi \in \Phi$ and every $k \in \{1 \ldots, p\}$ there is one and only one $l \in \{1, \ldots, q\}$ for which $h_{\phi(\omega^{-1}(l))} = k$, hence $B^{h_{\phi(\omega^{-1}(1))}\cdots h_{\phi(\omega^{-1}(q))}} = 0$. Consequently $B^{i_1\cdots i_q} = 0$ for all $i_1, \ldots, i_q \in \{1, \ldots, n\}$ such that $\{1, \ldots, p\} \subset \{i_1, \ldots, i_q\}$. This means that $L = 0$ on account of Lemmas 5.1 and 5.2, and the proof is complete. $\square$

Lemma 5.4. *If* $B^{\omega^{-1}(1)\dots\omega^{-1}(q)} = 0$ *for every* $\omega \in \Omega$, *then* $L = 0$.

P r o o f. We first show that if $i_1, \dots, i_q \in \{1, \dots, n\}$ are such that $i_k = k$ for $k \leqslant p$ and $i_k > p$ for $k > p$, then

$$B^{i_1\dots i_q}(X) = C^{1\dots q}(X^{i_{p+1}}, \dots, X^{i_q}) \tag{5.5}$$

for every $X \in A^n$. The proof of (5.5) is by induction on the number $N(i_{p+1}, \dots, i_q)$ of the elements of the set $\{i_{p+1}, \dots, i_q\}$. We fix $g_1, \dots, g_q \in \{1, \dots, n\}$ such that $g_k = k$ for $k \leqslant p$ and $g_k > p$ for $k > p$, and suppose (5.5) holds whenever $N(i_{p+1}, \dots, i_q) > N(g_{p+1}, \dots, g_q)$. Let $R \subset \{p+1, \dots, q\}$ be such that for each $k \in \{g_{p+1}, \dots, g_q\}$ there is one and only one $l \in R$ such that $k = g_l$. Next, let $h_1, \dots, h_n \in \{1, \dots, n\}$ be such that $h_m = g_m$ for every $m \in \{1, \dots, p\} \cup R$ and $h_k \neq h_l$ for all $k, l \in \{1, \dots, n\}$ such that $k \neq l$. Put

$$S_m = \begin{cases} \{h_m\} & \text{if } m \in \{1, \dots, p\} \cup R \cup \{q+1, \dots, n\}, \\ \{g_m, h_m\} & \text{if } m \in \{p+1, \dots, q\} \setminus R, \end{cases}$$

and define

$$f\colon\ \mathbb{R}^n \ni x \mapsto \Bigl(\sum_{s_1 \in S_1} x^{s_1}, \dots, \sum_{s_n \in S_n} x^{s_n}\Bigr) \in \mathbb{R}^n.$$

Consider the consequent of (3.1) with this f, $t = e$ and $u = e$. Comparing the parts of both sides which contain $e_1 \otimes \ldots \otimes e_q$ we obtain $\sum\limits_{s_1 \in S_1} \dots \sum\limits_{s_q \in S_q} B^{s_1\dots s_q}(X) = B^{1\dots q}\Bigl(\sum\limits_{s_1 \in S_1} X^{s_1}, \dots, \sum\limits_{s_n \in S_n} X^{s_n}\Bigr)$ for every $X \in A^n$. This may be rewritten as

$$\sum_{s_{p+1} \in S_{p+1}} \dots \sum_{s_q \in S_q} B^{1\dots p s_{p+1}\dots s_q}(X) = \sum_{s_{p+1} \in S_{p+1}} \dots \sum_{s_q \in S_q} C^{1\dots q}(X^{s_{p+1}}, \dots, X^{s_q}). \tag{5.6}$$

But if $s_{p+1} \in S_{p+1}, \dots, s_q \in S_q$ are such that there exists $r \in \{p+1, \dots, q\} \setminus R$ with the property that $s_r = h_r$, then $B^{1\dots p s_{p+1}\dots s_q}(X) = C^{1\dots q}(X^{s_{p+1}}, \dots, X^{s_q})$ on account of our assumption, because we have $N(s_{p+1}, \dots, s_q) > N(g_{p+1}, \dots, g_q)$. Subtracting all terms with such indices $s_{p+1}, \dots, s_q$ from each side of (5.6) gives the equality $B^{g_1\dots g_q}(X) = C^{1\dots q}(X^{g_{p+1}}, \dots, X^{g_q})$, which completes the proof of (5.5).

Let now $i_1, \dots, i_q \in \{1, \dots, n\}$ be such that for every $k \in \{1, \dots, p\}$ there is one and only one $l \in \{1, \dots, q\}$ for which $i_l = k$. There are $g_1, \dots, g_q \in \{1, \dots, n\}$ such that $g_k = k$ for $k \leqslant p$ and $g_k > p$ for $k > p$, as well as $\omega \in \Omega$ such that $g_k = i_{\omega(k)}$ for every $k \in \{1, \dots, q\}$. Applying (5.5) to $\omega \circ L$ instead of L and using (5.3) we obtain for every $X \in A^n$

$$\begin{aligned} B^{i_1\dots i_q}(X) &= \omega_A^{-1}(B_\omega^{g_1\dots g_q}(X)) = \omega_A^{-1}(C_\omega^{1\dots q}(X^{g_{p+1}}, \dots, X^{g_q})) = \omega_A^{-1}(B_\omega^{1\dots q}(Y)) \\ &= B^{\omega^{-1}(1)\dots\omega^{-1}(q)}(Y), \end{aligned}$$

where Y is an element of the set A^n such that $Y^{p+1} = X^{g_{p+1}}, \dots, Y^q = X^{g_q}$. This means that $L = 0$ on account of Lemma 5.3 and the proof is complete. $\square$

Lemma 5.5. *If $B^{\omega^{-1}_{k_1\dots k_p}(1)\dots\omega^{-1}_{k_1\dots k_p}(q)} = 0$ for all $k_1, \dots, k_p \in \{1, \dots, q\}$ such that $k_1 < \dots < k_p$, then $L = 0$.*

P r o o f. Let ω be an arbitrary permutation of the set $\{1, \dots, q\}$. Then there are $k_1, \dots, k_p \in \{1, \dots, q\}$ such that $k_1 < \dots < k_p$ and the permutations σ and τ of the sets $\{1, \dots, p\}$ and $\{p+1, \dots, q\}$, respectively, such that $\omega = \omega_{k_1\dots k_p} \circ (\sigma \cup \tau)^{-1}$, where

$$(\sigma \cup \tau)(m) = \begin{cases} \sigma(m) & \text{if } m \in \{1, \dots, p\}, \\ \tau(m) & \text{if } m \in \{p+1, \dots, q\}. \end{cases}$$

Put

$$f\colon \mathbb{R}^n \ni x \mapsto (x^{\sigma(1)}, \dots, x^{\sigma(p)}, x^{\tau(p+1)}, \dots, x^{\tau(q)}, x^{q+1}, \dots, x^n) \in \mathbb{R}^n.$$

Consider the consequent of (3.1) with this f, $t = e$ and $u = \operatorname{sgn} \sigma e$. Comparing the parts of both sides which contain $e_{\omega^{-1}_{k_1\dots k_p}(1)} \otimes \dots \otimes e_{\omega^{-1}_{k_1\dots k_p}(q)}$ we obtain

$$B^{\omega^{-1}(1)\dots\omega^{-1}(q)}(X) = \operatorname{sgn} \sigma B^{\omega^{-1}_{k_1\dots k_p}(1)\dots\omega^{-1}_{k_1\dots k_p}(q)}((T^A f)(X))$$

for every $X \in A^n$. But $B^{\omega^{-1}_{k_1\dots k_p}(1)\dots\omega^{-1}_{k_1\dots k_p}(q)} = 0$, so using Lemma 5.4 completes the proof. $\square$

P r o o f of Theorem 4.1. For every $D \in \mathrm{ED}^q_p(A)$ and every $X \in A^n$ we have

$$\overline{D}_{\mathbb{R}^n}(e)(X) = \sum_{\phi\in\Phi} \sum_{i_{p+1}=1}^{n} \dots \sum_{i_q=1}^{n} \operatorname{sgn} \phi D(X^{i_{p+1}}, \dots, X^{i_q}) \otimes e_{\phi(1)} \otimes \dots \otimes e_{\phi(p)} \otimes e_{i_{p+1}} \otimes \dots \otimes e_{i_q},$$

where Φ is the group of permutations of $\{1, \dots, p\}$. From this formula we see that for all $k_1, \dots, k_p, l_1, \dots, l_p \in \{1, \dots, q\}$ such that $k_1 < \dots < k_p$ and $l_1 < \dots < l_p$ the coordinate of $\omega^{-1}_{k_1\dots k_p} \circ \overline{D}$ at $e_{\omega^{-1}_{l_1\dots l_p}(1)} \otimes \dots \otimes e_{\omega^{-1}_{l_1\dots l_p}(q)}$ equals either

$$(\omega^{-1}_{l_1\dots l_q})_A(D(X^{p+1}, \dots, X^q)) \quad \text{if } (k_1, \dots, k_p) = (l_1, \dots, l_p)$$

or 0 if $(k_1, \dots, k_p) \neq (l_1, \dots, l_p)$. It follows immediately that if $D_{k_1\dots k_p} \in \mathrm{ED}^q_p(A)$ for all $k_1, \dots, k_p \in \{1, \dots, q\}$ such that $k_1 < \dots < k_p$, then the coordinate of $\sum\limits_{1\leqslant k_1<\dots<k_p\leqslant q} \omega^{-1}_{k_1\dots k_p} \circ \overline{D}_{k_1\dots k_p}$ at $e_{\omega^{-1}_{l_1\dots l_p}(1)} \otimes \dots \otimes e_{\omega^{-1}_{l_1\dots l_p}(q)}$ is equal to

$$(\omega^{-1}_{l_1\dots l_q})_A(D_{l_1\dots l_p}(X^{p+1}, \dots, X^q)).$$

For all $l_1, \ldots, l_p \in \{1, \ldots, q\}$ such that $l_1 < \ldots < l_p$ put

$$D_{l_1 \ldots l_p} = (\omega_{l_1 \ldots l_p})_A \circ C^{\omega^{-1}_{l_1 \ldots l_p}(1) \ldots \omega^{-1}_{l_1 \ldots l_p}(q)},$$

where $C^{\omega^{-1}_{l_1 \ldots l_p}(1) \ldots \omega^{-1}_{l_1 \ldots l_p}(q)}$ is defined by the coordinate $B^{\omega^{-1}_{l_1 \ldots l_p}(1) \ldots \omega^{-1}_{l_1 \ldots l_p}(q)}$ of L as in (5.1).

By the above, the proof will be completed as soon as we can show that

$$(\omega_{l_1 \ldots l_p})_A \circ C^{\omega^{-1}_{l_1 \ldots l_p}(1) \ldots \omega^{-1}_{l_1 \ldots l_p}(q)} \in \mathrm{ED}^q_p(A) \tag{5.7}$$

for all $l_1, \ldots, l_p \in \{1, \ldots, q\}$ such that $l_1 < \ldots < l_p$. Indeed, if we apply Lemma 5.5 to $L - \sum_{1 \leqslant k_1 < \ldots < k_p \leqslant q} \omega^{-1}_{k_1 \ldots k_p} \circ \overline{D}_{k_1 \ldots k_p}$ instead of L in this case, we obtain the desired equality $L - \sum_{1 \leqslant k_1 < \ldots < k_p \leqslant q} \omega^{-1}_{k_1 \ldots k_p} \circ \overline{D}_{k_1 \ldots k_p} = 0$.

Therefore it remains to prove (5.7).

Let $i, j \in \{1, \ldots, p\}$ be such that $i < j$. Put

$$f\colon\ \mathbb{R}^n \ni x \mapsto \Big(x^1, \ldots, x^{i-1}, x^i + \frac{(x^j)^2}{2}, x^{i+1}, \ldots, x^n\Big) \in \mathbb{R}^n.$$

Consider the consequent of (3.1) with this f, $t = e$, $u = e$. Comparing the parts of both sides which contain

$$e_{\omega^{-1}_{l_1 \ldots l_p}(1)} \otimes \ldots \otimes e_{\omega^{-1}_{l_1 \ldots l_p}(l_j - 1)} \otimes e_{\omega^{-1}_{l_1 \ldots l_p}(l_i)} \otimes e_{\omega^{-1}_{l_1 \ldots l_p}(l_j + 1)} \otimes \ldots \otimes e_{\omega^{-1}_{l_1 \ldots l_p}(q)}$$

we obtain for every $X \in A^n$

$$\begin{aligned} &Z^{l_j}_{X^j} \circ C^{\omega^{-1}_{\boldsymbol{l}}(1) \ldots \omega^{-1}_{\boldsymbol{l}}(q)} \\ &\quad + Z^{l_i}_{X^j} \circ C^{\omega^{-1}_{\boldsymbol{l}}(1) \ldots \omega^{-1}_{\boldsymbol{l}}(l_i - 1)\omega^{-1}_{\boldsymbol{l}}(l_j)\omega^{-1}_{\boldsymbol{l}}(l_i + 1) \ldots \omega^{-1}_{\boldsymbol{l}}(l_j - 1)\omega^{-1}_{\boldsymbol{l}}(l_i)\omega^{-1}_{\boldsymbol{l}}(l_j + 1) \ldots \omega^{-1}_{\boldsymbol{l}}(q)} = 0 \end{aligned}$$

where $\boldsymbol{l} = l_1 \ldots l_p$. Consider the consequent of (3.1) with

$$f\colon\ \mathbb{R}^n \ni x \mapsto (x^1, \ldots, x^{i-1}, x^j, x^{i-1}, \ldots, x^{j-1}, x^i, x^{j+1}, \ldots, x^n) \in \mathbb{R}^n,$$

$t = e$, $u = -e$. Comparing the parts of both sides which contain $e_{\omega^{-1}_{l_1 \ldots l_p}(1)} \otimes \ldots \otimes e_{\omega^{-1}_{l_1 \ldots l_p}(q)}$ we obtain

$$\begin{aligned} &C^{\omega^{-1}_{l_1 \ldots l_p}(1) \ldots \omega^{-1}_{l_1 \ldots l_p}(l_i - 1)\omega^{-1}_{l_1 \ldots l_p}(l_j)\omega^{-1}_{l_1 \ldots l_p}(l_i + 1) \ldots \omega^{-1}_{l_1 \ldots l_p}(l_j - 1)\omega^{-1}_{l_1 \ldots l_p}(l_i)\omega^{-1}_{l_1 \ldots l_p}(l_j + 1) \ldots \omega^{-1}_{l_1 \ldots l_p}(q)} \\ &\quad = -C^{\omega^{-1}_{l_1 \ldots l_p}(1) \ldots \omega^{-1}_{l_1 \ldots l_p}(q)}. \end{aligned}$$

Therefore

$$Z^{l_j}_{X^j} \circ C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)} = Z^{l_i}_{X^j} \circ C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)}.$$

Combining the both sides of this equality with $(\omega_{l_1\dots l_p})_A$ yields

$$(5.8)\quad Z^{j}_{X^j} \circ (\omega_{l_1\dots l_p})_A \circ C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)} = Z^{i}_{X^j} \circ (\omega_{l_1\dots l_p})_A \circ C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)}.$$

Since X^j in (5.8) may be any element of A, $(\omega_{l_1\dots l_p})_A \circ C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)}$ satisfies (3.2).

Let now $k \in \{p+1,\dots,q\}$. Put $U = \{x \in \mathbb{R}^n\colon\ x^k > 0\}$ and

$$f\colon\ U \ni x \mapsto \Big(x^1,\dots,x^{k-1},\frac{(x^k)^2}{2},x^{k+1},\dots,x^n\Big) \in \mathbb{R}^n.$$

Consider the consequent of (3.1) with this f, $t = e$ and $u = e$. Comparing the parts of both sides which contain $e_{\omega^{-1}_{l_1\dots l_p}(1)} \otimes \dots \otimes e_{\omega^{-1}_{l_1\dots l_p}(q)}$ we obtain for every $X \in T^A U$

$$(5.9)\quad Z^{\omega_{l_1\dots l_p}(k)}_{X^k}(C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)}(X^{p+1},\dots,X^q)) = C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)}\Big(X^{p+1},\dots,X^{k-1},\frac{(X^k)^2}{2},X^{k+1},\dots,X^q\Big).$$

In the same manner, with U replaced by $\{x \in \mathbb{R}^n\colon\ x^k < 0\}$, we see that (5.9) also holds for every $X \in T^A\{x \in \mathbb{R}^n\colon\ x^k < 0\}$, and so, by continuity, for every $X \in A^n$. Combining the both sides of (5.9) with $(\omega_{l_1\dots l_p})_A$ yields

$$(5.10)\quad Z^{k}_{X^k}((\omega_{l_1\dots l_p})_A(C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)}(X^{p+1},\dots,X^q))) = (\omega_{l_1\dots l_p})_A\Big(C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)}\Big(X^{p+1},\dots,X^{k-1},\frac{(X^k)^2}{2},X^{k+1},\dots,X^q\Big)\Big).$$

Now the polarization of (5.10) with respect to X^k leads to the conclusion that $(\omega_{l_1\dots l_p})_A \circ C^{\omega^{-1}_{l_1\dots l_p}(1)\dots\omega^{-1}_{l_1\dots l_p}(q)}$ satisfies (3.3). This completes the proof. □

References

[1] *J. Dębecki*: Canonical tensor fields of type $(p, 0)$ on Weil bundles. Ann. Pol. Math. *88* (2006), 271–278.

[2] *J. Dębecki*: Linear liftings of skew-symmetric tensor fields to Weil bundles. Czech. Math. J. *55* (2005), 809–816.

[3] *D. J. Eck*: Product-preserving functors on smooth manifolds. J. Pure Appl. Algebra *42* (1986), 133–140.

[4] *J. Grabowski, P. Urbański*: Tangent lifts of Poisson and related structures. J. Phys. A, Math. Gen. *28* (1995), 6743–6777.

[5] *G. Kainz, P. W. Michor*: Natural transformations in differential geometry. Czech. Math. J. *37* (1987), 584–607.

[6] *I. Kolář*: Weil Bundles as Generalized Jet Spaces. Handbook of Global Analysis (D. Krupka et al., eds.). Elsevier, Amsterdam, 2008, pp. 625–664.

[7] *I. Kolář*: On the natural operators on vector fields. Ann. Global Anal. Geom. *6* (1988), 109–117.

[8] *I. Kolář, P. W. Michor, J. Slovák*: Natural Operations in Differential Geometry. (corrected electronic version), Springer, Berlin, 1993.

[9] *O. O. Luciano*: Categories of multiplicative functors and Weil's infinitely near points. Nagoya Math. J. *109* (1988), 69–89.

[10] *W. M. Mikulski*: The linear natural operators lifting 2-vector fields to some Weil bundles. Note Mat. *19* (1999), 213–217.

Author's address: Jacek Dębecki, Instytut Matematyki Uniwersytetu Jagiellońskiego, ul. Łojasiewicza 6, 30-348 Kraków, Poland, e-mail: `jacek.debecki@im.uj.edu.pl`.