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SOME RELATIONS ON HUMBERT MATRIX POLYNOMIALS

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Abstract. The Humbert matrix polynomials were first studied by Khammash and Shehata (2012). Our goal is to derive some of their basic relations involving the Humbert matrix polynomials and then study several generating matrix functions, hypergeometric matrix representations, matrix differential equation and expansions in series of some relatively more familiar matrix polynomials of Legendre, Gegenbauer, Hermite, Laguerre and modified Laguerre. Finally, some definitions of generalized Humbert matrix polynomials also of two, three and several index are derived.

Keywords: hypergeometric matrix function; Humbert matrix polynomials; matrix functional calculus; generating matrix function; matrix differential equation

MSC 2010: 15A60, 33C55, 33C45, 33E20

1. Introduction and preliminaries

Special matrix functions are very close to statistics, Lie group theory and number theory [14]. In the recent work, matrix polynomials have significant emergent in [1]–[7], [9], [10], [12], [17]–[21], [24]–[32], [34]–[37]. Results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials in [8], [16], [38]. Humbert matrix polynomials have been introduced and studied in [22]. The reasons of interest in this family of Humbert matrix polynomials are their intrinsic mathematical importance.

Our main aim in this paper is to prove new properties the generalized hypergeometric matrix function and Humbert matrix polynomials. The structure of this paper is as follows. In Section 2, some basic relations involving the generalized hypergeometric matrix functions and convergence properties, integral representation, differential properties, and matrix differential equation are obtained. In Section 3, we give some new results involving the Humbert matrix polynomials and then establish

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several operational results, matrix differential equation, finite series representation, hypergeometric representations, four additional generating matrix functions and expansions of Humbert matrix polynomials in series of other polynomials which are best stated in terms of the generalized matrix polynomials. Relationships with other polynomial systems are also developed. Finally, we define the multiindex Humbert matrix polynomials and present several new results for matrix polynomials in Section 4.

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. Its two-norm will be denoted by $\|A\|$, and is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

where, for a vector $x \in \mathbb{C}^N$, $\|x\|_2 = (x^{\mathrm{T}}x)^{1/2}$ is the Euclidean norm of x.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$ and $\sigma(B) \subset \Omega$, such that $AB = BA$, then from the properties of the matrix functional calculus in [13] it follows that

$$f(A)g(B) = g(B)f(A). \tag{1.1}$$

We recall that the reciprocal gamma function denoted by $\Gamma^{-1}(z) = 1/\Gamma(z)$ is an entire function of the complex variable z and thus for any matrix A in $\mathbb{C}^{N\times N}$, $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if A is a matrix in $\mathbb{C}^{N\times N}$ such that

$$A + nI \quad \text{is an invertible matrix for all integers } n \geqslant 0 \tag{1.2}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$, then the Pochhammer symbol or shifted factorial is defined by [15]

$$\begin{aligned}(A)_n &= A(A+I)(A+2I)\dots(A+(n-1)I)\\ &= \Gamma(A+nI)\Gamma^{-1}(A); \quad n \geqslant 1,\ (A)_0 = I.\end{aligned} \tag{1.3}$$

For matrices $A(k,n)$ and $B(k,n)$ in $\mathbb{C}^{N\times N}$ where $n \geqslant 0$, $k \geqslant 0$ the following relations are satisfied according to Defez and Jódar in [11]:

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k),\\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k,n-mk); \quad m \in \mathbb{N}.\end{aligned} \tag{1.4}$$

Similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k), \tag{1.5}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk); \quad m\in\mathbb{N}.$$

Let us denote real numbers $M(A)$ and $m(A)$ by

$$M(A)=\max\{\Re(z)\colon\ z\in\sigma(A)\};\quad m(A)=\min\{\Re(z)\colon\ z\in\sigma(A)\}. \tag{1.6}$$

Note that, if k is large enough so that for $k>\|C\|$, then we have the relation which is given in Jódar and Cortés [18], [17] in the form

$$\|(C+kI)^{-1}\|\leqslant\frac{1}{k-\|C\|};\quad k>\|C\|. \tag{1.7}$$

The hypergeometric matrix function ${}_2F_1(A,B;C;z)$ has been given in [17] in the form

$${}_2F_1(A,B;C;z)=\sum_{k=0}^{\infty}\frac{(A)_k(B)_k((C)_k)^{-1}}{k!}z^k \tag{1.8}$$

where A, B and C are matrices of $\mathbb{C}^{N\times N}$ such that $C+nI$ is an invertible matrix for all integers $n\geqslant 0$. It has been shown by Jódar and Cortés [17] that the series is absolutely convergent for $|z|=1$ when

$$m(C)>M(A)+M(B).$$

If $CB=BC$ and C, B, $C-B$ are positive stable matrices, then, for $|z|<1$, an integral representation of (1.8) was given in the form [17]

$$\begin{aligned}{}_2F_1(A,B;C;z)=&\int_0^1(1-tz)^{-A}t^{B-I}(1-t)^{C-B-I}\,\mathrm{d}t\\&\times\Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C).\end{aligned} \tag{1.9}$$

For any matrix A in $\mathbb{C}^{N\times N}$, we will exploit the relation due to [17]

$$(1-z)^{-A}={}_1F_0(A;-;z)=\sum_{n=0}^{\infty}\frac{1}{n!}(A)_nz^n;\quad |z|<1. \tag{1.10}$$

Our main aim in the next section is to prove new properties of generalized hypergeometric matrix function.

2. Properties of generalized hypergeometric matrix function

The generalized hypergeometric matrix function is defined as follows:

Definition 2.1. Let A_1, A_2, A_3, B_1 and B_2 be matrices in $\mathbb{C}^{N\times N}$ such that

$$B_1 + nI \text{ and } B_2 + nI \text{ are invertible matrices for all integers } n \geqslant 0. \tag{2.1}$$

Then we define the generalized hypergeometric matrix function as

$${}_3F_2(A_1, A_2, A_3; B_1, B_2; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}(A_1)_k(A_2)_k(A_3)_k((B_1)_k)^{-1}((B_2)_k)^{-1}. \tag{2.2}$$

We are going to study the convergence properties of the hypergeometric matrix function. Note that if k is large enough so that $k > \|B_j\|$; $j = 1, 2$, then by the perturbation lemma, see [17], we can write

$$\Big\|\Big(\frac{1}{k}B_j + I\Big)^{-1}\Big\| \leqslant \frac{1}{1 - k^{-1}\|B_j\|} = \frac{k}{k - \|B_j\|}; \quad j = 1, 2$$

and

$$\begin{aligned} \|(B_j + kI)^{-1}\| &= \Big\|\frac{1}{k}\Big(\frac{1}{k}B_j + I\Big)^{-1}\Big\| = \frac{1}{k}\Big\|\Big(\frac{1}{k}B_j + I\Big)^{-1}\Big\| \\ &\leqslant \frac{1}{k - \|B_j\|}; \qquad k > \|B_j\|;\ j = 1, 2. \end{aligned} \tag{2.3}$$

Let us denote

$$T_j(k) = \|B_j^{-1}\|\|(B_j + I)^{-1}\| \ldots \|(B_j + (k-1)I)^{-1}\| \tag{2.4}$$

for $k \geqslant 1$ and $j = 1, 2$.

Note that from (1.3) we obtain

$$\|(A_i)_k\| \leqslant (\|A_i\|)_k; \quad i = 1, 2, 3. \tag{2.5}$$

Using (2.3), (2.4) and (2.5) for $k > \|B_j\|$ we have

$$\begin{aligned} &\Big\|\frac{z^k}{k!}(A_1)_k(A_2)_k(A_3)_k((B_1)_k)^{-1}((B_2)_k)^{-1}\Big\| \\ &\qquad \leqslant \frac{1}{k!}|z^k|\|(A_1)_k\|\|(A_2)_k\|\|(A_3)_k\|T_1(k)T_2(k) \\ &\qquad \leqslant \frac{1}{k!}|z^k|(\|A_1\|)_k(\|A_2\|)_k(\|A_3\|)_k T_1(k)T_2(k). \end{aligned} \tag{2.6}$$

Now we will investigate the convergence of the series

$$\sum_{k=0}^{\infty} \frac{1}{k!} |z^k| (\|A_1\|)_k (\|A_2\|)_k (\|A_3\|)_k T_1(k) T_2(k). \tag{2.7}$$

Using the ratio test and the relation (1.7), one gets

$$\begin{aligned}
(2.8)\quad \lim_{k\to\infty} & \left| \frac{(\|A_1\|)_{k+1}(\|A_2\|)_{k+1}(\|A_3\|)_{k+1} T_1(k+1) T_2(k+1) k!}{(\|A_1\|)_k (\|A_2\|)_k (\|A_3\|)_k T_1(k) T_2(k)(k+1)!} \frac{z^{k+1}}{z^k} \right| \\
& \leqslant \lim_{k\to\infty} \left| \frac{(\|A_1\|+k)(\|A_2\|+k)(\|A_3\|+k)\|(B_1+kI)^{-1}\|\|(B_2+kI)^{-1}\|}{(k+1)} \frac{z^{k+1}}{z^k} \right| \\
& \leqslant \lim_{k\to\infty} \frac{(\|A_1\|+k)(\|A_2\|+k)(\|A_3\|+k)}{(k-\|B_1\|)(k-\|B_2\|)(k+1)} |z|,
\end{aligned}$$

hence the power series (2.2) is absolutely convergent for $|z| < 1$ and diverges for $|z| > 1$. Analogously to Theorem 3 in [17], we can state

Theorem 2.1. *The generalized hypergeometric matrix function* ${}_3F_2(A_1, A_2, A_3; B_1, B_2; z)$ *is absolutely convergent for* $|z| = 1$ *when*

$$m(B_1) + m(B_2) > M(A_1) + M(A_2) + M(A_3) \tag{2.9}$$

where M *and* m *are defined in* (1.6).

The integral representation (1.9) of the generalized hypergeometric matrix function can be extended to obtain the following result:

Theorem 2.2. *Let* A_i, $1 \leqslant i \leqslant 3$ *and* B_j, $1 \leqslant j \leqslant 2$ *be matrices in* $\mathbb{C}^{N\times N}$ *such that* $B_j + kI$ *are invertible matrices for all integers* $k \geqslant 0$. *Suppose that the matrices* A_i, B_j *and* $B_j - A_i$ *are positive stable matrices. For* $|z| < 1$ *and* $|zt| < 1$ *it follows that*

$$\begin{aligned}
(2.10)\quad {}_3F_2(A_1, A_2, A_3; B_1, B_2; z) = & \int_0^1 t^{A_1 - I}(1-t)^{B_1 - A_1 - I} {}_2F_1(A_2, A_3; B_2; zt)\,\mathrm{d}t \\
& \times \Gamma^{-1}(A_1)\Gamma^{-1}(B_1 - A_1)\Gamma(B_1)
\end{aligned}$$

where $A_i B_j = B_j A_i$.

The proof is similar to the proof of Theorem 5 *in* [17].

Consider the differential operator $\theta = z(\mathrm{d}/\mathrm{d}z)$, $D_z = \mathrm{d}/\mathrm{d}z$, $\theta z^k = kz^k$. For commutative matrices, we have

$$\begin{aligned}
&\theta(\theta I + B_1 - I)(\theta I + B_2 - I)_3F_2\\
&\quad= \sum_{k=1}^{\infty} \frac{kz^k}{k!}(kI + B_1 - I)(kI + B_2 - I)(A_1)_k(A_2)_k(A_3)_k((B_1)_k)^{-1}((B_2)_k)^{-1}\\
&\quad= \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!}(A_1)_k(A_2)_k(A_3)_k((B_1)_{k-1})^{-1}((B_2)_{k-1})^{-1}.
\end{aligned}$$

Replacing k by $k+1$, we have

$$\begin{aligned}
&\theta(\theta I + B_1 - I)(\theta I + B_2 - I)_3F_2\\
&\quad= \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!}(A_1)_{k+1}(A_2)_{k+1}(A_3)_{k+1}((B_1)_k)^{-1}((B_2)_k)^{-1}\\
&\quad= z(\theta I + A_1)(\theta I + A_2)(\theta I + A_3)_3F_2.
\end{aligned}$$

Thus, we have shown that ${}_3F_2$ is a solution of the matrix differential equation

$$(\theta(\theta I + B_1 - I)(\theta I + B_2 - I) - z(\theta I + A_1)(\theta I + A_2)(\theta I + A_3))_3F_2 = \mathbf{0}$$

where $\mathbf{0}$ is a zero matrix or null matrix in $\mathbb{C}^{N\times N}$. This result is summarized below.

Theorem 2.3. *Let A_1, A_2, A_3, B_1 and B_2 be matrices in $\mathbb{C}^{N\times N}$ satisfying the spectral condition* (2.1) *and $B_jA_i = A_iB_j$; $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2$. Then ${}_3F_2$ satisfies the matrix differential equation*

$$(\theta(\theta I + B_1 - I)(\theta I + B_2 - I) - z(\theta I + A_1)(\theta I + A_2)(\theta I + A_3))_3F_2 = \mathbf{0} \tag{2.11}$$

or

$$\begin{aligned}
\Big((1-z)z^2\frac{\mathrm{d}^3}{\mathrm{d}z^3} &+ ((B_1 + B_2 + I) - (A_1 + A_2 + A_3 + 3)z)z\frac{\mathrm{d}^2}{\mathrm{d}z^2}\\
&+ (B_1B_2 - z(A_1A_2 + A_2A_3 + A_1A_3 + A_1 + A_2 + A_3 + I))\frac{\mathrm{d}}{\mathrm{d}z}\\
&- A_1A_2A_3\Big)_3F_2 = \mathbf{0}.
\end{aligned} \tag{2.12}$$

3. Properties of Humbert matrix polynomials

In this section, we deal with the Humbert matrix polynomials $h_n^A(x)$ that are defined by (see [22])

$$h_n^A(x) = \sum_{k=0}^{[n/3]} \frac{(-1)^k (3x)^{n-3k}}{k!(n-3k)!} (A)_{n-2k} \tag{3.1}$$

for a matrix A in $\mathbb{C}^{N\times N}$ such that

$$A + nI \quad \text{is an invertible matrix for all integers } n \geqslant 0. \tag{3.2}$$

Such matrix polynomials have the generating matrix function

$$(1-3xt+t^3)^{-A} = \sum_{n=0}^{\infty} h_n^A(x) t^n; \quad |3xt - t^3| < 1, \tag{3.3}$$

and the hypergeometric matrix representation for Humbert matrix polynomials

$$\begin{aligned} h_n^A(x) = \frac{(A)_n (3x)^n}{n!} {}_3F_2\Big(&-\frac{1}{3}nI, \frac{1}{3}(1-n)I, \frac{1}{3}(2-n)I; \\ &\frac{1}{2}(I-A-nI), \frac{1}{2}(2I-A-nI); \frac{1}{4x^3}\Big) \end{aligned} \tag{3.4}$$

where $(I-A-nI)/2$ and $(2I-A-nI)/2$ are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (2.1).

Next, let us recall the convergent properties and matrix differential equations of Humbert matrix polynomials which will be used in the following.

Theorem 3.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition* (3.2). *Then the power series* (3.1) *is absolutely convergent for $|1/4x^3| < 1$ and diverges for $|1/4x^3| > 1$.*

Theorem 3.2. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition* (3.2). *Then Humbert matrix polynomials are absolutely convergent for $|1/4x^3| = 1$ when*

$$\begin{aligned} &m\Big(\frac{1}{2}(2I-A-nI)\Big) + m\Big(\frac{1}{2}(I-A-nI)\Big) \\ &\qquad > M\Big(-\frac{1}{3}nI\Big) + M\Big(\frac{1}{3}(1-n)I\Big) + M\Big(\frac{1}{3}(2-n)I\Big) \end{aligned} \tag{3.5}$$

where M and m are defined in (1.6).

From the fundamental relation (3.3) by the usual method in such cases (differentiating (3.3) with respect to x and then with respect to t) the following recurrence formula may be easily obtained:

$$(3.6)\ (n+1)h_{n+1}^{A}(x) - 3x(A+nI)h_{n}^{A}(x) + (3A+(n-2)I)h_{n-2}^{A}(x) = \mathbf{0}, \quad n \geqslant 2,$$

see [22]. This formula affects only Humbert matrix polynomials with the same superior index. But another one can be readily written, connecting Humbert matrix polynomials with different indices, both superior and inferior, namely

$$(3.7)\qquad (n+1)h_{n+1}^{A}(x) + 3Ah_{n-2}^{A+I}(x) = 3Axh_{n}^{A+I}(x), \quad n \geqslant 2.$$

In [22], other recurrence formulae introduce the differential coefficient of the Humbert matrix polynomials:

$$(3.8)\qquad nh_{n}^{A}(x) + \frac{\mathrm{d}}{\mathrm{d}x}h_{n-2}^{A}(x) - x\frac{\mathrm{d}}{\mathrm{d}x}h_{n}^{A}(x) = \mathbf{0}, \quad n \geqslant 2,$$

$$(3.9)\qquad (3A+nI)h_{n}^{A}(x) = \frac{\mathrm{d}}{\mathrm{d}x}h_{n+1}^{A}(x) - 2x\frac{\mathrm{d}}{\mathrm{d}x}h_{n}^{A}(x)$$

and

$$(3.10)\qquad \frac{\mathrm{d}}{\mathrm{d}x}h_{n}^{A}(x) = 3Ah_{n-1}^{A+I}(x), \quad n \geqslant 1.$$

With help of these recurrence formulas, we can write the following equations: The first of them is

$$(3.11)\qquad \frac{\mathrm{d}}{\mathrm{d}x}h_{n+1}^{A}(x) = 3Ah_{n}^{A+I}(x).$$

Comparing (3.11) with (3.10), we obtain

$$(3.12)\qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}h_{n+2}^{A}(x) = 9A(A+I)h_{n}^{A+2I}(x).$$

Due to (3.12), it is easy to see that

$$(3.13)\qquad D^k h_{n+k}^{A}(x) = 3^k(A)_k h_{n}^{A+kI}(x); \quad D = \frac{\mathrm{d}}{\mathrm{d}x}.$$

From the relations (3.8) and (3.9), we easily get the relations

$$(3.14)\qquad (xD-n)h_{n}^{A}(x) = Dh_{n-2}^{A}(x), \quad n \geqslant 2$$

and

$$(2xDI + nI + 3A)h_n^A(x) = Dh_{n+1}^A(x). \tag{3.15}$$

Now iteratively applying the linear differential operator in (3.15) twice to (3.14) and using the simple relation

$$(axD + b)D^n = D^n(axD + b - na) \tag{3.16}$$

the matrix differential equation is given in the form

$$(2xDI + (n+3)I + 3A)(2xDI + nI + 3A)(xD - n)h_n^A(x) = D^3h_n^A(x). \tag{3.17}$$

From (3.17), we get the matrix differential equation of third order satisfied by Humbert matrix polynomials

$$\begin{aligned}(1 - 4x^3)D^3h_n^A(x) &- 6(3I + 2A)x^2D^2h_n^A(x)\\ &- \big(((n+5)I + 3A)((2-3n)I + 3A) + 10nI\big)xDh_n^A(x)\\ &+ n(nI + 3A)((n+3)I + 3A)h_n^A(x) = \mathbf{0}.\end{aligned} \tag{3.18}$$

This result is summarized in

Theorem 3.3. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition* (3.2). *Then the Humbert matrix polynomial is the solution of the matrix differential equation* (3.18).

A variety of interesting results can be easily deduced from Theorem 2.3. In the next result, the Humbert matrix polynomials appear as finite series solutions of the third order matrix differential equation.

Corollary 3.1. *The Humbert matrix polynomial is the solution of the matrix differential equation of the third order*

$$\begin{aligned}(1 - 4x^3)D^3h_n^A(x) &- 6(3I + 2A)x^2D^2h_n^A(x)\\ &- (((n+5)I + 3A)((2-3n)I + 3A) + 10nI)xDh_n^A(x)\\ &+ n(nI + 3A)((n+3)I + 3A)h_n^A(x) = \mathbf{0}.\end{aligned} \tag{3.19}$$

Proof. Put $z = 1/4x^3$, $A_1 = -nI/3$, $A_2 = (1-n)I/3$, $A_3 = (2-n)I/3$, $B_1 = (I - A - nI)/2$ and $B_2 = (2I - A - nI)/2$ in (2.12) and multiply by $(3x)^n$, then the proof of (3.19) follows directly. □

Now we obtain the finite series representation for Humbert matrix polynomials $h_n^A(x)$.

Theorem 3.4. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying* (3.2). *Then we have*

$$h_n^A(x)=\sum_{k=0}^{[n/2]}\sum_{s=0}^{n-2k}\frac{(A)_k(-k)_s(2A+2kI)_{n-2k-s}\left(\frac{3}{2}x\right)^{n-3s}}{k!s!(n-2k-s)!}. \tag{3.20}$$

P r o o f. From (3.3) and (1.5) we have

$$\begin{aligned}\sum_{n=0}^{\infty}h_n^A(x)t^n&=(1-3xt+t^3)^{-A}=\left(\left(1-\frac{3xt}{2}\right)^2-\left(\frac{3xt}{2}\right)^2+t^3\right)^{-A}\\&=\left(1-\frac{3xt}{2}\right)^{-2A}\left(1-\frac{(3xt)^2 2^{-2}-t^3}{(1-3xt2^{-1})^2}\right)^{-A}\\&=\left(1-\frac{3xt}{2}\right)^{-2A}\sum_{k=0}^{\infty}\frac{(A)_k}{k!}\left(\frac{(3xt)^2 2^{-2}-t^3}{(1-3xt2^{-1})^2}\right)^{k}\\&=\sum_{k=0}^{\infty}\frac{(A)_k}{k!}\left(\left(\frac{3xt}{2}\right)^2-t^3\right)^k\left(1-\frac{3xt}{2}\right)^{-2A-2kI}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(A)_k(2A+2kI)_n\left(\frac{3}{2}xt\right)^{n+2k}}{k!n!}\left(1-\frac{4t}{(3x)^2}\right)^k\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{(A)_k(-k)_s(2A+2kI)_n\left(\frac{3}{2}x\right)^{n+2k-2s}}{k!s!n!}t^{n+2k+s}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(A)_k(-k)_s(2A+2kI)_n\left(\frac{3}{2}x\right)^{n+2k-2s}}{k!s!n!}t^{n+2k+s}.\end{aligned}\tag{3.21}$$

Replacing n by $n-2k-s$, we get

$$\sum_{n=0}^{\infty}h_n^A(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\sum_{s=0}^{n-2k}\frac{(A)_k(-k)_s(2A+2kI)_{n-2k-s}\left(\frac{3}{2}x\right)^{n-3s}}{k!s!(n-2k-s)!}t^n. \tag{3.22}$$

Comparing the coefficients of t^n, we get the finite series representation (3.20) for Humbert matrix polynomials. $\square$

Theorem 3.5. *Let A, $2A$ and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the spectral condition* (3.2). *Then the Humbert matrix polynomials have the following four ad-*

ditional generating matrix functions:

$$(3.23)\quad \sum_{n=0}^{\infty}((A)_n)^{-1}h_n^A(x)t^n = \sum_{n=0}^{\infty}\frac{(3xt)^n}{n!} \times {}_1F_3\Big(A+nI; \frac{A+nI}{3}, \frac{A+(n+1)I}{3}, \frac{A+(n+2)I}{3}; -\Big(\frac{t}{3}\Big)^3\Big),$$

$$(3.24)\quad \sum_{n=0}^{\infty}(B)_n((A)_n)^{-1}h_n^A(x)t^n = \sum_{n=0}^{\infty}\frac{(3xt)^n(B)_n}{n!} \times {}_4F_3\Big(\frac{B+nI}{3}, \frac{B+(n+1)I}{3}, \frac{B+(n+2)I}{3}, A+nI; \frac{A+nI}{3}, \frac{A+(n+1)I}{3}, \frac{A+(n+2)I}{3}; -t^3\Big), \quad |-t^3|<1,$$

$$(3.25)\quad \sum_{n=0}^{\infty}[(2A)_n]^{-1}h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(-k)_r\big(\frac{3}{2}xt\big)^{n+2k}}{n!k!r!2^{2k}} \times \Big(\Big(A+\frac{1}{2}\Big)_k\Big)^{-1}((2A+(n+2k)I)_r)^{-1}\Big(\frac{4t}{(3x)^2}\Big)^r$$

and

$$(3.26)\quad \sum_{n=0}^{\infty}(B)_n((2A)_n)^{-1}h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(-k)_r\big(\frac{3}{2}xt\big)^{n+2k}}{n!k!r!2^{2k}}(B)_{n+2k} \times (B+(n+2k)I)_r\Big(\Big(A+\frac{1}{2}\Big)_k\Big)^{-1}((2A+(n+2k)I)_r)^{-1}\Big(\frac{4t}{(3x)^2}\Big)^r.$$

P r o o f. From (3.1) we get

$$(3.27)\quad \sum_{n=0}^{\infty}((A)_n)^{-1}h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{(n/3)}\frac{(-1)^k((A)_n)^{-1}(A)_{n-2k}(3x)^{n-3k}}{k!(n-3k)!}t^n.$$

Using the result (1.5), we have

$$(3.28)\quad \sum_{n=0}^{\infty}((A)_n)^{-1}h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k((A)_{n+3k})^{-1}(A)_{n+k}(3x)^n}{k!n!}t^{n+3k}.$$

Using (1.3) it is easy to show that

$$(3.29)\quad (A)_{n+k} = (A)_n(A+nI)_k, \quad (A)_{n+3k} = (A)_n(A+nI)_{3k}$$

and using Gauss's multiplication theorem, we can write

$$(A+nI)_{3k} = 3^{3k}\Big(\frac{A+nI}{3}\Big)_k\Big(\frac{A+(n+1)I}{3}\Big)_k\Big(\frac{A+(n+2)I}{3}\Big)_k. \tag{3.30}$$

From (3.28), (3.29) and (3.30), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}((A)_n)^{-1}h_n^A(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^{3k}k!n!}\bigg((-1)^k\Big(\Big(\frac{A+nI}{3}\Big)_k\Big)^{-1} \\
&\quad\times\Big(\Big(\frac{A+(n+1)I}{3}\Big)_k\Big)^{-1}\Big(\Big(\frac{A+(n+2)I}{3}\Big)_k\Big)^{-1}(A+nI)_k(3x)^n\bigg)t^{n+3k} \\
&= \sum_{n=0}^{\infty}\frac{(3xt)^n}{n!}\sum_{k=0}^{\infty}\frac{1}{k!3^{3k}}\bigg((-1)^k(A+nI)_k\Big(\Big(\frac{A+nI}{3}\Big)_k\Big)^{-1} \\
&\quad\times\Big(\Big(\frac{A+nI+I}{3}\Big)_k\Big)^{-1}\Big(\Big(\frac{A+nI+2I}{3}\Big)_k\Big)^{-1}\bigg)t^{3k},
\end{aligned}$$

which is equivalent to (3.23).

Next, for A and $B \in \mathbb{C}^{N\times N}$ we obtain another generating matrix function for the Humbert matrix polynomials. Indeed, using (3.29) and (3.30), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(B)_n((A)_n)^{-1}h_n^A(x)t^n \\
&\quad= \sum_{n=0}^{\infty}\sum_{k=0}^{(n/3)}\frac{(-1)^k(B)_n((A)_n)^{-1}(A)_{n-2k}(3x)^{n-3k}}{k!(n-3k)!}t^n \\
&\quad= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(B)_{n+3k}((A)_{n+3k})^{-1}(A)_{n+k}(3x)^n}{k!n!}t^{n+3k} \\
&\quad= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!n!}\big((-1)^k(B)_n(B+nI)_{3k}((A)_n)^{-1} \\
&\qquad\qquad\times((A+nI)_{3k})^{-1}(A)_n(A+nI)_k(3x)^n\big)t^{n+3k} \\
&\quad= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(B)_n\big(\frac13(B+nI)\big)_k\big(\frac13(B+(n+1)I)\big)_k\big(\frac13(B+(n+2)I)\big)_k}{k!n!} \\
&\qquad\times\Big(\Big(\frac{A+nI}{3}\Big)_k\Big)^{-1}\Big(\Big(\frac{A+(n+1)I}{3}\Big)_k\Big)^{-1}\Big(\Big(\frac{A+(n+2)I}{3}\Big)_k\Big)^{-1} \\
&\qquad\times(A+nI)_k(3x)^n t^{n+3k} \\
&\quad= \sum_{n=0}^{\infty}\frac{(B)_n(3xt)^n}{n!}{}_4F_3\Big(\frac{B+nI}{3},\frac{B+(n+1)I}{3},\frac{B+(n+2)I}{3}, \\
&\qquad\qquad A+nI;\frac{A+nI}{3},\frac{A+(n+1)I}{3},\frac{A+(n+2)I}{3};-t^3\Big),
\end{aligned}$$

which leads to (3.25). The proof of (3.26) and (3.27) can be done similarly to that of (3.24) and (3.25) by using (3.21). $\square$

Expansions of Humbert matrix polynomials in series of Legendre, Gegenbauer, Hermite, Laguerre and modified Laguerre matrix polynomials relevant to the present investigation are summarized in the following theorem.

Theorem 3.6. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition* (3.2). *Expansions of Humbert matrix polynomials in series of Legendre, Gegenbauer, Hermite, Laguerre and modified Laguerre matrix polynomials are given by*

$$h_n^A(x) = \sum_{k=0}^{[n/3]} \sum_{r=0}^{[(n-3k)/2]} \frac{(-1)^k (A)_{n-2k}(2n-4r-6k+1)}{k!r!\left(\frac{3}{2}\right)_{n-r-3k}} \times \left(\sqrt{2A}\right)^{3k-n} P_{n-2r-3k}(3x,A) \tag{3.31}$$

where $P_n(x,A)$ is the Legendre matrix polynomial [35]*;*

$$h_n^A(x) = \sum_{k=0}^{[n/3]} \sum_{r=0}^{[(n-3k)/2]} \frac{(-1)^k (A)_{n-2k}}{k!r!(n-2r-3k)!} \left(\sqrt{2A}\right)^{3k-n} H_{n-2r-3k}(3x,A) \tag{3.32}$$

where $H_n(x,A)$ stands for the Hermite matrix polynomial [10], [15]*;*

$$h_n^A(x) = \sum_{k=0}^{[n/3]} \sum_{r=0}^{[(n-3k)/2]} \frac{(-1)^k (A)_{n-2k}(A+(n-3k-2r)I)}{k!r!} \times \left((A)_{n+1-3k-r}\right)^{-1} C_{n-3k-2r}^A\left(\frac{3x}{2}\right) \tag{3.33}$$

where $C_n^A(x)$ stands for the Gegenbauer matrix polynomial [27]*;*

$$h_n^A(x) = \sum_{k=0}^{[n/3]} \sum_{r=0}^{n-3k} \frac{1}{k!(n-r-3k)!} \big((-1)^{k+r}\lambda^{-n+3k}(A)_{n-2k} \times (A+I)_{n-3k}((A+I)_r)^{-1}\big) L_r^{(A,\lambda)}(3x) \tag{3.34}$$

where $L_n^{(A,\lambda)}(x)$ stands for the Laguerre matrix polynomial [20]*;*

$$h_n^A(x) = \sum_{k=0}^{[n/3]} \sum_{r=0}^{n-3k} \frac{(-1)^k (A)_{n-2k}}{k!r!} \lambda^{-n+3k}(-A)_r f_{n-r-3k}^{(A,\lambda)}(3x) \tag{3.35}$$

where $f_n^{(A,\lambda)}(3x)$ stands for the modified Laguerre matrix polynomial (see [23], [33]*).*

P r o o f. Using the relation (3.1), we have

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]} \frac{(-1)^k (A)_{n-2k}(3x)^{n-3k}}{k!(n-3k)!} t^n. \tag{3.36}$$

From (1.5) and (3.36) we get

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (A)_{n+k}(3x)^{n}}{k!n!} t^{n+3k}. \tag{3.37}$$

In [35], the expansion of $x^n I$ in a series of Legendre matrix polynomials has been given in the form

$$\frac{(3x)^n}{n!} I = \left(\sqrt{2A}\right)^{-n} \sum_{r=0}^{[n/2]} \frac{(2n-4r+1)}{r!\left(\frac{3}{2}\right)_{n-r}} P_{n-2r}(3x, A), \tag{3.38}$$

hence we get

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[n/2]} \frac{(-1)^k (A)_{n+k}(2n-4r+1)}{k!r!\left(\frac{3}{2}\right)_{n-r}} \times \left(\sqrt{2A}\right)^{-n} P_{n-2r}(3x, A) t^{n+3k}. \tag{3.39}$$

Replacing n by $n-3k$, we get

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\sum_{r=0}^{[(n-3k)/2]} \frac{(-1)^k (A)_{n-2k}(2n-4r-6k+1)}{k!r!\left(\frac{3}{2}\right)_{n-r-3k}} \times \left(\sqrt{2A}\right)^{3k-n} P_{n-2r-3k}(3x, A) t^{n}. \tag{3.40}$$

Comparing the coefficients of t^n, we obtain (3.31).

In [15], the expansion of $x^n I$ in a series of Hermite matrix polynomials has been given in the form

$$\left(x\sqrt{2A}\right)^n = \sum_{r=0}^{[n/2]} \frac{n!}{r!(n-2r)!} H_{n-2r}(x, A), \tag{3.41}$$

hence we get

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[n/2]} \frac{(-1)^k (A)_{n+k}}{k!r!(n-2r)!} \left(\sqrt{2A}\right)^{-n} H_{n-2r}(3x, A) t^{n+3k}. \tag{3.42}$$

Replacing n by $n-3k$, we get

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\sum_{r=0}^{[(n-3k)/2]} \frac{(-1)^k (A)_{n-2k}}{k!r!(n-2r-3k)!} \times \left(\sqrt{2A}\right)^{3k-n} H_{n-2r-3k}(3x,A)t^n. \tag{3.43}$$

Comparing the coefficients of t^n, we obtain (3.32).

Using the result (1.17) in [16], [27], we get

$$(3x)^n I = n! \sum_{r=0}^{[n/2]} \frac{(A+(n-2r)I)((A)_{n-r+1})^{-1}}{r!} C_{n-2r}^A\left(\frac{3}{2}x\right). \tag{3.44}$$

Using (3.37) and (3.44), we have

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[n/2]} \frac{(-1)^k (A)_{n+k}}{k!r!}(A+(n-2r)I) \times ((A)_{n-r+1})^{-1} C_{n-2r}^A\left(\frac{3}{2}x\right)t^{n+3k}. \tag{3.45}$$

Replacing n by $n-3k$, we get

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\sum_{r=0}^{[(n-3k)/2]} \frac{(-1)^k (A)_{n-2k}}{k!r!}(A+(n-2r-3k)I) \times ((A)_{n-r-3k+1})^{-1} C_{n-2r-3k}^A\left(\frac{3}{2}x\right)t^n. \tag{3.46}$$

Comparing the coefficients of t^n, we obtain (3.33).

Using the results (1.13) in [20], [25], we have

$$(3x)^n I = n! \sum_{r=0}^{n} \frac{(-1)^r \lambda^{-n}}{(n-r)!}(A+I)_n((A+I)_r)^{-1} L_r^{(A,\lambda)}(3x). \tag{3.47}$$

From (3.37) and (3.47) we get

$$\sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^{k+r}\lambda^{-n}(A)_{n+k}}{k!(n-r)!}(A+I)_n \times ((A+I)_r)^{-1} L_r^{(A,\lambda)}(3x)t^{n+3k}. \tag{3.48}$$

Replacing n by $n-3k$, we get

$$(3.49)\qquad \sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\sum_{r=0}^{n-3k} \frac{(-1)^{k+r}\lambda^{-n+3k}(A)_{n-2k}}{k!(n-r-3k)!}(A+I)_{n-3k} \times ((A+I)_r)^{-1}L_r^{(A,\lambda)}(3x)t^n.$$

Comparing the coefficients of t^n, we obtain (3.34).

Using the result (1.15) in [23], we have

$$(3.50)\qquad (3\lambda x)^n I = n!\sum_{r=0}^{n} \frac{(-A)_r}{r!} f_{n-r}^{(A,\lambda)}(3x).$$

From (3.37) and (3.50) we have

$$(3.51)\qquad \sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^k(A)_{n+k}}{k!r!}\lambda^{-n}(-A)_r f_{n-r}^{(A,\lambda)}(3x)t^{n+3k}.$$

Replacing n by $n-3k$, we get

$$(3.52)\qquad \sum_{n=0}^{\infty} h_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\sum_{r=0}^{n-3k} \frac{(-1)^k(A)_{n-2k}}{k!r!}\lambda^{-n+3k}(-A)_r f_{n-r-3k}^{(A,\lambda)}(3x)t^n.$$

Comparing the coefficients of t^n, we obtain (3.35). Thus the proof is completed. $\square$

Theorem 3.7. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition* (3.2). *The Humbert matrix polynomials satisfy the formula*

$$(3.53)\qquad \sum_{n_1+n_2+\ldots+n_k=n} h_{n_1}^A(x)h_{n_2}^A(x)\ldots h_{n_k}^A(x) = \sum_{s=0}^{[n/3]} \frac{(-1)^s(kA)_{n-2s}(3x)^{n-3s}}{s!(n-3s)!}$$

for $k\in\mathbb{N}$.

Proof. Using the power series of $(1-3xt+t^3)^{-kA}$ for $|3xt-t^3|<1$ and making the necessary arrangements, we have

$$(3.54)\qquad \begin{aligned}(1-3xt+t^3)^{-kA} &= \sum_{n=0}^{\infty} \frac{(kA)_n(3x-t^2)^n}{n!}t^n \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s(kA)_n(3x)^{n-s}}{s!(n-s)!}t^{n+2s} \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{[n/3]} \frac{(-1)^s(kA)_{n-2s}(3x)^{n-3s}}{s!(n-3s)!}t^n.\end{aligned}$$

In addition, we can write

$$(3.55)\quad (1-3xt+t^3)^{-kA}=\sum_{n=0}^{\infty}\Bigg(\sum_{n_1+n_2+\ldots+n_k=n} h_{n_1}^{A}(x)h_{n_2}^{A}(x)\ldots h_{n_k}^{A}(x)\Bigg)t^n.$$

From (3.54) and (3.55) one can see that the proof is completed. □

Theorem 3.8. *For $k\in\mathbb{N}$ and $A_1,A_2,\ldots,A_k$ being matrices in $\mathbb{C}^{N\times N}$ satisfying the spectral condition* (3.2), *we have the relation*

$$(3.56)\quad \sum_{n_1=0}^{n}\ldots\sum_{n_{k-1}=0}^{n-n_1-n_2-\ldots-n_{k-2}} h^{A_1}_{n-n_1-n_2-\ldots-n_{k-1}}(x)h^{A_2}_{n_1}(x)\ldots h^{A_k}_{n_{k-1}}(x)$$
$$=\sum_{s=0}^{[n/3]}\frac{(-1)^s(A_1+A_2+\ldots+A_k)_{n-2s}(3x)^{n-3s}}{s!(n-3s)!},$$

where the matrices are assumed to be commutative.

P r o o f. Using the power series of $(1-3xt+t^3)^{-(A_1+A_2+\ldots+A_k)}$ for $|3xt-t^3|<1$ and (1.5), we obtain

$$(3.57)\quad (1-3xt+t^3)^{-(A_1+A_2+\ldots+A_k)}=\sum_{n=0}^{\infty}\frac{(A_1+A_2+\ldots+A_k)_n(3x-t^2)^n}{n!}t^n$$
$$=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s(A_1+A_2+\ldots+A_k)_n(3x)^{n-s}}{s!(n-s)!}t^{n+2s}$$
$$=\sum_{n=0}^{\infty}\sum_{s=0}^{[n/3]}\frac{(-1)^s(A_1+A_2+\ldots+A_k)_{n-2s}(3x)^{n-3s}}{s!(n-3s)!}t^n.$$

On the other hand, we get

$$(3.58)\ (1-3xt+t^3)^{-(A_1+A_2+\ldots+A_k)}$$
$$=(1-3xt+t^3)^{-A_1}(1-3xt+t^3)^{-A_2}\ldots(1-3xt+t^3)^{-A_k}$$
$$=\Bigg(\sum_{n_1=0}^{\infty}h^{A_1}_{n_1}(x)t^{n_1}\Bigg)\Bigg(\sum_{n_2=0}^{\infty}h^{A_2}_{n_2}(x)t^{n_2}\Bigg)\ldots\Bigg(\sum_{n_k=0}^{\infty}h^{A_k}_{n_k}(x)t^{n_k}\Bigg)$$
$$=\sum_{n=0}^{\infty}\sum_{n_1=0}^{n}\ldots\sum_{n_{k-1}=0}^{n-n_1-n_2-\ldots-n_{k-2}} h^{A_1}_{n-n_1-n_2-\ldots-n_{k-1}}(x)h^{A_2}_{n_1}(x)\ldots h^{A_k}_{n_{k-1}}(x)t^n.$$

Thus the proof is completed. □

Theorem 3.9. *Let $k \in \mathbb{N}$ and let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition* (3.2). *The Humbert matrix polynomials satisfy the relation*

$$h_n^A(x_1+x_2+\ldots+x_k)=\sum_{s=0}^{[n/3]}\frac{(-1)^s(A)_{n-2s}(3(x_1+x_2+\ldots+x_k))^{n-3s}}{s!(n-3s)!} \tag{3.59}$$

Proof. For $|3(x_1+x_2+\ldots+x_k)t-t^3|<1$, using (1.2), we can write

$$\begin{aligned}(1-3(x_1+x_2+\ldots+x_k)t+t^3)^{-kA}\\ &=\sum_{n=0}^{\infty}\frac{(A)_n(3(x_1+x_2+\ldots+x_k)-t^2)^n}{n!}t^n\\ &=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s(A)_n(3(x_1+x_2+\ldots+x_k))^{n-s}}{s!(n-s)!}t^{n+2s}\\ &=\sum_{n=0}^{\infty}\sum_{s=0}^{[n/3]}\frac{(-1)^s(kA)_{n-2s}(3(x_1+x_2+\ldots+x_k))^{n-3s}}{s!(n-3s)!}t^n.\end{aligned} \tag{3.60}$$

On the other hand, we get

$$(1-3(x_1+x_1+\ldots+x_k)t+t^3)^{-A}=\sum_{n=0}^{\infty}h_n^A(x_1+x_2+\ldots+x_k)t^n. \tag{3.61}$$

Combining (3.60) and (3.61), the proof is completed. □

4. Generalized Humbert matrix polynomials

The purpose of this section is to introduce a new matrix polynomial representing a generalization of the Humbert matrix polynomials in (3.1). For a positive integer m, let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (3.2). Then we define the generalized Humbert matrix polynomials by the generating matrix function in the form

$$(1-mxt+t^m)^{-A}=\sum_{n=0}^{\infty}h_{n,m}^A(x)t^n, \quad |mxt-t^m|<1 \tag{4.1}$$

where

$$h_{n,m}^A(x)=\sum_{k=0}^{[n/m]}\frac{(-1)^k(mx)^{n-mk}}{k!(n-mk)!}(A)_{n-(m-1)k}. \tag{4.2}$$

In a forthcoming work, we will consider the problems of a unified approach to the theory of new orthogonal matrix polynomials by the technique discussed in this paper. The notation will be used here are that implied by the following generalized Humbert matrix polynomials and generating matrix function definitions: For the generalized Humbert matrix polynomials of index two, three and p in terms of series they are represented as follows:

$$h^A_{n,m}(x,y) = \sum_{k=0}^{[n/3]} \sum_{r=0}^{[m/3]} \frac{(-1)^{k+r}(3x)^{n-3k}(3y)^{m-3r}}{k!r!(n-3k)!(m-3r)!}(A)_{n+m-2(k+r)}, \tag{4.3}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h^A_{n,m}(x,y)t^n u^m = (1-3xt-3yu+t^3+u^3)^{-A};$$

$$|3xt+3yu-t^3-u^3|<1,$$

$$h^A_{n,m,p}(x,y,z) = \sum_{k=0}^{[n/3]} \sum_{r=0}^{[m/3]} \sum_{s=0}^{[p/3]} \frac{(-1)^{k+r+s}(3x)^{n-3k}(3y)^{m-3r}(3z)^{p-3s}}{k!r!s!(n-3k)!(m-3r)!(p-3s!)!} \times (A)_{n+m+p-2(k+r+s)}, \tag{4.4}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} h^A_{n,m,p}(x,y,z)t^n u^m v^p = (1-3xt-3yu-3zv+t^3+u^3+v^3)^{-A};$$

$$|3xt+3yu+3zv-t^3-u^3-v^3|<1$$

and

$$h^A_{n_1,n_2,\ldots n_p}(x) = \sum_{k_i=0,\, i=1,2,\ldots,p}^{[n_i/3]} \frac{(-1)^{\sum k_i}(3x)^{\sum n_i - 3\sum k_i}}{\prod_{i=1}^{p}(k_i)!\prod_{i=1}^{p}(n_i-3k_i)!}(A)_{\sum n_i - 2\sum k_i}, \tag{4.5}$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_p=0}^{\infty} h^A_{n_1,n_2,\ldots n_p}(x)t_1^{n_1}t_2^{n_2}\ldots t_p^{n_p}$$

$$= (1-3x(t_1+t_2+\ldots+t_p)+(t_1^3+t_2^3+\ldots+t_p^3))^{-A},$$

where $|3x(t_1+t_2+\ldots+t_p)-(t_1^3+t_2^3+\ldots+t_p^3)|<1$ and $\sum$ denotes $\sum\limits_{i=1}^{p}$.

Finally, we consider $P_n(m,x,y,c,A)$ which is defined by

$$(c-mxt+yt^m)^{-A} = \sum_{n=0}^{\infty} P_n(m,x,y,c,A)t^n, \tag{4.6}$$

where m is a positive integer and all parameters satisfy the condition $|mxt-yt^m|<|c|$ (see [4]). This leads us to define a new class of matrix polynomials $P_n(m,x,y,c,A)$

by the relation

$$(1 - axt + bx^l t^m)^{-A} = \sum_{n=0}^{\infty} P_n^l(m, x, a, b, A) t^n \tag{4.7}$$

where a, b, m and l are parameters satisfying the condition $|axt - bx^l t^m| < 1$.

From (4.7) we have

$$P_n^l(m, x, a, b, A) = \sum_{k=0}^{[n/m]} \frac{(-1)^k a^{n-km} b^k x^{n-(m-l)k}}{k!(n - mk)!} (A)_{n-(m-1)k}. \tag{4.8}$$

A generalization of various matrix polynomials mentioned above is provided by the following definition. This definition includes Gegenbauer, Legendre, Chebyeshev, Pincherle, Kinney and Humbert matrix polynomials.

$$(c - axt + bt^m(2x - 1)^d)^{-A} = \sum_{n=0}^{\infty} P_{n,m,a,b,c,d}^A(x) t^n, \tag{4.9}$$
$$|axt - bt^m(2x - 1)^d| < |c|$$

where m and a are positive integers and the other parameters are unrestricted in general.

From (4.9) we get

$$P_{n,m,a,b,c,d}^A(x) = \sum_{k=0}^{[n/m]} \frac{1}{k!(n - mk)!} \big((-1)^k c^{-A-nI+(m-1)kI} \times (ax)^{n-(m-l)k} (b(2x - 1)^d)^k\big)(A)_{n-(m-1)k}. \tag{4.10}$$

Setting $m = 3$, $c = 1$ in (4.6), we get the series representation in (3.1). Also, if we set $a = m$, $b = c = 1$ in (4.7), we get (4.1).

In conclusion, we remark that it would be easy to extend these properties to certain polynomials with two variables; for instance, just as with the generalized Humbert matrix polynomials, we can start from the expansion

$$(1 - 3xt - 3yu + t^3 + u^3)^{-A} = \sum_{m,n=0}^{\infty} h_{m,n}^A(x, y) t^n u^m, \tag{4.11}$$
$$|3xt + 3yu - t^3 - u^3| < 1$$

and obtain formulae such as

$$\frac{\partial}{\partial x} h_{m,n+1}^A(x, y) = \frac{\partial}{\partial y} h_{m+1,n}^A(x, y), \tag{4.12}$$

$$\frac{\partial}{\partial x} h_{m,n-2}^A(x, y) - x \frac{\partial}{\partial x} h_{m,n}^A(x, y) + n h_{m,n}^A(x, y) = \mathbf{0}, \quad n \geqslant 2 \tag{4.13}$$

and

$$\frac{\partial}{\partial y} h^A_{m-2,n}(x,y) - y\frac{\partial}{\partial y} h^A_{m,n}(x,y) + m h^A_{m,n}(x,y) = \mathbf{0}, \quad n \geqslant 2. \tag{4.14}$$

There are many ways of investigating the generalized classes of Humbert matrix polynomials. Starting from the modified forms of the generating matrix function of Humbert matrix polynomials is one of the direct methods clearly offering some directions to develop more researches and studies in that area.

5. Open problem

One can use the same class of new integral representations, operational methods and the property of orthogonality for the generalized Humbert matrix polynomials, from which a variety of interesting results follows as special cases. Hence, new results and further applications can be obtained. Further results and applications will be discussed in a forthcoming work.

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