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A FIEDLER-LIKE THEORY FOR THE PERTURBED LAPLACIAN

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To the memory of Miroslav Fiedler, whose work and kindness has been an inspiration to many mathematicians all over the world.

Abstract. The perturbed Laplacian matrix of a graph G is defined as $L^D = D - A$, where D is any diagonal matrix and A is a weighted adjacency matrix of G. We develop a Fiedler-like theory for this matrix, leading to results that are of the same type as those obtained with the algebraic connectivity of a graph. We show a monotonicity theorem for the harmonic eigenfunction corresponding to the second smallest eigenvalue of the perturbed Laplacian matrix over the points of articulation of a graph. Furthermore, we use the notion of Perron component for the perturbed Laplacian matrix of a graph and show how its second smallest eigenvalue can be characterized using this definition.

Keywords: perturbed Laplacian matrix; Fiedler vector; algebraic connectivity; graph partitioning

MSC 2010: 05C50, 15B57, 05C22

1. Motivation

The object of spectral graph theory is to determine structural properties of a graph G from the spectrum of the matrices one may associate with G. The matrices that are well known and for which there are many results are the adjacency matrix A and the combinatorial Laplacian matrix $L = d - A$, where d is the diagonal degree matrix.

The normalized Laplacian matrix $\mathcal{L} = d^{-1/2}Ld^{-1/2}$, popularized by Chung [4] in the 90s, has a more recent record and fewer results are known. In particular,

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given a graph G, one may ask whether the matrix $\mathcal{L}$ gives additional or distinct information from what L gives. Bapat, Kirkland and Pati [1] defined the perturbed Laplacian matrix of a graph G as $\not{L} = D - A$, where D is any diagonal matrix and A is a weighted adjacency matrix of G. We note that $\mathcal{L}$ is a perturbed Laplacian matrix.

As in [1], the goal of this paper is also to study the second smallest eigenvalue of $\not{L}$ and its eigenvector. Here we develop a Fiedler-like theory for this matrix, leading to results that are of the same type as those obtained with the algebraic connectivity of a graph G. In order to state the kind of results we discuss in this paper, we recount some of Fiedler's findings on the second smallest eigenvalue of the Laplacian L, called the algebraic connectivity, one of the most studied spectral parameters. Indeed, the pioneering work of Fiedler [7], [6] is qualified as a mathematical *gold strike* by Nikiforov [12] due to its impact in many areas of pure and applied science.

An eigenvector associated with the algebraic connectivity is called a Fiedler vector. Labeling the vertices of G by $v_1, v_2, \ldots, v_n$ and denoting a Fiedler vector by $y = [y_i]$, the coordinates of y can be assigned to the vertices of G naturally: the coordinate y_i labels the vertex v_i. This assignment has been called the *characteristic valuation* and Fiedler noticed that it induces partitions of the vertices of G that are natural connected clusters, important for applications and for characterizing the graph structure. If G is a connected graph, a vertex v is called an *articulation point* if the graph $G \setminus v$ is disconnected. A *block* in G is any maximal induced connected subgraph with no articulation points.

Fiedler's monotonicity theorem [7] shows (among other results) that exactly one of the following cases occurs.

Case A: There is exactly one block C in G which contains the positive as well as negative vertices. Every other block is either a positive block, or a negative block, or a zero block.

Case B: No block of G contains positive as well as negative vertices. In this case, there exists a unique characteristic vertex z. This vertex z is a point of articulation. Each block (with the exception of z) is either a positive block, or a negative block, or a zero block.

The word monotonicity is justified because Fiedler also shows that the characteristic valuation through articulation points is a monotone sequence.

In this paper we show a property of the harmonic eigenfunction of the second smallest eigenvalue of $\not{L}$ (see the next section for the definitions) over the points of articulation which, as in the work of Fiedler, enables us to classify every graph into two distinct families.

In the late 90s, Kirkland, Neumann, Shader [9] and Fallat [8] used Fiedler's theory and the Perron values of matrices associated with the components $G \setminus \{v\}$ (v an articulation point) in order to characterize the algebraic connectivity of trees [9] and for graphs with articulation points [8].

In this paper, using the notion of a Perron component for the perturbed Laplacian matrix of a graph, we provide a characterization of the second smallest eigenvalue in terms of this definition. Moreover, we introduce the notion of a *perturbed bottleneck matrix* of a branch of a tree, which allows us to describe the second smallest eigenvalue and the characteristic vertices of trees as a function of its Perron values.

As a general outcome of this note, we show that the eigenvector of the second smallest eigenvalue of a perturbed Laplacian matrix gives partition properties that are similar to the properties that the Fiedler vector gives to a graph. The characterizations we provide are also of the same type as Kirkland et al. gave. We point out, however, that the two eigenvectors do not give the same information. The following example shows a graph whose sign partitions provided by the eigenvectors are distinct.

Example 1.1. Consider P_7 the path on 7 vertices. Its Laplacian matrix has one eigenvector for the algebraic connectivity whose coordinates are approximately

$$[-1\ \ -0.8\ \ -0.44\ \ 0\ \ 0.44\ \ 0.8\ \ 1]^{\mathrm{T}}.$$

On the other hand, if we consider one perturbed Laplacian matrix given by

$$\begin{bmatrix} 4 & -4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 8 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

we obtain one eigenfunction of the second smallest eigenvalue of the above matrix given by (approximately)

$$[-0.71\ \ -0.64\ \ -0.5\ \ -0.64\ \ -0.02\ \ 0.6\ \ 1]^{\mathrm{T}}.$$

Clearly the sign partition does not induce the same set of vertices of P_7.

Saying that, it is reasonable to ask when, for a given graph G, the sign partition (given by the signs of the eigenvector entries) provided by eigenvectors of the algebraic connectivity and the second smallest eigenvalue of $Ł$ are always the same. A more precise question is the following.

> Let G be a graph with Laplacian matrix L and sign partition obtained by a Fiedler vector. Is there an eigenfunction of the second smallest eigenvalue of $\pounds$ that gives the same sign partition?

We give sufficient conditions to answer this question affirmatively. In the last section of the paper we provide results for graphs satisfying both the Cases A and B of Fiedler's monotonicity theorem.

The main results of this paper are the mathematical proofs of the fact that the eigenvectors of the second smallest eigenvalues of L and $\pounds$ give similar type of information. We emphasize, however, that it is not true that the matrices L and $\pounds$ (and in particular the normalized $\mathcal{L}$) provide the same kind of information for a given graph. As an example, we point out that the Laplacian energy and the normalized Laplacian energy have distinct behavior. Moreover, in light of the last example, the sign partition provided by the eigenvectors may have distinct behavior as well. In any event, since the algebraic connectivity and its Fiedler vector has led to many important applications, we hope that the theory presented here may also lead to important and perhaps different findings.

2. Notation and known results

In this paper we denote by $G = G(V, E)$ a connected weighted graph, which is a graph with vertex set V and edge set E, where each edge is associated with a positive number (the weight of the edge).

Given a connected weighted graph $G = (V, E)$ on n vertices, the adjacency matrix of G is the order n matrix defined by

$$A(v_i, v_j) = \begin{cases} w_{ij}, & \text{the weight of the edge } v_i v_j, \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

For any real diagonal matrix D, as defined in [1], the *perturbed Laplacian matrix* of G, denoted by $\pounds$, is given by $\pounds = D - A$.

We notice that if $D(i,i) = \sum\limits_{i \neq j} w_{ij} := w_i$ for each $1 \leqslant i \leqslant n$, then $\pounds$ is simply $L = D - A$, the combinatorial Laplacian matrix for G, while when $D(i,i) = -w_i$ for each $1 \leqslant i \leqslant n$, we have $\pounds = -Q = -(D + A)$, where Q is the signless Laplacian. When D is the null matrix, $\pounds = -A$, the adjacency matrix. One more important instance of the perturbed Laplacian worth mentioning is the (weighted) *normalized*

Laplacian matrix $\mathcal{L}(G)$ defined by Chung [5] as

$$\mathcal{L}(v_i, v_j) = \begin{cases} 1, & v_i = v_j \text{ and } w_i \neq 0; \\ \dfrac{-w_{ij}}{\sqrt{w_i w_j}}, & \text{whenever } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

As G is connected, it does not have isolated vertices and in this case, the diagonal degree matrix d is invertible and $\mathcal{L}$ and L are related by the formula

$$\mathcal{L} = d^{-1/2} L d^{-1/2},$$

which may be written as $\mathcal{L} = d^{-1/2} L d^{-1/2} = d^{-1/2}(d - A)d^{-1/2} = I - d^{-1/2} A d^{-1/2}$, a perturbed Laplacian matrix.

The normalized Laplacian eigenvalues were first studied by Chung in [4]. More recently, Butler [2] and Cavers [3] have provided numerous spectral properties of $\mathcal{L}$ and among them the fact that, like the algebraic connectivity, the second smallest eigenvalue of $\mathcal{L}$ is nonzero if and only if the graph is connected.

Chung has also defined the notion of an eigenfunction as follows. Let g denote a function which assigns to each vertex v of G a real value $g(v)$. We may view g as a column vector and whenever $\mathcal{L}g = \lambda g$, we call g an eigenfunction of $\mathcal{L}$. Further, the harmonic eigenfunction of λ for $\mathcal{L}$ is defined as $f = d^{-1/2} g$. We point out that the sign partition induced by the harmonic eigenfunction for the normalized Laplacian has been studied by Li, Li and Fan in [10].

We consider the perturbed Laplacian matrix $\not{\mathcal{L}}$ of a connected weighted graph G and denote the eigenvalues of $\not{\mathcal{L}}$ by $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$. We note that, since G is connected, $\not{\mathcal{L}}$ is irreducible and by the Perron-Frobenius theory, it follows that the smallest eigenvalue of $\not{\mathcal{L}}$ is simple and has a corresponding eigenvector with all entries positive.

To be consistent with Chung's notation, we call g an eigenfunction of $\not{\mathcal{L}}$ for the eigenvalue λ whenever $\not{\mathcal{L}} g = \lambda g$. We denote by z the eigenfunction associated with the smallest eigenvalue of $\not{\mathcal{L}}$ and we define the harmonic eigenfunction of λ_i for $i \geqslant 2$ as

$$f = \frac{g}{z} = \Big[\frac{g(v_1)}{z(v_1)}, \ldots, \frac{g(v_n)}{z(v_n)}\Big].$$

The vectors f and g for λ_2 were studied in [1] and it turns out that, in the context of normalized Laplacian, they are compatible with the definition of eigenfunction and harmonic eigenfunction provided by Chung in [4].

In this paper we further investigate the harmonic eigenfunction, providing a monotonicity theorem for this function in the next section. This result turns out to be a fundamental tool for our approach to characterizing and computing λ_2. Since the perturbed Laplacian encompasses both the Laplacian and normalized Laplacian as well as adjacency matrices, the general framework of the theory presented here may be useful in many particular situations. We point out, however, that some of the results have a restrictive hypothesis. Typically, it is required that the smallest eigenvalue of $\mathcal{L}$ is zero, which is a reasonable condition, but it rules out the adjacency matrix, in general.

3. Monotonicity theorem

In this section we show a property of the harmonic eigenfunction of λ_2 over the points of articulation of a graph. We shall provide a monotonicity theorem for such a harmonic eigenfunction. This enables us to classify every graph into one of two distinct families.

Let us recall that a block of a graph is a maximal induced connected subgraph not containing a point of articulation. Hence, a block is either a maximal 2-connected subgraph, or a bridge (with its ends) or an isolated vertex. We say that a block is positive, negative or null if, over that block, f is positive, negative or null, respectively. If over a block, f assumes positive and negative values, then we call it a mixed block.

Let g be an eigenfunction of $\mathcal{L}$ corresponding to λ_2. A vertex v is called a *characteristic vertex* of G if $g(v) = 0$ and if there is a vertex w adjacent to v such that $g(w) \neq 0$. An edge $\{u, v\}$ is called a *characteristic edge* of G if $g(u)g(v) < 0$. We define the *characteristic set* as the collection of all characteristic vertices and characteristic edges of G. Here $\varrho(M)$ denotes the Perron value of the matrix M.

Remark 3.1. Let g be an eigenfunction of $\mathcal{L}$ corresponding to λ_2 and f its harmonic eigenfunction. Then $\operatorname{sign}(f(v)) = \operatorname{sign}(g(v))$ for each vertex v of G.

The next lemma follows from Lemma 3 in [1] adapted to our notation.

Lemma 3.2. *Let G be a connected weighted graph and g an eigenfunction of $\mathcal{L}$ corresponding to λ_2. Let W be a nonempty set of vertices of G such that $g(u) = 0$ for all $u \in W$ and suppose $G \setminus W$ is disconnected with $t \geqslant 2$ components $C_1, C_2, \ldots, C_t$ such that $g(C_i) \neq 0$. Let $\mathcal{L}(C_i)$ be the principal submatrix of $\mathcal{L}$ corresponding to the component C_i. Then each C_i satisfies either $g(C_i) > 0$ or $g(C_i) < 0$ with $\lambda_2 = 1/(\varrho(\mathcal{L}(C_i)^{-1}))$ in either case.*

The next lemma follows from Lemma 4 in [1].

Lemma 3.3. *Let G be a connected weighted graph. Let g be an eigenfunction of $\not{\mathcal{L}}$ corresponding to λ_2. Suppose that the characteristic set contains an edge. Then the vertices v such that $g(v) > 0$ induce a connected subgraph.*

The next lemma follows from part (ii) of Lemma 5 in [1].

Lemma 3.4. *Let G be a connected weighted graph and let g be an eigenfunction of $\not{\mathcal{L}}$ corresponding to λ_2. Let S be its characteristic set. Then either S is a single vertex or S is contained in a block of G.*

The next lemma follows from Remark 3.1 and Corollary 16 in [1]. It is worth mentioning that this result was proved under the hypothesis that the smallest eigenvalue of $\not{\mathcal{L}}$ is zero even though it is not explicitly written in the original statement.

Lemma 3.5. *Let v be a point of articulation of G and C a hanging component at v. Assume that $f(C) > 0$ for some harmonic eigenfunction f of $\not{\mathcal{L}}$ corresponding to λ_2 and assume that the smallest eigenvalue of $\not{\mathcal{L}}$ is zero. Let u be any vertex in C. Then $f(v) < f(u)$.*

A path is said to be pure if it contains at most two articulation points in each block.

Theorem 3.6. *Let G be a connected graph with perturbed Laplacian matrix $\not{\mathcal{L}}$. Let g be an eigenfunction of $\not{\mathcal{L}}$ corresponding to λ_2. Let f be the corresponding harmonic eigenfunction. Then only one of the following cases can occur.*

Case 1: *G has no mixed block, i.e. a block with both positive and negative values of g. In this case, there is a unique point of articulation z having $f(z) = 0$ and a nonzero neighbour. Each block (with the exception of the vertex z) is either a positive block, or a negative block, or a zero block. Let P be a pure path which starts at z. If the smallest eigenvalue of $\not{\mathcal{L}}$ is zero, then over the points of articulation of G in P (with the exception of z), f forms either an increasing, or decreasing, or a zero sequence. Every path containing both positive and negative vertices passes through z.*

Case 2: *G has a unique block B_0 which is mixed. In this case, each of the remaining blocks is positive, negative or null. If the smallest eigenvalue of $\not{\mathcal{L}}$ is zero, then each pure path P starting in B_0 and containing only one vertex $v \in B_0$ has the property that over the points of articulation contained in P, f forms either an increasing, or decreasing, or a zero sequence according to whether $f(v) > 0$, $f(v) < 0$ or $f(v) = 0$. In the last case, $f \equiv 0$ along the path.*

P r o o f. First, for Case 1. If no block is mixed, then there is one block with a positive vertex and one block with a negative vertex. Furthermore, since the intersections of blocks have only articulation points and no block is mixed, it follows that there exists an articulation point z with $f(z) = 0$. Now, applying Lemma 3.2 with $W = \{z\}$, we obtain that each component is either null, positive or negative. Therefore, there is no other vertex $v \neq z$ having $f(v) = 0$ and a nonzero neighbour. This shows the first part of Case 1.

Also, if P contains another vertex v with $f(v) = 0$, by the previous argument we can see that $f \equiv 0$ over the vertices of P. On the other hand, if P has a vertex v with $f(v) \neq 0$, then we denote by $z = v_0, v_1, \ldots, v_s$ the points of articulation in P in the order they appear. If $f(v) > 0$, then from Lemma 3.5 we obtain $f(v_i) < f(v_{i+1})$, $i = 0, \ldots, s-1$. If $f(v) < 0$, then the same argument applied to the eigenfunction $-f$ shows that this forms a decreasing sequence.

Now we proceed to proving Case 2. If G has only one block, then we are done. If the characteristic set contains an edge, then Lemma 3.3 ensures that the remaining blocks are either positive or negative. Otherwise, Lemma 3.4 implies that only one block contains the characteristic set. Now by Lemma 3.2 and Lemma 3.3 all the remaining blocks are either null, positive or negative. This completes the first part of Case 2.

Finally, denote by $v = v_0, v_1, \ldots, v_s$ the points of articulation in P in the order they appear. If $f(v) > 0$, then Lemma 3.5 we obtain $f(v_i) < f(v_{i+1})$, $i = 0, \ldots, s-1$. If $f(v) < 0$, then the same argument applied to the eigenfunction $-f$ shows that this forms a decreasing sequence. If $f(v) = 0$, then only the component at v that contains non-null vertices is the component containing the mixed block, otherwise it would contradict Lemma 3.2. Therefore, we obtain that $f \equiv 0$ over the vertices of P. This concludes the proof. $\square$

Remark 3.1 and Theorem 3.6 give us the following result.

Theorem 3.7. *Let G be a connected graph and let g be the eigenfunction of $D\!\!\!\!L$ corresponding to λ_2. Then only one of the following cases can occur.*

Case 1: *There is no mixed block. In this case, there is a unique point of articulation z having $g(z) = 0$ and a nonzero neighbour. Each block (removing vertex z) is either a positive block, or a negative block, or a zero block.*

Case 2: *There is a unique block B_0 which is mixed. In this case, each of the remaining blocks is either positive, negative or null.*

Theorem 3.7 does not claim the monotonicity property for the eigenfunction as Theorem 3.6 does for the harmonic eigenfunction. In fact, it is not always the case that the eigenfunction has this property. The next example shows a graph without this property.

Example 3.8. For the graph shown in Figure 1 the spectrum of its normalized Laplacian matrix has $\lambda_2 = 0.140\,851\,868\,4$ and its eigenfunction g is given by

$$\begin{matrix} [0.332\,146\,838\,0 & 0.403\,562\,860\,1 & 0.353\,716\,996\,1 & -0.353\,717\,003\,3 \\ -0.403\,562\,857\,4 & -0.332\,146\,834\,1 & -0.184\,119\,332\,7 & -0.184\,119\,332\,7 \\ -0.184\,119\,332\,7 & 0.184\,119\,332\,7 & 0.184\,119\,332\,7 & 0.184\,119\,332\,7]^{\mathrm{T}}. \end{matrix}$$

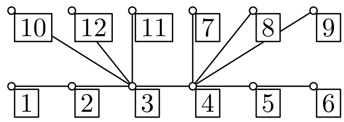

Figure 1. The graph whose monotonicity does not hold for the eigenfunction.

Since the graph is a tree, every vertex is a point of articulation and every edge with its vertices is a block. We notice that Case 2 of Theorem 3.6 holds for this graph. Consider the pure path $P = \{1, 2, 3\}$. It is easy to compute the harmonic eigenfunction f. Satisfies $f(1) > f(2) > f(3)$ in accordance to Theorem 3.6. On the other hand, the eigenfunction g satisfies $g(3) < g(2)$ and $g(2) > g(1)$. Hence, the monotonicity property does not hold for the eigenfunction.

4. Characterizing the second smallest eigenvalue

Despite giving classification for graphs and a good insight into the behavior of the harmonic eigenfunction, Theorems 3.6 and 3.7 do not give us information about λ_2 itself. However, there is a different characterization for Cases 1 and 2 such that the information about λ_2 arises.

More precisely, in this section we are interested in describing λ_2 in terms of the Perron value of special matrices. These results were inspired by [9].

Let G be a connected graph, $\not{\mathcal{L}}$ its perturbed Laplacian matrix and v a cut vertex of G, with $C_0, C_1, \ldots, C_r$ as all connected components of the graph $G \setminus v$. For each component C_i, let $\not{\mathcal{L}}(C_i)$ be the principal submatrix of $\not{\mathcal{L}}$ corresponding to the vertices of C_i. Whenever these matrices are nonsingular with inverses $M_i = \not{\mathcal{L}}^{-1}(C_i)$, we call them the *perturbed bottleneck matrices of C_i at v*. Each such inverse is entry-wise positive, and so by Perron's Theorem, it has a simple positive dominant eigenvalue called the Perron value, and a corresponding eigenvector with all entries positive called the Perron vector. A component C_j is said to be the *Perron component at v* if its Perron value is maximal among $C_0, C_1, \ldots, C_r$.

If G satisfies Case 1 of Theorem 3.7, then we say that G is a Type 1 graph. If G satisfies Case 2 of Theorem 3.7, then we say that G is a Type 2 graph.

If G is a Type 1 graph, then the only null vertex adjacent to a non-null vertex (see Theorem 3.7) is said to be the *characteristic vertex of G*. In the context of the combinatorial Laplacian matrix, it is shown in [11] that the characteristic vertices of a tree are independent of the eigenvector associated with the algebraic connectivity.

For Type 1 graphs it is easy to characterize $\lambda_2(\cal{\underline{L}})$ and the following theorem is just a recast of the results in [1] using our framework.

Theorem 4.1. *Let G be a graph and g an eigenfunction of λ_2. Then G is a Type 1 graph with characteristic vertex v if and only if there are at least two Perron components at v. In this case, $\lambda_2 = 1/\varrho(\cal{\underline{L}}(C)^{-1})$ for each Perron component C at v.*

Proof. The first part follows from parts (i) and (iv) of Theorem 7 in [1] and $\lambda_2 = 1/\varrho(\cal{\underline{L}}(C)^{-1})$ follows from Lemma 3.2 by taking $W = \{v\}$. $\square$

The previous theorem may be seen as a natural generalization of the result in [9] where, in the context of the Laplacian matrix, the authors characterized the algebraic connectivity for Type 1 trees.

A Type 1 graph (or a Type 2 graph) which is a tree is called a Type 1 tree (a Type 2 tree, respectively). Now, we want to find some characterization for λ_2 of Type 2 trees using the perturbed Laplacian matrix. However, a different method must be used. As the next theorem shows, these matrices are more complicated than those in [9].

First, we need some definitions. Let T be a tree. We call a branch of T at v any of the connected components of $T \setminus v$ obtained from T by deleting the vertex v and its edges. If T is a Type 2 tree, by Theorem 3.7 the only mixed block is formed by only two adjacent vertices. For a Type 2 tree, we say that two vertices i and j are characteristic vertices if and only if they are adjacent and satisfy $\operatorname{sign}(g(i)) \neq \operatorname{sign}(g(j))$.

Consider the matrix $\Delta L \Delta$, where L is the combinatorial Laplacian matrix and Δ is a positive diagonal matrix. Since $\Delta L \Delta = \Delta D \Delta - \Delta A \Delta$, it is clear that this is a special kind of perturbed Laplacian matrix. From now on, we assume that the perturbed Laplacian matrix is of the form $\Delta L \Delta$.

Theorem 4.2. *Let T be a Type 2 tree on n vertices with perturbed Laplacian matrix $\cal{\underline{L}} = \Delta L \Delta$ and let i and j be adjacent vertices of T with edge weight w. Then i and j are the characteristic vertices of T if and only if there exists a $\gamma \in (0,1)$ such that*

$$\varrho\Big(M_1 - \frac{1}{w}\gamma D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}}D_1^{-1}\Big) = \varrho\Big(M_2 - \frac{1}{w}(1-\gamma) D_2^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}}D_2^{-1}\Big) = \frac{1}{\lambda_2},$$

where M_1 is the perturbed bottleneck matrix for the branch at j containing i, and D_1 is the submatrix of Δ corresponding to this branch, M_2 is the perturbed bottleneck matrix for the branch at i containing j, and D_2 is the submatrix of Δ corresponding to this branch.

To prove Theorem 4.2 we need more information about the matrices M_1 and M_2. Therefore, the proof is given in the next section.

5. Perturbed bottleneck matrix of trees

In the previous section we pointed out that in order to characterize λ_2 of trees, it is necessary to study the perturbed bottleneck matrices. Hence, in this section we perform a more careful analysis of the structure of these matrices with the expectation of characterizing λ_2 and proving Theorem 4.2. Along the way we also provide a simple way to characterize Type 1 and Type 2 trees in terms of Perron components.

First, we define the set $P_{i,j,k}$ as the set of edges of T which are on both the path from vertex i to vertex k and the path from vertex j to vertex k. The following lemma was obtained by Kirkland in [9], where he investigated Perron components of trees using the Laplacian matrix.

Lemma 5.1. *Consider a tree T with n vertices. Denote by L_k the principal submatrix of the Laplacian matrix $L(T)$ obtained by deleting the k-th column and the k-th row from $L(T)$. Then the entry (i,j) of L_k^{-1} is equal to $\sum\limits_{e\in P_{i,j,k}} 1/w(e)$, where $w(e)$ is the weight of the edge e.*

Likewise, for a perturbed Laplacian matrix $\not{L}$, we can describe the entries of $\not{L}_k^{-1}$ in terms of $P_{i,j,k}$ as the following result shows.

Lemma 5.2. *Consider a tree T with n vertices with perturbed Laplacian matrix $\not{L} = \Delta L \Delta$. Then the entry (i,j) of $\not{L}_k^{-1}$ is equal to $(d_i d_j)^{-1} \sum\limits_{e\in P_{i,j,k}} 1/w(e)$, where d_i is the diagonal entry of Δ corresponding to the vertex v_i.*

P r o o f. We observe that since Δ is a diagonal matrix, we have

$$\not{L}_k = (\Delta L \Delta)_k = \Delta_k L_k \Delta_k.$$

Thus, it is straightforward to obtain $\not{L}_k^{-1} = \Delta_k^{-1} L_k^{-1} \Delta_k^{-1}$. By applying Lemma 5.1, we obtain that the (i,j) entry of $\not{L}_k^{-1}$ is equal to

$$(\Delta_k^{-1} L_k^{-1} \Delta_k^{-1})_{i,j} = (\Delta_k^{-1})_{i,i} \sum_{e\in P_{i,j,k}} \frac{1}{w(e)} (\Delta_k^{-1})_{j,j} = \frac{1}{d_i d_j} \sum_{e\in P_{i,j,k}} \frac{1}{w(e)}.$$

□

Since now we have a good description of the perturbed bottleneck of a component, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. As usual, e_i denotes the canonical vector with nonzero entry on the i-th position. Then we can write the perturbed Laplacian matrix of T in the format

$$\tilde{L} = \begin{bmatrix} M_1^{-1} & -d_i d_j w e_k e_1^{\mathrm{T}} \\ -d_i d_j w e_1 e_k^{\mathrm{T}} & M_2^{-1} \end{bmatrix},$$

where the last row of M_1^{-1} represents the vertex i and the first row of M_2^{-1} represents the vertex j.

First, we suppose that i and j are characteristic vertices of T. Hence, the block composed by them and its incident edge is the unique mixed block. By Theorem 3.7 we have that the vertices in the branch at i containing j have the sign of the vertex j, whereas the vertex in the branch at j containing i have the sign of the vertex i. Hence, this two branches have opposite signs. Moreover, the theorem ensures that we can write the eigenvector associated with λ_2 as $v = [-v_1^{\mathrm{T}} | v_2^{\mathrm{T}}]^{\mathrm{T}}$, where both v_1 and v_2 are positive vectors. Here $\mathbf{1}$ denotes the vector of ones. Since $\mathbf{1}^{\mathrm{T}} \Delta^{-1} v = 0$, we have $\mathbf{1}^{\mathrm{T}} D_1^{-1} v_1 = \mathbf{1}^{\mathrm{T}} D_2^{-1} v_2$.

From the equation $\tilde{L} v = \lambda_2 v$, if we set $\alpha = w e_1^{\mathrm{T}} v_2$ and $\beta = w e_k^{\mathrm{T}} v_1$, we find that

$$-M_1^{-1} v_1 - \alpha d_i d_j e_k = -\lambda_2 v_1,$$

which we can rewrite as

$$\frac{v_1}{\lambda_2} = M_1 v_1 - \frac{\alpha}{\lambda_2} d_i d_j M_1 e_k.$$

Using Lemma 5.2, we conclude that $M_1 e_k = (d_i w)^{-1} D_1^{-1} \mathbf{1}$, because $\sum\limits_{e \in P_{a,i,j}} 1/w(e) = 1/w$ for any vertex a in the branch at j containing i. Hence, we have

$$\frac{v_1}{\lambda_2} = M_1 v_1 - \frac{\alpha d_j}{\lambda_2 w} D_1^{-1} \mathbf{1}. \tag{5.1}$$

Now we multiply (5.1) by e_k^{T} to obtain

$$\frac{e_k^{\mathrm{T}} v_1}{\lambda_2} = e_k^{\mathrm{T}} \Big(M_1 v_1 - \frac{\alpha d_j}{\lambda_2 w} D_1^{-1} \mathbf{1} \Big) = \frac{1}{d_i w} \mathbf{1}^{\mathrm{T}} D_1^{-1} v_1 - \frac{\alpha d_j}{\lambda_2 w d_i}.$$

Hence,

$$\frac{\beta}{\lambda_2} = \frac{1}{d_i} \mathbf{1}^{\mathrm{T}} D_1^{-1} v_1 - \frac{\alpha d_j}{\lambda_2 d_i},$$

which can be rewritten as

$$\frac{1}{\beta d_i + \alpha d_j}\mathbf{1}^{\mathrm{T}} D_1^{-1} v_1 = \frac{1}{\lambda_2}.$$

Now, we substitute for $1/\lambda_2$ in (5.1) to obtain

$$\begin{aligned}\frac{v_1}{\lambda_2} &= M_1 v_1 - \frac{\alpha d_j}{w(\beta d_i + \alpha d_j)}\mathbf{1}^{\mathrm{T}} D_1^{-1} v_1 D_1^{-1}\mathbf{1} \\ &= M_1 v_1 - \frac{\alpha d_j}{w(\beta d_i + \alpha d_j)} D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1} v_1.\end{aligned}$$

Therefore, we have

$$\frac{v_1}{\lambda_2} = \Big(M_1 - \frac{\alpha d_j}{w(\beta d_i + \alpha d_j)} D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1}\Big) v_1.$$

The same calculation for the matrix M_2 gives the relation

$$\frac{v_2}{\lambda_2} = \Big(M_2 - \frac{\beta d_i}{w(\beta d_i + \alpha d_i)} D_2^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_2^{-1}\Big) v_2.$$

Now, from Lemma 5.2, we conclude that $\sum\limits_{e\in P_{a,b,j}} 1/w(e) \geqslant 1/w$ and $\sum\limits_{e\in P_{a,b,i}} 1/w(e) \geqslant 1/w$. Hence, we have $M_1 \geqslant w^{-1} D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1}$ and $M_2 \geqslant w^{-1} D_2^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_2^{-1}$. If we define $\gamma = \alpha d_j/(\beta d_i + \alpha d_j)$ and notice that $\gamma \in (0,1)$, we conclude that v_1 is a positive eigenvector of the positive matrix $M_1 - w^{-1}\gamma D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1}$ and that v_2 is a positive eigenvector of the matrix $M_2 - w^{-1}(1-\gamma) D_2^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_2^{-1}$. Therefore, from the Perron-Frobenius theory we have

$$\varrho\Big(M_1 - \frac{1}{w}\gamma D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1}\Big) = \varrho\Big(M_2 - \frac{1}{w}(1-\gamma) D_2^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_2^{-1}\Big) = \frac{1}{\lambda_2},$$

as required.

Conversely, assume that there is a $\gamma \in (0,1)$ that satisfies

$$\varrho\Big(M_1 - \frac{1}{w}\gamma D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1}\Big) = \varrho\Big(M_2 - \frac{1}{w}(1-\gamma) D_2^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_2^{-1}\Big) = \frac{1}{\lambda_2}.$$

Let v_1 and v_2 be the Perron vectors of $M_1 - w^{-1}\gamma D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1}$ and $M_2 - w^{-1} \times (1-\gamma) D_2^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_2^{-1}$, respectively. Then we can compute

$$\begin{aligned}\frac{e_k^{\mathrm{T}} v_1}{\lambda_2} &= e_k^{\mathrm{T}}\Big(M_1 - \frac{1}{w}\gamma D_1^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1}\Big) v_1 \\ &= \Big(\frac{1}{d_i w}\mathbf{1}^{\mathrm{T}} D_1^{-1} - \gamma\frac{1}{d_i w}\mathbf{1}^{\mathrm{T}} D_1^{-1}\Big) v_1 \\ &= (1-\gamma)\frac{1}{d_i w}\mathbf{1}^{\mathrm{T}} D_1^{-1} v_1.\end{aligned}$$

Also, we can choose normalized eigenvectors v_1 and v_2 such that $\mathbf{1}^{\mathrm{T}} D_1^{-1} v_1 = \mathbf{1}^{\mathrm{T}} D_2^{-1} v_2$, and then we can write

$$\frac{e_k^{\mathrm{T}} v_1}{\lambda_2} = (1-\gamma) \frac{1}{d_i w} \mathbf{1}^{\mathrm{T}} D_2^{-1} v_2. \tag{5.2}$$

Similarly, using the same procedure, we can compute

$$e_1^{\mathrm{T}} \Big(M_2 - \frac{1}{w} (1-\gamma) D_2^{-1} \mathbf{1} \mathbf{1}^{\mathrm{T}} D_2^{-1} \Big) v_2$$

and obtain the relation

$$\frac{e_1^{\mathrm{T}} v_2}{\lambda_2} = \gamma \frac{1}{d_j w} \mathbf{1}^{\mathrm{T}} D_1^{-1} v_1. \tag{5.3}$$

Using relation (5.2) in the equation $(M_2 - (1/w)(1-\gamma) D_2^{-1} \mathbf{1}\mathbf{1}^{\mathrm{T}} D_2^{-1}) v_2 = v_2/\lambda_2$, we obtain

$$\begin{aligned} \frac{1}{\lambda_2} v_2 &= M_2 v_2 - \frac{1}{w} (1-\gamma) D_2^{-1} \mathbf{1}\mathbf{1}^{\mathrm{T}} D_2^{-1} v_2 \\ &= M_2 v_2 - \frac{d_i}{\lambda_2} D_2^{-1} \mathbf{1} e_k^{\mathrm{T}} v_1. \end{aligned}$$

By applying Lemma 5.2, we get the relation $M_2 e_1 = (d_j w)^{-1} D_2^{-1} \mathbf{1}$ and

$$\frac{1}{\lambda_2} v_2 = M_2 v_2 - \frac{w d_i d_j}{\lambda_2} M_2 e_1 e_k^{\mathrm{T}} v_1,$$

which is equivalent to

$$\lambda_2 v_2 = M_2^{-1} v_2 + w d_i d_j e_1 e_k^{\mathrm{T}} v_1. \tag{5.4}$$

In the same way, we can use relation (5.3) and rewrite the equation

$$\Big(M_1 - \frac{1}{w} \gamma D_1^{-1} \mathbf{1}\mathbf{1}^{\mathrm{T}} D_1^{-1} \Big) v_1 = \frac{1}{\lambda_2} v_1$$

as

$$\lambda_2 v_1 = M_1^{-1} v_1 + w d_i d_j e_k e_1^{\mathrm{T}} v_2. \tag{5.5}$$

Therefore, equations (5.4) and (5.5) show that the vector $v = [-v_1 | v_2]^{\mathrm{T}}$ satisfies $L v = \lambda_2 v$. This proves the result. $\square$

6. Characteristic vertices via Perron values

The results in this section describe the characteristic vertices of trees, so characterize their type in a way similar to as in [9], in terms of the Perron branches of the trees.

Lemma 6.1. *If T is a Type 2 tree with characteristic vertices i and j, then i and j are adjacent and the branch at i containing vertex j is the unique Perron branch at i, while the branch at j containing vertex i is the unique Perron branch at j.*

The proof is similar to that of Corollary 1.1 in [9].

For square nonnegative matrices A and B (not necessarily of the same order), we use the notation $A \leqslant B$ (or $B \geqslant A$) to express that there are permutation matrices P and Q such that PAP^{T} is entrywise dominated by a principal submatrix of QBQ^{T}, with strict inequality in at least one position in the case that A and B have the same order. It is worth to recall that according to the Perron-Frobenius theory whenever we have a nonnegative matrix C such that $A - C$ and $B - C$ are positive, then $\varrho(A - C) < \varrho(B - C)$.

The following result provides a simple way how to characterize Type 1 and Type 2 trees as an alternative to Theorem 3.7.

Theorem 6.2. *Let T be a tree. T is a Type 1 tree if and only if there is only one vertex at which there are at least two Perron branches. Tree T is a Type 2 tree if and only if at each vertex there is a unique Perron branch.*

P r o o f. First, assume that there is only one vertex at which there are two or more Perron branches. Then by Theorem 4.1, T is a Type 1 tree. Conversely, assume that T is a Type 1 tree with characteristic vertex v. Take any branch at some vertex $u \neq v$. Let P be the branch at u containing v and let Q be any other branch at u. Let C be the component at v that contains u. In light of Lemma 5.2, we can see that $L(Q)^{-1} \leqslant L(C)^{-1} \leqslant L(P)^{-1}$ with the strict inequality for at least one entry. Hence, we conclude that $\varrho(L(Q)^{-1}) < \varrho(L(P)^{-1})$ and there is only one Perron component at u.

If T is a Type 2 tree, then by Lemma 6.1 there is a pair of adjacent vertices i and j such that there is a unique Perron branch at each of them. If we consider a vertex different from i and j, then we can use the same argument from the previous part to conclude that there is only one Perron branch at this vertex. Finally, assume that at each vertex there is a unique Perron branch. If T is not a Type 2 tree, then we have a contradiction with Theorem 4.1. This completes the proof. $\square$

Corollary 6.3. *Let T be a tree and u a vertex which is not its characteristic vertex. Then the unique Perron branch at u is the branch containing the characteristic set of T.*

7. Sign partition of graphs

In the sense of the Laplacian matrix, paper [9] describes Type I and II trees and for the case of the perturbed Laplacian matrix this paper introduces the analogues as Type 1 and 2 trees. In this section, we will make some remarks about this analogy in order to understand if the information given by both eigenvectors (from λ_2 and $a(G)$) are really the same.

We will refer to Laplacian Perron components in the sense of [9], where their definition comes from the Laplacian matrix. Further, we refer to a perturbed Perron component for the definition in the sense of the perturbed Laplacian matrix. If G satisfies Case B of Fiedler's monotonicity theorem, then we say that G is a Type I graph.

Relating this concepts of Perron components we are able to describe case when the eigenspaces of λ_2 and $a(G)$ give the same sign partition to a graph. The next results explain how it may occur, giving an explicit description of such eigenspaces.

Theorem 7.1. *Let $G=(V,E)$ be a Type 1 graph and let v be its characteristic vertex with $C_0, C_1, \ldots, C_r$ as perturbed Perron components. If $C_0, C_1, \ldots, C_r$ are also Laplacian Perron components, then G is a Type I graph and there is a base of eigenvectors of λ_2 and a base of eigenvectors of $a(G)$ with the same sign over V.*

P r o o f. First, assume that the perturbed Laplacian matrix is in the form

$$\text{Ł} = \begin{bmatrix} \text{Ł}(C_0) & 0 & \ldots & 0 & c_0 \\ 0 & \text{Ł}(C_1) & \ldots & 0 & c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \text{Ł}(C_r) & c_r \\ (c_0)^{\mathrm{T}} & (c_1)^{\mathrm{T}} & \ldots & (c_r)^{\mathrm{T}} & d_v \end{bmatrix}. \tag{7.1}$$

Let $y^{(0)}, y^{(1)}, \ldots, y^{(r)}$ be the set of Perron vectors for the set of matrices $\text{Ł}(C_0)^{-1}$, $\text{Ł}(C_1)^{-1}, \ldots, \text{Ł}(C_r)^{-1}$ such that $\mathbf{1}^{\mathrm{T}} y^{(i)} = 1$. Define for $i=1,\ldots,r$ the vector

$$Y_i = \begin{cases} y^{(0)}(v), & v \in C_0, \\ -y^{(i)}(v), & v \in C_i, \\ 0, & \text{otherwise.} \end{cases} \tag{7.2}$$

It is easy to see that $\mathbf{1}^{\mathrm{T}} Y_i = 0$, and each of $Y_1, Y_2, \ldots, Y_r$ is a set of linearly independent vectors. Thus, $1/\varrho(\mathcal{L}(C_0)^{-1})$ is the eigenvalue associated with $y^{(i)}$ for $i = 1, \ldots, r$. Then we have

$$\frac{Y_i^{\mathrm{T}} L(D) Y_i}{Y_i^{\mathrm{T}} Y_i} = \frac{1}{\varrho(\mathcal{L}(C_0)^{-1})}.$$

Using Theorem 4.1, we obtain

$$\frac{Y_i^{\mathrm{T}} \mathcal{L} Y_i}{Y_i^{\mathrm{T}} Y_i} = \lambda_2$$

for $i = 1, \ldots, r$, therefore $Y_1, Y_2, \ldots, Y_r$ is a set of linearly independent eigenvectors associated with λ_2.

Now let Z be an eigenvector of $\mathcal{L}$ corresponding to λ_2 such that G is a Type 1 graph with characteristic vertex v. From the relation $\mathcal{L}Z = \lambda_2 Z$ it follows for each component C_i that $\mathcal{L}(C_i)Z(C_i) = \lambda_2 Z(C_i)$. Since $\mathcal{L}(C_i)y^{(i)} = \lambda_2 y^{(i)}$, by the Perron-Frobenius theorem, we have that $1/\varrho(\mathcal{L}(C_0)^{-1})$ is a simple eigenvalue of $\mathcal{L}(C_i)^{-1}$. Hence, it follows that $Z(C_i)$ is a scalar multiple of $y^{(i)}$. That implies that Z is a linear combination of Y_i. Therefore, $Y_1, Y_2, \ldots, Y_r$ is a base for the eigenspace associated with λ_2 if and only if G is a Type 1 graph with v as a characteristic vertex.

On the other hand, let $f^{(0)}, f^{(1)}, \ldots, f^{(r)}$ be the set of Perron vectors for the set of submatrices of the combinatorial Laplacian $\mathcal{L}(C_0)^{-1}, \mathcal{L}(C_1)^{-1}, \ldots, \mathcal{L}(C_r)^{-1}$ such that $\mathbf{1}^{\mathrm{T}} f^{(i)} = 1$. Define for $i = 1, \ldots, r$ the vector

$$F_i = \begin{cases} f^{(0)}(v), & v \in C_0, \\ -f^{(i)}(v), & v \in C_i, \\ 0, & \text{otherwise.} \end{cases} \tag{7.3}$$

It is easy to see that $\mathbf{1}^{\mathrm{T}} F_i = 0$, and each of $F_0, F_1, \ldots, F_r$ is a set of linearly independent vectors. Now, using the same considerations as before and the characterization given in [9], we can conclude that $F_1, F_2, \ldots, F_r$ is a base for the eigenspace associated with $a(G)$ if and only if G is a Type I graph with a characteristic vertex v.

Hence, both the bases $Y_1, Y_2, \ldots, Y_r$ and $F_1, F_2, \ldots, F_r$ provide the same signs over the vertices of G and the result follows. □

Corollary 7.2. *Let G be a Type 1 graph and let v be its characteristic vertex with $C_0, C_1, \ldots, C_r$ as perturbed Perron components. Let $y^{(0)}, y^{(1)}, \ldots, y^{(r)}$ be the set of Perron vectors for the set of matrices $\mathcal{L}(C_0)^{-1}, \mathcal{L}(C_1)^{-1}, \ldots, \mathcal{L}(C_r)^{-1}$ such* *that* $\mathbf{1}^{\mathrm{T}} y^{(i)} = 1$. *Define for* $i = 1, \ldots, r$ *the vector*

$$Y_i = \begin{cases} y^{(0)}(v), & v \in C_0, \\ -y^{(i)}(v), & v \in C_i, \\ 0, & \textit{otherwise}. \end{cases} \tag{7.4}$$

Then $Y_1, Y_2, \ldots, Y_r$ *is a base for the eigenspace associated with* λ_2.

The last corollary ensures that the characteristic vertex of a Type 1 graph is independent of the choice of the eigenvector that defines the eigenfunction. In the context of the Laplacian matrix, this is similar to the results in [11], where it is shown that the characteristic vertices of a tree are independent of the eigenvector associated with the algebraic connectivity.

Further, it is possible to show a similar result for Type 2 trees.

Theorem 7.3. *Let* $G = (V, E)$ *be a Type* 2 *tree, let* $\{u, v\}$ *be a characteristic edge and let the component* C_1 *at* u *containing* v *and the component* C_2 *at* v *containing* u *be the perturbed Perron components. If* C_1 *and* C_2 *are also Laplacian Perron components, then* G *is a Type* II *graph with an eigenvector of* λ_2 *and an eigenvector of* $a(G)$ *with the same signs over* V.

P r o o f. It follows straightforward from Theorem 4.2 and Case 2 of Theorem 3.7. □

In light of Theorem 6.2 and Corollary 6.3, trees that fulfil Theorems 7.1 and 7.3 must have the same Perron and perturbed Perron components at any vertex. In any event, whenever the Perron components for L and Ł are the same, it seems that the sign partition provided by the eigenvector of their second smallest eigenvalue coincide. Thus, we finish the paper by reformulating the question posed at the beginning of the paper.

> Let G be a graph with Laplacian matrix L and perturbed Laplacian matrix Ł. Let v be an articulation point of G. Are the Perron components at v for both L and Ł the same?

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