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# POLYCYCLIC GROUPS WITH AUTOMORPHISMS OF ORDER FOUR

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*Abstract.* In this paper, we study the structure of polycyclic groups admitting an automorphism of order four on the basis of Neumann's result, and prove that if $\alpha$ is an automorphism of order four of a polycyclic group $G$ and the map $\varphi\colon G\to G$ defined by $g^{\varphi}=[g,\alpha]$ is surjective, then $G$ contains a characteristic subgroup $H$ of finite index such that the second derived subgroup $H''$ is included in the centre of $H$ and $C_H(\alpha^2)$ is abelian, both $C_G(\alpha^2)$ and $G/[G,\alpha^2]$ are abelian-by-finite. These results extend recent and classical results in the literature.

*Keywords*: polycyclic group; regular automorphism; surjectivity

*MSC 2010*: 20E36

## 1. INTRODUCTION AND MAIN RESULTS

An automorphism $\alpha$ of a group $G$ is called regular if it has no nontrivial fixed points.

The classical result concerning a regular automorphism of order 2 in a finite group is due to Burnside [1]: a finite group $G$ admits a regular automorphism of order 2 if and only if $G$ is abelian of odd order.

For a regular automorphism of order 3 of an arbitrary group, Neumann [8] proved the following result.

**Proposition 1.1.** *Let $G$ be a group and $\alpha$ a regular automorphism of order* 3 *of $G$. If the map $\varphi\colon G\to G$ $(g\mapsto[g,\alpha])$ is surjective, then $G$ is nilpotent of class at most* 2.

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For a regular automorphism of prime order of a finite group, Thompson [13] proved that if a finite group $G$ has a regular automorphism of prime order, then $G$ is nilpotent. Deeper results concerning regular automorphisms can be found in [5] and [12].

Abandoning the condition of regularity, we consider an arbitrary automorphism of order 2 of an arbitrary group, and easily obtain the following result.

**Proposition 1.2.** *Let $G$ be a group and $\alpha$ an automorphism of order 2 of $G$. If the map $\varphi\colon G \to G$ ($g \mapsto [g,\alpha]$) is surjective, then $G$ is abelian.*

Proof. Since $\varphi$ is surjective, for all $x \in G$ there exists some $g \in G$ such that $x = [g,\alpha] = g^{-1}g^{\alpha}$. Moreover,

$$x^{\alpha} = (g^{-1}g^{\alpha})^{\alpha} = (g^{-1})^{\alpha}g^{\alpha^2} = (g^{\alpha})^{-1}g = x^{-1}.$$

Thus for any $g_1, g_2 \in G$, we have $(g_1^{-1}g_2^{-1})^{\alpha} = (g_1^{-1}g_2^{-1})^{-1} = g_2g_1$ and $(g_1^{-1}g_2^{-1})^{\alpha} = (g_1^{-1})^{\alpha}(g_2^{-1})^{\alpha} = g_1g_2$. Obviously, $g_1g_2 = g_2g_1$. Hence $G$ is abelian. □

In [11], we treat the general case, and obtain the result that if $\alpha$ is an automorphism of prime order of a polycyclic group $G$ and the map $\varphi\colon G \to G$ defined by $g^{\varphi} = [g,\alpha]$ is surjective, then $G$ is nilpotent-by-finite. In this paper, we are interested in an arbitrary automorphism of order 4 of a polycyclic group, and obtain the following result which extends Kovács' result [6].

**Theorem 1.1.** *Let $G$ be a polycyclic group and $\alpha$ an automorphism of order 4 of $G$. If the map $\varphi\colon G \to G$ ($g \mapsto [g,\alpha]$) is surjective, then $G$ contains a characteristic subgroup $H$ of finite index such that*

(i) *the second derived subgroup $H''$ is included in the centre of $H$,*
(ii) *$C_H(\alpha^2)$ is abelian.*

Let $\alpha$ be an automorphism of a group $G$. Obviously, the centralizer $C_G(\alpha)$ and the commutator subgroup $[G,\alpha] = \langle g^{-1}g^{\alpha};\ g \in G\rangle$ are normal $\alpha$-invariant subgroups of $G$. It is well known that the centralizer $C_G(\alpha)$ in some sense has consequences on $G$, and in particular on $G/[G,\alpha]$. For example, Endimioni and Moravec [3] proved that if $\alpha$ is an automorphism of a polycyclic group $G$ and $C_G(\alpha)$ is finite, then $G/[G,\alpha]$ is finite.

In this paper, we are interested in the case where the automorphism $\alpha$ is of order 4, and decide the structure of $C_G(\alpha^2)$ and $G/[G,\alpha^2]$. In Section 3, we shall see that Lemma 3.7 in some sense generalizes Endimioni and Moravec's result mentioned above.

**Theorem 1.2.** *Let $G$ be a polycyclic group and $\alpha$ an automorphism of order 4 of $G$. If the map $\varphi\colon G \to G$ $(g \mapsto [g,\alpha])$ is surjective, then the following hold:*

(i) *$C_G(\alpha^2)$ is abelian-by-finite,*
(ii) *$G/[G,\alpha^2]$ is abelian-by-finite.*

## 2. Proof of Theorem 1.1

**Lemma 2.1.** *Let $G$ be a group and $\alpha$ an automorphism of finite order $n$ of $G$. If the map $\varphi\colon G \to G$ $(g \mapsto [g,\alpha])$ is surjective, then for all $x \in G$, we have $xx^{\alpha}x^{\alpha^2}\dots x^{\alpha^{n-1}} = 1$.*

Proof. Since the map $\varphi$ is surjective, for all $x \in G$ there exists some $g \in G$ such that $x = [g,\alpha]$. Thus

$$xx^{\alpha}x^{\alpha^2}\dots x^{\alpha^{n-1}} = [g,\alpha][g,\alpha]^{\alpha}[g,\alpha]^{\alpha^2}\dots[g,\alpha]^{\alpha^{n-1}} = g^{-1}g^{\alpha^n} = g^{-1}g = 1.$$

The proof is completed. □

**Lemma 2.2.** *Let $G$ be a polycyclic group and $\alpha$ an automorphism of finite order $n$ of $G$. If the map $\varphi\colon G \to G$ $(g \mapsto [g,\alpha])$ is surjective, then $G$ contains a characteristic subgroup $H$ of finite index such that for each prime $q \nmid n$*

(i) $\bigcap\limits_{t>0} H^{q^t} = 1$,
(ii) *for any positive integer $t$, the automorphism $\overline{\alpha}_t$ induced by $\alpha$ on $H/H^{q^t}$ is regular.*

Proof. (i) By a result of Shmel'kin [10], $G$ contains a normal subgroup $N$ of finite index which is a residually finite $q$-group for every prime $q$. Denote by $e$ the exponent of $G/N$. Then $G^e \leqslant N$. Put $H = G^e$. Then $G$ contains a characteristic subgroup $H$ of finite index such that $H$ is a residually finite $q$-group for every prime $q$. Thus for $q \nmid n$ and positive integer $t$, $H/H^{q^t}$ is a finite $q$-group and $\bigcap\limits_{t>0} H^{q^t} = 1$.

(ii) Consider an element $\bar{h} \in H/H^{q^t}$ such that $\bar{h}^{\overline{\alpha}_t} = \bar{h}$. By Lemma 2.1, for all $h \in H \leqslant G$ we have $hh^{\alpha}h^{\alpha^2}\dots h^{\alpha^{n-1}} = 1$. In $H/H^{q^t}$, the following relations hold:

$$\bar{h}\bar{h}^{\overline{\alpha}_t}\bar{h}^{\overline{\alpha}_t^2}\dots\bar{h}^{\overline{\alpha}_t^{n-1}} = \bar{h}^n = 1.$$

Since $q \nmid n$, it follows that $\bar{h} = 1$, as required. □

P r o o f of Theorem 1.1. (i) Choose $q \neq 2$. According to (ii) of Lemma 2.2, $G$ contains a characteristic subgroup $H$ of finite index such that $H/H^{q^t}$ has a regular automorphism of order dividing 4. By Kovács' result [6], the second derived subgroup $(H/H^{q^t})''$ is included in the centre of $H/H^{q^t}$. It follows that $[(H/H^{q^t})'', H/H^{q^t}] = 1$. In other words, $[H''H^{q^t}, H] \leqslant H^{q^t}$. This implies that

$$[H'', H] \leqslant [H''H^{q^t}, H] \leqslant H^{q^t}.$$

Consequently,

$$[H'', H] \leqslant \bigcap_{t>0} H^{q^t} = 1.$$

That is, $H'' \leqslant Z(H)$.

(ii) Set $\varphi = \alpha^2$. It suffices to prove that $C_H(\varphi)$ is abelian. Choose $q \neq 2$ and consider $C_{H/H^{q^t}}(\bar{\varphi})$.

If $C_{H/H^{q^t}}(\overline{\varphi}) = 1$, then $\overline{\varphi}$ is a regular automorphism of order 2 of $H/H^{q^t}$. By Burnside's result [1], $H/H^{q^t}$ is abelian. Hence for any $\bar{h}_1, \bar{h}_2 \in \overline{H} = H/H^{q^t}$ we have $[\bar{h}_1, \bar{h}_2] = 1$. Namely, $[h_1, h_2] \in H^{q^t}$. But $\bigcap_{t>0} H^{q^t} = 1$, thus $[h_1, h_2] = 1$ for any $h_1, h_2 \in H$. This shows that $H$ is abelian. It follows that $C_H(\varphi)$ is abelian. If $C_{H/H^{q^t}}(\overline{\varphi}) \neq 1$, then $C_{H/H^{q^t}}(\overline{\varphi})$ is $\overline{\alpha}$-invariant, and thus $\overline{\alpha}$ is an automorphism of order 1 or 2 of $C_{H/H^{q^t}}(\overline{\varphi})$. Observe that

$$C_{C_{H/H^{q^t}}(\overline{\varphi})}(\overline{\alpha}) \leqslant C_{H/H^{q^t}}(\overline{\alpha}) = 1,$$

and we have that $\overline{\alpha}$ is a regular automorphism of $C_{H/H^{q^t}}(\overline{\varphi})$. Since $C_{H/H^{q^t}}(\overline{\varphi}) \neq 1$, $\overline{\alpha}$ is a regular automorphism of order 2 of $C_{H/H^{q^t}}(\overline{\varphi})$. By Burnside's result [1], $C_{H/H^{q^t}}(\overline{\varphi})$ is abelian. Noticing that

$$C_H(\varphi)/C_H(\varphi) \cap H^{q^t} \simeq C_H(\varphi)H^{q^t}/H^{q^t} \leqslant C_{H/H^{q^t}}(\overline{\varphi}),$$

we obtain that $C_H(\varphi)/C_H(\varphi) \cap H^{q^t}$ is abelian. So, for any elements $\bar{h}_1, \bar{h}_2 \in C_H(\varphi)/C_H(\varphi) \cap H^{q^t}$ we have $[\bar{h}_1, \bar{h}_2] = 1$. It follows that $[h_1, h_2] \in C_H(\varphi) \cap H^{q^t} \leqslant H^{q^t}$. But $\bigcap_{t>0} H^{q^t} = 1$, thus $[h_1, h_2] = 1$ for any $h_1, h_2 \in C_H(\varphi)$. That is to say, $C_H(\varphi)$ is abelian. The proof is completed. $\square$

## 3. Proof of Theorem 1.2

**Lemma 3.1.** *Let $G$ be a polycyclic group and $\alpha$ an automorphism of finite order $n$ of $G$. If the map $\varphi\colon\ G \to G$ $(g \mapsto [g,\alpha])$ is surjective, then $C_G(\alpha)$ is finite.*

Proof. By Lemma 2.1, for all $x \in C_G(\alpha) \leqslant G$ we have

$$xx^{\alpha}x^{\alpha^2}\dots x^{\alpha^{n-1}} = x^n = 1.$$

This implies that the exponent of $C_G(\alpha)$ is finite. Hence $C_G(\alpha)$ is finite. □

**Lemma 3.2.** *Let $\alpha$ be an automorphism of a polycyclic group $G$ and $H$ a normal $\alpha$-invariant finite subgroup of $G$. If $C_{G/H}(\overline{\alpha})$ is abelian-by-finite, then $C_G(\alpha)$ is abelian-by-finite.*

Proof. Notice that $C_G(\alpha)/C_G(\alpha)\cap H \simeq C_G(\alpha)H/H \leqslant C_{G/H}(\overline{\alpha})$, and we have $C_G(\alpha)$ is finite-by-abelian-by-finite. We may assume that $B$ is a normal finite-by-abelian subgroup of $C_G(\alpha)$ and $C_G(\alpha)/B$ is finite. By Theorem 1.3.4 of [7], $C_G(\alpha)$ has a normal torsion-free subgroup $A$ of finite index. Hence

$$C_G(\alpha)/A\cap B \leqslant C_G(\alpha)/A \times C_G(\alpha)/B$$

is finite. Since $A$ is torsion-free, $A\cap B$ is abelian. Thus $C_G(\alpha)$ is abelian-by-finite. □

Proof of (i) of Theorem 1.2. Set $\varphi = \alpha^2$. It suffices to prove that $C_G(\varphi)$ is abelian-by-finite. Denote by $T$ the torsion subgroup of $G$. If $C_{G/T}(\overline{\varphi}) = 1$, then $\overline{\varphi}$ is a regular automorphism of order 2 of $G/T$. By Theorem 1.1 of [2], $G/T$ is abelian-by-finite. Hence $C_{G/T}(\overline{\varphi})$ is abelian-by-finite. Since $T$ is finite, it follows from Lemma 3.2 that $C_G(\varphi)$ is abelian-by-finite. If $C_{G/T}(\overline{\varphi}) \neq 1$, then $C_{G/T}(\overline{\varphi})$ is $\overline{\alpha}$-invariant, and thus $\overline{\alpha}$ is an automorphism of order 1 or 2 of $C_{G/T}(\overline{\varphi})$. By Lemma 3.1 and (ii) of Lemma 2.4 of [2], $C_{G/T}(\overline{\alpha})$ is finite. Since

$$C_{C_{G/T}(\overline{\varphi})}(\overline{\alpha}) \leqslant C_{G/T}(\overline{\alpha}),$$

$C_{C_{G/T}(\overline{\varphi})}(\overline{\alpha})$ is finite. If $\overline{\alpha}$ is an automorphism of order 1 of $C_{G/T}(\overline{\varphi})$, then $C_{G/T}(\overline{\varphi})$ is finite. By Theorem 1.1 of [2], $G/T$ is abelian-by-finite. Hence $C_{G/T}(\overline{\varphi})$ is abelian-by-finite. By Lemma 3.2, $C_G(\varphi)$ is abelian-by-finite. If $\overline{\alpha}$ is an automorphism of order 2 of $C_{G/T}(\overline{\varphi})$, it follows from Theorem 1.1 of [2] that $C_{G/T}(\overline{\varphi})$ is abelian-by-finite. By Lemma 3.2, $C_G(\varphi)$ is abelian-by-finite. □

**Lemma 3.3.** *Let $\alpha$ be an automorphism of a polycyclic group $G$ and $H$ a normal $\alpha$-invariant finite subgroup of $G$. If $C_G(\alpha)$ is abelian-by-finite, then $C_{G/H}(\overline{\alpha})$ is abelian-by-finite.*

P r o o f. Set $C/H = C_{G/H}(\overline{\alpha})$ and consider the map $\varphi\colon C \to H$ defined by $g^{\varphi} = g^{-1}g^{\alpha}$. On the one hand, for any $x \in C_G(\alpha)$, we have

$$(xg)^{\varphi} = (xg)^{-1}(xg)^{\alpha} = g^{-1}x^{-1}x^{\alpha}g^{\alpha} = g^{-1}g^{\alpha}.$$

This shows that $\varphi$ maps all elements of the coset $C_G(\alpha)g$ to $g^{-1}g^{\alpha}$. On the other hand, if $g_1^{-1}g_1^{\alpha} = g_2^{-1}g_2^{\alpha}$, then

$$(g_1g_2^{-1})^{\alpha} = g_1g_2^{-1},$$

and thus $g_1g_2^{-1} \in C_G(\alpha)$. This implies that $C_G(\alpha)g_1 = C_G(\alpha)g_2$. It follows that $|C : C_G(\alpha)| \leqslant |H| < \infty$.

We may assume that $A$ is a characteristic abelian subgroup of $C_G(\alpha)$ of finite index without loss of generality. Since $C/A/C_G(\alpha)/A \simeq C/C_G(\alpha)$ and $C_G(\alpha)/A$ is finite, one has that $C/A$ is finite. Noticing that

$$C_{G/H}(\overline{\alpha})/AH/H = C/H/AH/H \simeq C/AH \leqslant C/A,$$

we have that $C_{G/H}(\overline{\alpha})/AH/H$ is finite. Since $AH/H \simeq A/A\cap H$ is abelian, $C_{G/H}(\overline{\alpha})$ is abelian-by-finite. □

**Lemma 3.4.** *Let $\alpha$ be an automorphism of order 2 of an abelian group $A$. Then for any $x \in A$, the element $x^2$ can be written in the form $x^2 = ay^{-1}y^{\alpha}$, where $a \in C_A(\alpha)$ and $y \in A$.*

P r o o f. Consider an element $x \in A$ and put $a = xx^{\alpha}$. Then $a \in C_A(\alpha)$. Obviously, $x^{-2}a = x^{-1}x^{\alpha} \in [A,\alpha]$. Set $u = x^{-2}a$. Then $x^2 = au^{-1}$, with $a \in C_A(\alpha)$ and $u^{-1} \in [A,\alpha]$. Since $A$ is abelian, $u^{-1}$ is of the form $y^{-1}y^{\alpha}$. The proof is completed. □

**Lemma 3.5.** *Let $\alpha$ be an automorphism of order 2 of a polycyclic group $G$ and $A$ a torsion-free normal $\alpha$-invariant abelian subgroup of $G$. If $C_G(\alpha)$ is abelian-by-finite, then $C_{G/A}(\overline{\alpha})$ is abelian-by-finite.*

P r o o f. Put $A_0 = A^2$ and denote by $\overline{\alpha}_0$ the automorphism induced by $\alpha$ on $G/A_0$. Consider an element $x \in G$ such that $xA_0 \in C_{G/A_0}(\overline{\alpha}_0)$. Then $x^{\alpha} = xh$ for some $h \in A_0$. By Lemma 3.4, $h = ay^{-1}y^{\alpha}$, where $a \in C_A(\alpha)$ and $y \in A$. Thus $x^{\alpha} =$

$xay^{-1}y^{\alpha}$. Note that $x^{\alpha^2} = xa^2y^{-1}y^{\alpha^2}$, and we have $a^2 = 1$. Since $A$ is torsion-free, $a = 1$. Clearly, $x^{\alpha} = xy^{-1}y^{\alpha}$. It follows that $(xy^{-1})^{\alpha} = xy^{-1}$, hence $v = xy^{-1} \in C_G(\alpha)$ and so $x = vy \in C_G(\alpha)A$. This implies that

$$C_{G/A_0}(\overline{\alpha}_0) \leqslant C_G(\alpha)A/A_0 = C_G(\alpha)A_0/A_0 \cdot A/A_0.$$

Set $B = C_G(\alpha) \cap A_0$. Then $C_G(\alpha)/B \simeq C_G(\alpha)A_0/A_0$. We can assume $A_1$ is a characteristic abelian subgroup of $C_G(\alpha)$ of finite index. Observe that

$$(C_G(\alpha)/B)/(A_1/A_1 \cap B) \simeq (C_G(\alpha)/B)/(A_1B/B) \simeq C_G(\alpha)/A_1B \leqslant C_G(\alpha)/A_1,$$

and we have that $C_G(\alpha)/B$ is abelian-by-finite. It is easy to see that $C_{G/A_0}(\overline{\alpha}_0)$ is abelian-by-finite. But $\overline{\alpha}_0$ induces the automorphism $\overline{\alpha}$ on $G/A$ and $A/A_0$ is finite. Consequently, it follows from Lemma 3.3 that $C_{G/A}(\overline{\alpha})$ is abelian-by-finite. □

**Lemma 3.6.** *Let $\alpha$ be an automorphism of a group $G$. If $[G, \alpha]$ is finite, then the index of $C_G(\alpha)$ in $G$ is finite.*

Proof. Denote by $n$ the order of $[G, \alpha]$ and consider $n+1$ elements $g_1, g_2, \ldots, g_{n+1}$ in $G$. Therefore, among the elements

$$g_1^{-1}g_1^{\alpha}, g_2^{-1}g_2^{\alpha}, \ldots, g_{n+1}^{-1}g_{n+1}^{\alpha}$$

at least two coincide. If $g_i^{-1}g_i^{\alpha} = g_j^{-1}g_j^{\alpha}$ $(i, j \in \{1, 2, \ldots, n+1\},\ i \neq j)$, then $(g_ig_j^{-1})^{\alpha} = g_ig_j^{-1}$, and so $g_ig_j^{-1} \in C_G(\alpha)$. Hence $|G : C_G(\alpha)| \leqslant n$. □

**Lemma 3.7.** *Let $\alpha$ be an automorphism of order 2 of a polycyclic group $G$. If $C_G(\alpha)$ is abelian-by-finite, then $G/[G, \alpha]$ is abelian-by-finite.*

Proof. We proceed by induction on the Hirsch length $h(G)$ of $G$. The result is obvious when $h(G) = 0$, so suppose that $h(G) > 0$. By Lemma 3.6, if $[G, \alpha]$ is finite, then $G/C_G(\alpha)$ is finite. It follows that $G/C_G(\alpha)[G, \alpha]$ is finite. Since $C_G(\alpha)$ is abelian-by-finite, $G/[G, \alpha]$ is abelian-by-finite. Therefore, we can assume that $[G, \alpha]$ is infinite. By Theorem 5.4.15 of [9], $[G, \alpha]$ contains a non-trivial torsion-free characteristic abelian subgroup $A$. It follows from Lemma 3.5 that $C_{G/A}(\overline{\alpha})$ is abelian-by-finite. Since $h(G/A) < h(G)$, we deduce from the inductive hypothesis that $G/A/[G/A, \overline{\alpha}]$ is abelian-by-finite. But $[G/A, \overline{\alpha}] = [G, \alpha]/A$ and $(G/A)/([G, \alpha]/A) \simeq G/[G, \alpha]$, hence $G/[G, \alpha]$ is abelian-by-finite. □

Proof of (ii) of Theorem 1.2. Set $\varphi = \alpha^2$. By Part (i) of Theorem 1.2, $C_G(\varphi)$ is abelian-by-finite. Thus the second part of Theorem 1.2 follows from Lemma 3.7. □

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