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PARALLEL AND TOTALLY GEODESIC HYPERSURFACES OF 5-DIMENSIONAL 2-STEP HOMOGENEOUS NILMANIFOLDS

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Abstract. In this paper we study parallel and totally geodesic hypersurfaces of two-step homogeneous nilmanifolds of dimension five. We give the complete classification and explicitly describe parallel and totally geodesic hypersurfaces of these spaces. Moreover, we prove that two-step homogeneous nilmanifolds of dimension five which have one-dimensional centre never admit parallel hypersurfaces. Also we prove that the only two-step homogeneous nilmanifolds of dimension five which admit totally geodesic hypersurfaces have three-dimensional centre.

Keywords: hypersurface; totally geodesic hypersurface; parallel geodesic hypersurfaces; two-step homogeneous nilmanifold

MSC 2010: 53C42, 53C30

1. Introduction

Parallel submanifolds play an important role in geometry and general relativity. A submanifold is called parallel if its second fundamental form is covariantly constant and it is called totally geodesic if its second fundamental form vanishes identically. Consequently, parallel submanifolds can be considered as a natural extension of totally geodesic submanifolds.

A classification of totally geodesic and parallel submanifolds of a certain manifold helps to enrich our knowledge and understanding of the geometry of this manifold (see [14]). Therefore, the classification of these submanifolds is very important and attracts special attention of geometers. For example, parallel and totally geodesic submanifolds of some (pseudo)-Riemannian real space forms with different (indexes and) dimensions have been classified by several authors (see for instance [11], [10], [18], [20]).

Homogeneous spaces, which are manifolds for which the action of the isometry group is transitive, generalize spaces of constant sectional curvature [12] (see also [3], [1], [2], [5], [6], [16] for some examples of these spaces). Thus it is interesting to choose these spaces as ambient spaces. Concerning this choice, the following results are known. Complete classification of parallel and totally geodesic surfaces in all three-dimensional Riemannian homogeneous spaces is given in [4], [15], [14]. Parallel and totally geodesic surfaces in three-dimensional Lorentzian homogeneous spaces are classified in [9], [8]. In [7], complete classification of parallel hypersurfaces of four-dimensional oscillator groups, equipped with a one-parameter family of left-invariant Lorentzian metrics, is given. In [12], totally geodesic hypersurfaces of four-dimensional generalized symmetric spaces are classified.

Hence, a natural problem is now to give a classification of parallel and totally geodesic hypersurfaces of homogeneous spaces with dimension 5. In the present paper, we deal with this problem for two-step homogeneous nilmanifolds of dimension 5.

A two-step homogeneous nilmanifold is a two-step nilpotent Lie group N which is equipped with a left-invariant metric g (see [13]). These spaces play an important role in Lie groups, geometrical analysis and mathematical physics. All homogeneous nilmanifolds (not necessary two-step) of dimension three and four, up to isometry, are classified in [17]. In [13], the classification of two-step homogeneous nilmanifolds of dimension five is given, and Randers metrics of Berwald type on these spaces have been studied in [19]. Our aim in the present paper is to classify parallel and totally geodesic hypersurfaces of these spaces.

The structure of the paper is as follows. In Section 2 we report the classification and the curvature tensor of two-step homogeneous nilmanifolds of dimension five. In Section 3 we first recall some facts and definitions about parallel and totally geodesic hypersurfaces. Then we give the complete classification of parallel and totally geodesic hypersurfaces of these spaces and describe some of the results of this classification which are related to the dimension of these spaces.

2. Two-step homogeneous nilmanifolds of dimension five

Let N be a simply connected two-step nilpotent Lie group of dimension five and $\mathcal{N}$ be its corresponding Lie algebra. We denote by $\langle\,,\rangle$ an inner product on $\mathcal{N}$ which is induced by a left invariant Riemannian metric g on N and we adopt the following conventions for the curvature tensor R:

$$R(X_i, X_j) = \nabla_{[X_i,X_j]} - [\nabla_{X_i}, \nabla_{X_j}], \quad R_{ijkt} = \langle R(X_i, X_j)X_k, X_t\rangle,$$

where ∇ is the Levi-Cività connection and X_i, X_j, X_k, X_t are left-invariant vector fields on N.

Here we recall the classification of simply connected two-step homogeneous nilmanifolds of dimension 5 which is given in [13], and their invariant curvatures which are explicitly given in [19].

(A_1) Lie algebras with 1-dimensional centre: For this type there exists an orthonormal basis $\{X_1, \dots, X_5\}$ of $\mathcal{N}$ such that the nonzero brackets are

$$[X_1, X_2] = \lambda X_5, \quad [X_3, X_4] = \mu X_5,$$

where $\lambda \geqslant \mu > 0$ and $\{X_5\}$ is a basis for the centre of $\mathcal{N}$. The nonzero connection components and the nonzero curvature components are given by

$$\begin{aligned}
&\nabla_{X_1} X_2 = -\nabla_{X_2} X_1 = \frac{\lambda}{2} X_5, \quad && \nabla_{X_1} X_5 = \nabla_{X_5} X_1 = \frac{-\lambda}{2} X_2,\\
&\nabla_{X_2} X_5 = \nabla_{X_5} X_2 = \frac{\lambda}{2} X_1, && \nabla_{X_3} X_4 = -\nabla_{X_4} X_3 = \frac{\mu}{2} X_5,\\
&\nabla_{X_3} X_5 = \nabla_{X_5} X_3 = -\frac{\mu}{2} X_4, && \nabla_{X_4} X_5 = \nabla_{X_5} X_4 = \frac{\mu}{2} X_3,
\end{aligned} \tag{2.1}$$

and $R_{1212} = -3\lambda^2/4$, $R_{1234} = -\lambda\mu/2$, $R_{1515} = R_{2525} = \lambda^2/4$, $R_{3434} = -3\mu^2/4$, $R_{3535} = R_{4545} = \mu^2/4$.

(A_2) Lie algebras with 2-dimensional centre: For this type there exists an orthonormal basis $\{X_1, \dots, X_5\}$ of $\mathcal{N}$ such that the nonzero brackets are

$$[X_1, X_2] = \lambda X_4, \quad [X_1, X_3] = \mu X_5,$$

where $\lambda \geqslant \mu > 0$ and $\{X_4, X_5\}$ is a basis for the centre of $\mathcal{N}$. The nonzero connection components and the nonzero curvature components are given by

$$\begin{aligned}
&\nabla_{X_1} X_2 = -\nabla_{X_2} X_1 = \frac{\lambda}{2} X_4, \quad && \nabla_{X_1} X_3 = -\nabla_{X_3} X_1 = \frac{\mu}{2} X_5,\\
&\nabla_{X_1} X_4 = \nabla_{X_4} X_1 = -\frac{\lambda}{2} X_2, && \nabla_{X_1} X_5 = \nabla_{X_5} X_1 = -\frac{\mu}{2} X_3,\\
&\nabla_{X_2} X_4 = \nabla_{X_4} X_2 = \frac{\lambda}{2} X_1, && \nabla_{X_3} X_5 = \nabla_{X_5} X_3 = \frac{\mu}{2} X_1,
\end{aligned} \tag{2.2}$$

and $R_{1212} = -3\lambda^2/4$, $R_{1313} = -3\mu^2/4$, $R_{1414} = R_{2424} = \lambda^2/4$, $R_{2345} = -\lambda\mu/4$, $R_{3535} = R_{1515} = \mu^2/4$.

(A_3) Lie algebras with 3-dimensional centre: For this type there exists an orthonormal basis $\{X_1, \dots, X_5\}$ of $\mathcal{N}$ such that the nonzero bracket is

$$[X_1, X_2] = \lambda X_3,$$

where $\lambda > 0$ and $\{X_3, X_4, X_5\}$ is a basis for the centre of $\mathcal{N}$. The nonzero connection components and the nonzero curvature components are given by

$$\begin{aligned}\nabla_{X_1}X_2 = -\nabla_{X_2}X_1 = \frac{\lambda}{2}X_3, \quad \nabla_{X_1}X_3 = \nabla_{X_3}X_1 = -\frac{\lambda}{2}X_2,\\ \nabla_{X_2}X_3 = \nabla_{X_3}X_2 = \frac{\lambda}{2}X_1,\end{aligned} \tag{2.3}$$

and $R_{1212} = -3\lambda^2/4$, $R_{1313} = R_{2323} = \lambda^2/4$.

3. Parallel and totally geodesic hypersurfaces of two-step homogeneous nilmanifolds of dimension five

Let $F\colon M^n \to N^{n+1}$ be an isometric immersion of Riemannian manifolds $(M, \langle\,,\rangle)$ and $(N, \langle\,,\rangle)$. Denote by ξ a unit normal vector field on the hypersurface M and by ∇^M and ∇ the Levi-Civita connections of M and N, respectively. Let us define the shape operator S by $SX = -\nabla_X\xi$ and identify vector fields tangent to M with their images under $\mathrm{d}F$. Then the formula of Gauss is given by

$$\nabla_X Y = \nabla^M_X Y + h(X,Y)\xi, \tag{3.1}$$

where X and Y are vector fields tangent to M and h is the second fundamental form defined by $h(X,Y) = \langle SX, Y\rangle$. If R is the Riemann-Christoffel curvature of N, then the Codazzi equation can be expressed by

$$\langle R(X,Y)Z, \xi\rangle = (\nabla^M h)(X,Y,Z) - (\nabla^M h)(Y,X,Z), \tag{3.2}$$

where X, Y, Z and W are vector fields tangent to M and $(\nabla^M h)$ is defined by

$$(\nabla^M h)(X,Y,Z) = X(h(Y,Z)) - h(\nabla^M_X Y, Z) - h(Y, \nabla^M_X Z).$$

We say that M^n is totally geodesic in N^{n+1} if $h = 0$, and that M^n is parallel in N^{n+1} if $\nabla^M h = 0$.

In order to classify parallel hypersurfaces of two-step homogeneous nilmanifolds of dimension five, we prove the following result.

Lemma 3.1. *Let $F\colon M^4 \to (N,g)$ be a parallel hypersurface of a two-step homogeneous nilmanifold of dimension five. Also let ξ be a unit normal vector field on M and let $\{X_1, \ldots, X_5\}$ be the orthonormal field on N. Then every point of M has a neighbourhood $U \subseteq M$ on which ξ has one of the following forms:*

(a) $\xi = \pm X_5$, *where N is of the type* (A_1),

(b) $\xi = \pm X_1$, *where N is of the type* (A_2),

(c) $\xi = \pm X_1, \pm X_2, \pm X_3, \pm X_4, \pm X_5$, $\sin\theta X_1 + \cos\theta X_2$ *or* $\xi = \cos\theta X_4 + \sin\theta X_5$, *where N is of the type* (A_3) *and* $\theta\colon U \to \mathbb{R}$ *is a function.*

P r o o f. Assume that $\xi = \sum_{i=1}^{5} a_i X_i$, where $a_i\colon U \subseteq M \to \mathbb{R}$ are functions. Then the vector fields

$$\begin{aligned}
V_1 &= a_1X_2 - a_2X_1, & V_2 &= a_1X_3 - a_3X_1, & V_3 &= a_1X_4 - a_4X_1,\\
V_4 &= a_1X_5 - a_5X_1, & V_5 &= a_2X_3 - a_3X_2, & V_6 &= a_2X_4 - a_4X_2,\\
V_7 &= a_2X_5 - a_5X_2, & V_8 &= a_3X_4 - a_4X_3, & V_9 &= a_3X_5 - a_5X_3,\\
V_{10} &= a_4X_5 - a_5X_4
\end{aligned}$$

are tangent to the hypersurface M^3. Since M is a parallel hypersurface we have $\nabla^M h = 0$. Thus (3.2) gives us

$$\langle R(V_i, V_j)V_k, \xi\rangle = 0, \quad i, j, k \in \{1, \dots, 10\}. \tag{3.3}$$

In order to obtain suitable forms of ξ we apply (3.3) to the types (A_1), (A_2) and (A_3), which are given in Section 2.

Type (A_1): For this type $0 = \langle R(V_1, V_2)V_1, \xi\rangle = a_1a_3(a_1^2 + a_2^2)3\lambda^2/4$ gives us three subcases $a_1 = 0$, $a_3 = 0$ and $a_1 = a_2 = 0$.

Case 1: $a_1 = 0$. In this case $0 = R(V_1, V_5)V_1, \xi\rangle = a_2^3a_3 3\lambda^2/4$ implies two subcases $a_2 = 0$ and $a_3 = 0$.

Case 1.1: $a_2 = 0$. In this case $0 = \langle R(V_6, V_{10})V_6, \xi\rangle = a_4^3a_5\lambda^2/4$ yields two subcases $a_4 = 0$ and $a_5 = 0$.

Case 1.1.1: $a_4 = 0$. In this case from $0 = \langle R(V_2, V_5)V_8, \xi\rangle = a_3^4\lambda\mu/2$ we have $\xi = \pm X_5$.

Case 1.1.2: $a_5 = 0$. In this case $0 = \langle R(V_2, V_6)V_8, \xi\rangle = a_3a_4(a_3^2 + a_4^2)\lambda\mu/2$ gives us either $a_3 = 0$ or $a_4 = 0$ or $a_3 = a_4 = 0$. If $a_3 = 0$, then we have $0 = \langle R(V_3, V_6)V_8, \xi\rangle = a_4^4\lambda\mu/2$. If $a_4 = 0$, then we have $0 = \langle R(V_2, V_5)V_8, \xi\rangle = a_3^4\lambda\mu/2$. If $a_3 = a_4 = 0$, then we have $\xi = 0$. Therefore, these three cases give us the contradiction $\xi = 0$ (since $\langle \xi, \xi\rangle = 1$).

Case 1.2: $a_3 = 0$. In this case $0 = \langle R(V_1, V_8)V_{10}, \xi\rangle = a_2^2a_4a_5(-\lambda\mu/4)$ yields three subcases $a_2 = 0$, $a_4 = 0$ and $a_5 = 0$.

Case 1.2.1: $a_2 = 0$. In this case $0 = \langle R(V_4, V_6)V_9, \xi\rangle = a_5^2a_4^2\lambda\mu/2$ gives us either $a_4 = 0$ or $a_5 = 0$. If $a_4 = 0$, then we have $\xi = \pm X_5$. If $a_5 = 0$, then from $0 = \langle R(V_3, V_6)V_8, \xi\rangle = a_4^4\lambda\mu/2$ we have the contradiction $\xi = 0$.

Case 1.2.2: $a_4 = 0$. In this case we obtain $0 = \langle R(V_1, V_5)V_{10}, \xi\rangle = a_2^3a_5\lambda\mu/4$. If $a_2 = 0$, then we have $\xi = \pm X_5$. If $a_5 = 0$, then from $0 = \langle R(V_1, V_5)V_6, \xi\rangle = a_2^4(-\lambda\mu/4)$ we obtain the contradiction $\xi = 0$.

Case 1.2.3: $a_5 = 0$. In this case $0 = \langle R(V_1, V_5)V_6, \xi\rangle = a_2^2(a_4^2 + a_2^2)(-\mu\lambda/4)$ gives us either $a_2 = 0$ or $a_2 = a_4 = 0$. If $a_2 = 0$, then we have $0 = \langle R(V_3, V_6)V_8, \xi\rangle = a_4^4\lambda\mu/2$.

If $a_2 = a_4 = 0$, then a similar argument yields $\xi = 0$. Therefore, both cases imply the contradiction $\xi = 0$.

Case 2: $a_3 = 0$. In this case from $0 = \langle R(V_2, V_3)V_2, \xi\rangle = a_1^3 a_4(-3\mu^2/4)$ we have two subcases $a_1 = 0$ and $a_4 = 0$.

Case 2.1: $a_1 = 0$. This case coincides with the case 1.2.

Case 2.2: $a_4 = 0$. In this case $0 = \langle R(V_5, V_7)V_5, \xi\rangle = a_5 a_2^3 \mu^2/4$ gives us two subcases $a_2 = 0$ and $a_5 = 0$.

Case 2.2.1: $a_2 = 0$. In this case $0 = \langle R(V_1, V_2)V_3, \xi\rangle = a_1^4(-\lambda\mu/4)$ gives us $\xi = \pm X_5$.

Case 2.2.2: $a_5 = 0$. In this case $0 = \langle R(V_2, V_1)V_6, \xi\rangle = a_1 a_2(a_1^2 + a_2^2)\lambda\mu/4$ yields that either $a_1 = 0$ or $a_2 = 0$ or $a_1 = a_2 = 0$. If $a_1 = 0$, then we have $0 = \langle R(V_1, V_5)V_6, \xi\rangle = a_2^4(-\lambda\mu/4)$. If $a_2 = 0$, then $0 = \langle R(V_1, V_2)V_3, \xi\rangle = a_1^4(-\lambda\mu/4)$ gives us $\xi = 0$. If $a_1 = a_2 = 0$, then we have $\xi = 0$. Therefore, these three cases give us the contradiction $\xi = 0$.

Case 3: $a_1 = a_2 = 0$. This case coincides with the case 1.1.

Type (A_2): For this type from $0 = \langle R(V_9, V_4)V_4, \xi\rangle = -a_5^3 a_3 \mu^2$ we obtain two subcases $a_5 = 0$ and $a_3 = 0$.

Case 1: $a_5 = 0$. In this case from $0 = \langle R(V_7, V_6)V_6, \xi\rangle = a_2^2 a_3 a_4(-\lambda\mu/2)$ we have three subcases $a_2 = 0$, $a_4 = 0$ and $a_3 = 0$.

Case 1.1: $a_2 = 0$. In this case $0 = \langle R(V_9, V_8)V_5, \xi\rangle = a_3^4(-\lambda\mu/4)$ gives us $a_3 = 0$. Thus we obtain $0 = \langle R(V_2, V_1)V_4, \xi\rangle = a_1^3 a_4(-\lambda\mu/4)$. If $a_4 = 0$, then we have $\xi = \pm X_1$. If $a_1 = 0$, then $0 = \langle R(V_6, V_8)V_{10}, \xi\rangle = a_4^4 \lambda\mu/4$ gives us the contradiction $\xi = 0$.

Case 1.2: $a_4 = 0$. In this case $0 = \langle R(V_9, V_8)V_5, \xi\rangle = a_3^2(a_3^2 + a_2^2)(-\mu\lambda/4)$ yields that either $a_3 = 0$ or $a_3 = a_2 = 0$. If $a_3 = 0$, then we have $0 = \langle R(V_7, V_6)V_5, \xi\rangle = a_2^4(-\lambda\mu/4)$. If $a_3 = a_2 = 0$, then we have $\xi = \pm X_1$. Therefore, both cases imply that $\xi = \pm X_1$.

Case 1.3: $a_3 = 0$. In this case from $0 = \langle R(V_1, V_2)V_2, \xi\rangle = a_1^3 a_2(-3\mu^2/4)$ we have two subcases $a_1 = 0$ and $a_2 = 0$.

Case 1.3.1: $a_2 = 0$. In this case $0 = \langle R(V_{10}, V_6)V_8, \xi\rangle = a_4^4(-\lambda\mu/4)$ gives us $\xi = \pm X_1$.

Case 1.3.2: $a_1 = 0$. In this case $0 = \langle R(V_1, V_6)V_1, \xi\rangle = \lambda^2 a_2^3 a_4$ gives us either $a_2 = 0$ or $a_3 = 0$. If $a_2 = 0$, then we have $0 = \langle R(V_{10}, V_6)V_8, \xi\rangle = a_4^4(-\lambda\mu/4)$. If $a_4 = 0$, then we have $0 = \langle R(V_5, V_6)V_7, \xi\rangle = a_2^4(-\lambda\mu/2)$. Therefore, both cases yield the contradiction $\xi = 0$.

Case 2: $a_3 = 0$. In this case from $0 = \langle R(V_1, V_2)V_2, \xi\rangle = a_2 a_1^3(-3\mu^2/4)$ we have subcases $a_1 = 0$ and $a_2 = 0$.

Case 2.1: $a_1 = 0$. In this case $0 = \langle R(V_8, V_{10})V_{10}, \xi\rangle = a_4^2 a_5 a_2(-\lambda\mu/2)$ yields three subcases $a_4 = 0$, $a_5 = 0$ and $a_2 = 0$.

Case 2.1.1: $a_4 = 0$. In this case $0 = \langle R(V_7, V_5)V_6, \xi\rangle = a_5^2 a_2^2 \lambda\mu/4$ implies that either $a_5 = 0$ or $a_2 = 0$. If $a_2 = 0$, then we have $0 = \langle R(V_7, V_9)V_{10}, \xi\rangle = a_5^4\lambda\mu/4$. If $a_5 = 0$, then we obtain $0 = \langle R(V_6, V_5)V_7, \xi\rangle = a_2^4\lambda\mu/4$. Therefore, both cases yield the contradiction $\xi = 0$.

Case 2.1.2: $a_5 = 0$. In this case $0 = \langle R(V_1, V_6)V_1, \xi\rangle = \lambda^2 a_4 a_2^3$ implies that either $a_2 = 0$ or $a_4 = 0$. If $a_2 = 0$, then we have $0 = \langle R(V_6, V_{10})V_8, \xi\rangle = a_4^4\lambda\mu/4$. If $a_4 = 0$, then we obtain $0 = \langle R(V_6, V_5)V_7, \xi\rangle = a_2^4\lambda\mu/4$. Therefore, both cases imply the contradiction $\xi = 0$.

Case 2.1.3: $a_2 = 0$. In this case $0 = \langle R(V_6, V_8)V_{10}, \xi\rangle = a_4^2(a_5^2 + a_4^2)\mu\lambda/4$ implies that either $a_4 = 0$ or $a_5 = a_4 = 0$. If $a_4 = 0$, then we have $0 = \langle R(V_7, V_9)V_{10}, \xi\rangle = a_5^4\lambda\mu/4$. If $a_4 = a_5 = 0$, then we have $\xi = 0$. Therefore, both cases imply the contradiction $\xi = 0$.

Case 2.2: $a_2 = 0$. In this case $0 = \langle R(V_6, V_9)V_{10}, \xi\rangle = a_4 a_5(a_5^2 + a_4^2)\lambda\mu/4$ yields that either $a_4 = 0$ or $a_5 = 0$ or $a_4 = a_5 = 0$. If $a_4 = 0$, then we have $\langle R(V_7, V_9)V_{10}, \xi\rangle = a_5^4\lambda\mu/4$. If $a_5 = 0$, then we have $0 = \langle R(V_6, V_8)V_{10}, \xi\rangle = a_4^4\lambda\mu/4$. Therefore, these three cases imply that $\xi = \pm X_1$.

Type (A_3): For this type $0 = \langle R(V_1, V_2)V_4, \xi\rangle = \lambda^2 a_1 a_2 a_3 a_5$ gives us four subcases $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ and $a_5 = 0$.

Case 1: $a_1 = 0$. In this case from $0 = \langle R(V_1, V_5)V_1, \xi\rangle = \lambda^2 a_2^3 a_3$ we have two subcases $a_2 = 0$ and $a_3 = 0$.

Case 1.1: $a_2 = 0$. In this case $0 = \langle R(V_2, V_8)V_2, \xi\rangle = a_3^3 a_4(-\lambda^2/4)$ gives us either $a_3 = 0$ or $a_4 = 0$. If $a_3 = 0$, then $\langle \xi, \xi\rangle = 1$ implies that $\xi = \cos\theta X_4 + \sin\theta X_5$. If $a_4 = 0$, then $0 = \langle R(V_2, V_9)V_2, \xi\rangle = a_3^3 a_5(-\lambda^2/4)$ yields that either $\xi = \pm X_5$ or $\xi = \pm X_3$.

Case 1.2: $a_3 = 0$. In this case $0 = \langle R(V_5, V_6)V_5, \xi\rangle = a_2^3 a_4(-\lambda^2/4)$ implies that either $a_2 = 0$ or $a_4 = 0$. If $a_2 = 0$, then we have $\xi = \cos\theta X_4 + \sin\theta X_5$. If $a_4 = 0$, then we have $0 = \langle R(V_4, V_7)V_4, \xi\rangle = a_5^3 a_2 3\lambda^2/4$, which gives us either $\xi = \pm X_5$ or $\xi = \pm X_2$.

Case 2: $a_2 = 0$. In this case $0 = \langle R(V_1, V_2)V_1, \xi\rangle = \lambda^2 a_1^3 a_3$ gives us two subcases $a_1 = 0$ and $a_3 = 0$.

Case 2.1: $a_1 = 0$. This case coincides with the Case 1.1.

Case 2.2: $a_3 = 0$. In this case $0 = \langle R(V_1, V_3)V_1, \xi\rangle = a_1^3 a_4 3\lambda^2/4$ yields that either $a_1 = 0$ or $a_4 = 0$. If $a_1 = 0$, then we have $\xi = \cos\theta X_4 + \sin\theta X_5$. If $a_4 = 0$, then $\langle R(V_1, V_4)V_1, \xi\rangle = a_1^3 a_5 3\lambda^2/4 = 0$ implies that either $\xi = \pm X_5$ or $\xi = \pm X_1$.

Case 3: $a_3 = 0$. In this case $0 = \langle R(V_2, V_4)V_2, \xi\rangle = a_1^3 a_5(-\lambda^2/4)$ implies two subcases $a_1 = 0$ and $a_5 = 0$.

Case 3.1: $a_1 = 0$. This case coincides with the Case 1.2.

Case 3.2: $a_5 = 0$. In this case $0 = \langle R(V_1, V_3)V_1, \xi\rangle = a_1 a_4(a_1^2 + a_2^2)3\lambda^2/4$ implies that either $a_1 = 0$ or $a_4 = 0$ or $a_1 = a_2 = 0$. If $a_1 = 0$, then $0 = \langle R(V_1, V_6)V_1, \xi\rangle = a_2^3 a_4 3\lambda^2/4 = 0$ implies that either $\xi = \pm X_4$ or $\xi = \pm X_2$. If $a_4 = 0$, then we have $\xi = \sin\theta X_1 + \cos\theta X_2$. If $a_1 = a_2 = 0$, then we have $\xi = \pm X_4$.

Case 4: $a_5 = 0$. In this case $0 = \langle R(V_5, V_6)V_6, \xi\rangle = a_2 a_4^2 a_3(-\lambda^2/4)$ gives us three subcases $a_2 = 0$, $a_4 = 0$ and $a_3 = 0$.

Case 4.1: $a_2 = 0$. In this case $0 = \langle R(V_1, V_3)V_1, \xi\rangle = a_1^3 a_4 3\lambda^2/4$ yields that either $a_1 = 0$ or $a_4 = 0$. If $a_1 = 0$, then $0 = \langle R(V_5, V_8)V_5, \xi\rangle = a_3^3 a_4(-\lambda^2/4)$ gives us that either $\xi = \pm X_4$ or $\xi = \pm X_3$. If $a_4 = 0$, then $\langle R(V_1, V_2)V_5, \xi\rangle = -\lambda^2 a_1^2 a_3^2$ gives us either $\xi = \pm X_1$ or $\xi = \pm X_3$.

Case 4.2: $a_4 = 0$. In this case $0 = \langle R(V_1, V_5)V_1, \xi\rangle = \lambda^2 a_2 a_3(a_1^2 + a_2^2)$ implies that either $a_2 = 0$ or $a_3 = 0$ or $a_1 = a_2 = 0$. If $a_2 = 0$, then $0 = \langle R(V_2, V_5)V_5, \xi\rangle = -\lambda^2 a_1 a_3^3$ gives us that either $\xi = \pm X_3$ or $\xi = \pm X_1$. If $a_3 = 0$, then $\xi = \sin\theta X_1 + \cos\theta X_2$. If $a_1 = a_2 = 0$, then we have $\xi = \pm X_3$.

Case 4.3: $a_3 = 0$. This case coincides with the Case 3.2. □

Theorem 3.2. *Let $F\colon M^4 \to (N, g)$ be a parallel hypersurface of a two-step homogeneous nilmanifold with dimension five. Then there exist local coordinates (w_1, w_2, w_3, w_4) on M such that the immersion with respect to these coordinates, up to isometry, is given by one of the following expressions:*

$$
\begin{aligned}
&(1)\ F(w_1, \ldots, w_4) = (0, w_1, w_2, w_3, w_4),\\
&(2)\ F(w_1, \ldots, w_4) = (w_1, 0, w_2, w_3, w_4),\\
&(3)\ F(w_1, \ldots, w_4) = (w_1, w_2, w_3, 0, w_4),\\
&(4)\ F(w_1, \ldots, w_4) = (w_1, w_2, w_3, w_4, 0),\\
&(5)\ F(w_1, \ldots, w_4) = (\cos\theta w_1, -\sin\theta w_1, w_2, w_3, w_4),\\
&(6)\ F(w_1, \ldots, w_4) = (w_1, w_2, w_3, -\sin\theta w_4, \cos\theta w_4),
\end{aligned}
\tag{3.4}
$$

where θ is a real constant. Conversely, all these hypersurfaces are parallel.

Proof. Assume that M is a parallel hypersurface. Then ξ has one of the forms given in (a), (b) and (c) of Lemma 3.1. Let us start with (a), i.e., $\xi = \pm X_5$. Then the following vector fields span the tangent space to M at each point:

$$Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = X_3, \quad Y_4 = X_4. \tag{3.5}$$

It follows from (2.1) and (3.5) that the nonzero connection components are $\nabla_{Y_1} Y_2 = -\nabla_{Y_2} Y_1 = \xi\lambda/2$ and $\nabla_{Y_3} Y_4 = -\nabla_{Y_4} Y_3 = \xi\mu/2$. Thus by using the formula of

Gauss (3.1), the second fundamental form is determined by $h(Y_1,Y_2) = -h(Y_2,Y_1) = \lambda/2$ and $h(Y_3,Y_4) = -h(Y_4,Y_3) = \mu/2$, where the remaining cases are zero. Therefore, the symmetry condition for h yields the contradiction $\lambda = \mu = 0$. If we consider (c) of Lemma 3.1, for $\xi = \pm X_3$, then by a similar argument we have the contradiction $\lambda = 0$.

Let us consider (b) of Lemma 3.1, i.e., $\xi = \pm X_1$. Then the following vector fields span the tangent space to M at each point:

$$Y_2 = X_2, \quad Y_3 = X_3, \quad Y_4 = X_4, \quad Y_5 = X_5. \tag{3.6}$$

From the equations (2.2) and (3.6) we can see that the nonzero connection components are $\nabla_{Y_2}Y_4 = \nabla_{Y_4}Y_2 = \xi\lambda/2$ and $\nabla_{Y_3}Y_5 = \nabla_{Y_5}Y_3 = \xi\mu/2$. Thus by using the formula of Gauss, the second fundamental form is determined by $h(Y_2,Y_4) = h(Y_4,Y_2) = \lambda/2$ and $h(Y_3,Y_5) = h(Y_5,Y_3) = \mu/2$, where the remaining cases are zero. Hence, $\nabla^M h = 0$ and the hypersurface is parallel. In order to obtain this hypersurface we put $\partial w_1 = Y_2$, …, $\partial w_4 = Y_5$ and denote by

$$F\colon\ M \to N\colon\ (w_1,\dots,w_4) \mapsto (F_1(w_1,\dots,w_4),\dots,F_5(w_1,\dots,w_4))$$

the immersion of the hypersurface. Thus by (3.6) the derivatives of F are

$$\begin{aligned}
\Big(\frac{\partial F_1}{\partial w_1},\dots,\frac{\partial F_5}{\partial w_1}\Big) &= (0,1,0,0,0), \quad &\Big(\frac{\partial F_1}{\partial w_2},\dots,\frac{\partial F_5}{\partial w_2}\Big) &= (0,0,1,0,0),\\
\Big(\frac{\partial F_1}{\partial w_3},\dots,\frac{\partial F_5}{\partial w_3}\Big) &= (0,0,0,1,0), \quad &\Big(\frac{\partial F_1}{\partial w_4},\dots,\frac{\partial F_5}{\partial w_4}\Big) &= (0,0,0,0,1).
\end{aligned}$$

From these equations, we immediately obtain

$$F_1 = c_1, \quad F_2 = w_1 + c_2, \quad F_3 = w_2 + c_3, \quad F_4 = w_3 + c_4, \quad F_5 = w_4 + c_5, \tag{3.7}$$

where $c_1,\dots,c_5$ are real constants. This hypersurface is isometric with the case (1) of the theorem.

Let us consider (c) of Lemma 3.1 for $\xi = \pm X_2$. Then the following vector fields span the tangent space to M at each point:

$$Y_1 = X_1, \quad Y_2 = X_3, \quad Y_3 = X_4, \quad Y_4 = X_5, \tag{3.8}$$

and by (2.3) the nonzero connection components are $\nabla_{Y_1}Y_2 = \xi(-\lambda/2)$ and $\nabla_{Y_2}Y_1 = \xi(-\lambda/2)$. Thus by the formula of Gauss, the second fundamental form is determined by $h(Y_1,Y_2) = h(Y_2,Y_1) = -\lambda/2$, where the remaining cases are zero. Thus the hypersurface is parallel and if we put $\partial w_1 = Y_1$, …, $\partial w_4 = Y_4$ and use (3.8) we obtain $F_1 = w_1 + c_1$, $F_2 = c_2$, $F_3 = w_2 + c_3$, $F_4 = w_3 + c_4$, $F_5 = w_4 + c_5$, where $c_1, \ldots, c_5$ are real constants. This hypersurface is isometric with the case (2) of the theorem.

Let us consider (c) of Lemma 3.1 with $\xi = \pm X_4$ and $\xi = \pm X_5$. Then by a similar argument as above, in both cases for all $i, j \in \{1, \ldots, 4\}$ we obtain $h(Y_i, Y_j) = 0$. Thus these hypersurfaces are parallel and they are given, respectively, by

$$F_1 = w_1 + c_1, \quad F_2 = w_2 + c_2, \quad F_3 = w_3 + c_3, \quad F_4 = c_4, \quad F_5 = w_4 + c_5, \tag{3.9}$$

and

$$F_1 = w_1 + c_1, \quad F_2 = w_2 + c_2, \quad F_3 = w_3 + c_3, \quad F_4 = w_4 + c_4, \quad F_5 = c_5, \tag{3.10}$$

where $c_1, \ldots, c_5$ are real constants. These hypersurfaces are isometric with the cases (3) and (4) of the theorem.

Let us consider (c) of Lemma 3.1 with $\xi = \sin\theta X_1 + \cos\theta X_2$. Then the following vector fields span the tangent space to M at each point:

$$Y_1 = \cos\theta X_1 - \sin\theta X_2, \quad Y_2 = X_3, \quad Y_3 = X_4, \quad Y_4 = X_5. \tag{3.11}$$

A direct computation, using (2.3) and (3.11), gives

$$\nabla_{Y_i} Y_1 = -Y_i(\theta)\xi, \quad \nabla_{Y_1} Y_2 = \frac{-\lambda}{2}\xi, \quad \nabla_{Y_2} Y_1 = -\Big(Y_2(\theta) + \frac{\lambda}{2}\Big)\xi,$$

where $i = 1, 3, 4$ and the other connection components are zero. Thus from the formula of Gauss, the second fundamental form is determined by $h(Y_i, Y_1) = -Y_i(\theta)$, $h(Y_1, Y_2) = -\lambda/2$ and $h(Y_2, Y_1) = -Y_2(\theta) - \lambda/2$, where $i = 1, 3, 4$ and the remaining cases are zero. Hence, by applying the symmetry condition for h we have $Y_1(\theta) = 0$, …, $Y_4(\theta) = 0$, which implies that θ is a real constant and the hypersurface is parallel. In order to obtain this immersion we put $\partial w_1 = Y_1$, …, $\partial w_4 = Y_4$ and use (3.11), which implies that

$$\begin{aligned}
&\Big(\frac{\partial F_1}{\partial w_1}, \ldots, \frac{\partial F_5}{\partial w_1}\Big) = (\cos\theta, -\sin\theta, 0, 0, 0), && \Big(\frac{\partial F_1}{\partial w_2}, \ldots, \frac{\partial F_5}{\partial w_2}\Big) = (0, 0, 1, 0, 0),\\
&\Big(\frac{\partial F_1}{\partial w_3}, \ldots, \frac{\partial F_5}{\partial w_3}\Big) = (0, 0, 0, 1, 0), && \Big(\frac{\partial F_1}{\partial w_4}, \ldots, \frac{\partial F_5}{\partial w_4}\Big) = (0, 0, 0, 0, 1).
\end{aligned}$$

From these equations we obtain

$$\begin{gathered}
F_1 = \cos\theta w_1 + c_1, \quad F_2 = -\sin\theta w_1 + c_2, \quad F_3 = w_2 + c_3,\\
F_4 = w_3 + c_4, \quad F_5 = w_4 + c_5,
\end{gathered}$$

where $c_1, \dots, c_5$ are real constants. This hypersurface is isometric with the case (5) of the theorem.

Let us consider (c) of Lemma 3.1 with $\xi = \cos\theta X_4 + \sin\theta X_5$. Then the following vector fields span the tangent space to M at each point:

$$Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = X_3, \quad Y_4 = -\sin\theta X_4 + \cos\theta X_5, \tag{3.12}$$

and by using (2.3) and (3.12) we obtain

$$\begin{gathered} \nabla_{Y_i} Y_4 = -Y_i(\theta)\xi, \quad \nabla_{Y_1} Y_2 = -\nabla_{Y_2} Y_1 = \frac{\lambda}{2} Y_3, \\ \nabla_{Y_1} Y_3 = \nabla_{Y_3} Y_1 = \frac{-\lambda}{2} Y_2, \quad \nabla_{Y_3} Y_2 = \nabla_{Y_2} Y_3 = \frac{\lambda}{2} Y_1, \end{gathered} \tag{3.13}$$

where $i = 1, \dots, 4$ and the remaining connection components are zero. Thus by the Gauss formula (3.1), the second fundamental form is determined by $h(Y_i, Y_4) = -Y_i(\theta)$, where $i = 1, \dots, 4$ and the remaining cases are zero. Since h is symmetric, we have $Y_1(\theta) = 0$, …, $Y_4(\theta) = 0$, which implies that θ is a real constant and the hypersurface is parallel. Thus if we put $\partial w_1 = Y_1$, …, $\partial w_4 = Y_4$ and use (3.12) we obtain

$$\begin{gathered} F_1 = w_1 + c_1, \quad F_2 = w_2 + c_2, \quad F_3 = w_3 + c_3, \\ F_4 = -\sin\theta w_4 + c_4, \quad F_5 = \cos\theta w_4 + c_5, \end{gathered} \tag{3.14}$$

where $c_1, \dots, c_5$ are real constants. This hypersurface is isometric with the case (6) of the theorem.

The converse of theorem can be obtained by a straightforward computation. $\square$

Since every totally geodesic hypersurface is parallel, we obtain the following result.

Theorem 3.3. *Let $F\colon M^4 \to (N, g)$ be a totally geodesic hypersurface of a two-step homogeneous nilmanifold of dimension five. Then there exist local coordinates (w_1, w_2, w_3, w_4) on M^4 such that the immersion with respect to these coordinates, up to isometry, is given by one of the following expressions:*

$$\begin{gathered} F(w_1, \dots, w_4) = (w_1, w_2, w_3, 0, w_4), \\ F(w_1, \dots, w_4) = (w_1, w_2, w_3, w_4, 0), \\ F(w_1, \dots, w_4) = (w_1, w_2, w_3, -\sin\theta w_4, \cos\theta w_4), \end{gathered} \tag{3.15}$$

where θ is a real constant. Conversely, these hypersurfaces are totally geodesic.

P r o o f. Assume that M is a totally geodesic hypersurface. Then it is sufficient to choose the hypersurfaces given in Theorem 3.2, for which we have $h(Y_i, Y_j) = 0$, where $i, j \in \{1, \ldots, 4\}$. Thus we obtain the hypersurfaces (3.9), (3.10) and (3.14), which are isometric with the hypersurfaces given in the system (3.15). The converse can be verified by a straightforward computation. $\square$

Theorems 3.2 and 3.3 give us the following result:

Theorem 3.4. (I) *Two-step homogeneous nilmanifolds of dimension five which have one-dimensional centre can never admit parallel hypersurfaces.*

(II) *The only two-step homogeneous nilmanifolds of dimension five which admit totally geodesic hypersurfaces have three-dimensional centre.*

P r o o f. Suppose that M is parallel in N. Then by Theorem 3.2 it can be expressed by one of the hypersurfaces given in system (3.4), which are obtained from the unit normal vector fields belonging to the types (A_2) and (A_3) of N, i.e., N has two- or three-dimensional centre.

To prove (II) we assume that M is totally geodesic in N. Then by Theorem 3.3 it can be expressed by one of the hypersurfaces given in the system (3.15). By Theorem 3.2, these hypersurfaces are obtained from $\xi = \pm X_4$, $\xi = \pm X_5$ and $\xi = \cos\theta X_4 + \sin\theta X_5$ which belong to the type (A_3) of N, i.e., N has three-dimensional centre. $\square$

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