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PRACTICAL ULAM-HYERS-RASSIAS STABILITY FOR NONLINEAR EQUATIONS

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Abstract. In this paper, we offer a new stability concept, practical Ulam-Hyers-Rassias stability, for nonlinear equations in Banach spaces, which consists in a restriction of Ulam-Hyers-Rassias stability to bounded subsets. We derive some interesting sufficient conditions on practical Ulam-Hyers-Rassias stability from a nonlinear functional analysis point of view. Our method is based on solving nonlinear equations via homotopy method together with Bihari inequality result. Then we consider nonlinear equations with surjective asymptotics at infinity. Moore-Penrose inverses are used for equations defined on Hilbert spaces. Specific practical Ulam-Hyers-Rassias results are derived for finite-dimensional equations. Finally, two examples illustrate our theoretical results.

Keywords: practical Ulam-Hyers-Rassias stability; nonlinear equation

MSC 2010: 46T20, 47H99, 47J05

1. Introduction

In 1940, Ulam [22] gave a talk about the stability theory of functional equations in a conference at Wisconsin University. The Ulam problem is: *Under what conditions does there exist an additive mapping near an approximately additive mapping?* Thereafter, Hyers [11] answered the Ulam problem in Banach spaces, which was called Ulam-Hyers stability. In 1978, Rassias [18] introduced a generalization of the Ulam-Hyers stability of mappings by considering variables, which was named Ulam-Hyers-Rassias stability. Ulam's stability problem attracted many famous researchers,

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for example Cădariu [7] and Jung [13], among others. For more recent contributions on this topic, one can see [2], [6], [9], [10], [12], [15], [16], [17], [19], [23], [24] and references therein.

This phenomenon can be described by some practical problems: a travel by a space vehicle between two points, an aircraft or a missile which may oscillate around a mathematically unstable course yet its performance may be acceptable, a chemical process of keeping the temperature within certain bounds, etc. Thus, the notion of practical stability of nonlinear equations [14], [20] has attracted more and more attention under such significant considerations. However, there are only few papers concerning the Ulam-Hyers-Rassias stability for nonlinear equations on bounded subsets. Motivated by [14], [18] we consider more general stability, practical Ulam-Hyers-Rassias stability, for nonlinear equations. Now we are ready to formulate our problem.

Let X and Y be Banach spaces. Consider a mapping $F \in C^1(X, Y)$. The problem of a practical Ulam-Hyers-Rassias stability (PUHRS for short) of F can be formulated as follows:

PUHRS: There is a function $\varphi \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ with $\varphi(r, 0) = 0$ for any $r \in \mathbb{R}_+$, nondecreasing in each variable such that for any $x \in X$ and $y \in F(X)$ there is an $x_y \in X$ such that $F(x_y) = y$ and $|x - x_y| \leqslant \varphi(|x|, |y - F(x)|)$.

There is no sense in involving also $|y|$ in the function φ, since

$$|y| \leqslant |y - F(x)| + |F(x)|$$

and $|F(x)|$ is controlled by $|x|$ in many cases. If $\varphi(r_1, r_2)$ is independent of r_1, so $\varphi(r_1, r_2) = \varphi(r_2)$, then we get the Ulam-Hyers-Rassias stability (UHRS for short) of F.

The meaning of PUHRS consists in a restriction of UHRS to bounded subsets: If F is PUHRS then it holds:

For any $M > 0$ there is a nondecreasing function $\varphi_M \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\varphi_M(0) = 0$ such that for any $x \in X$, $|x| \leqslant M$, and $y \in F(X)$, $|y| \leqslant M$, there is an $x_y \in X$ such that $F(x_y) = y$ and $|x - x_y| \leqslant \varphi_M(|y - F(x)|)$.

Indeed, we take $\varphi_M(r) = \varphi(M, r)$. Moreover, we may take $\varphi_M(r) = c_M r$ for $c_M > 0$ in many reasonable cases (see Corollary 2.3 below).

In what follows, we give results answering these interesting questions from a nonlinear functional analysis point of view.

2. Main results

First, we recall the following Bihari inequality [5].

Theorem 2.1. *If $w(t)$ is a nonnegative continuous function such that*

$$w(t) \leqslant \alpha + \beta \int_0^t g(w(s))\,\mathrm{d}s$$

with constants $\alpha \geqslant 0$, $\beta \geqslant 0$ and $g : \mathbb{R}_+ \to (0,\infty)$ nondecreasing continuous, then

$$w(t) \leqslant G^{-1}(G(\alpha) + \beta t)$$

for all $t \geqslant 0$ for which $G(\alpha) + \beta t$ belongs to the domain of G^{-1} and $G(x) = \int_1^x (1/g(u))\,\mathrm{d}u$.

Now we suppose that

(i) there is a mapping $R\colon X \to L(Y, X)$ such that
  (1) R is locally Lipschitz, i.e., for any $x \in X$, there is an open neighbourhood U_x of x and a constant L_x such that $\|R(x_1) - R(x_2)\| \leqslant L_x|x_1 - x_2|$ for all $x_1, x_2 \in U_x$,
  (2) $R(x)$ is a right inverse of $DF(x)$ for all $x \in X$, i.e., $DF(x)R(x) = \mathbb{I}_Y$ for all $x \in X$, where $\mathbb{I}_Y\colon Y \to Y$ is the identity map on Y,

(ii) there is a continuous nondecreasing function $g\colon \mathbb{R}_+ \to (0,\infty)$ such that

$$\|R(x)\| \leqslant g(|x|)$$

for any $x \in X$. Now we have the following result.

Theorem 2.2. *Assume* (i) *and* (ii). *If*

$$\int_1^\infty \frac{1}{g(u)}\,\mathrm{d}u = \infty, \tag{2.1}$$

then F is PUHRS.

P r o o f. Let $x \in X$ and $y \in Y$. Setting $e := F(x) - y$, we get

$$F(x) = y + e. \tag{2.2}$$

We plug (2.2) into the homotopy (see [1] for more complex homotopy theory)

$$F(z(t)) = y + (1-t)e, \quad z(0) = x, \quad t \in [0,1]. \tag{2.3}$$

Assuming that $z \in C^1([0,1], X)$ and differentiating (2.3), we obtain

$$DF(z(t))z'(t) = -e, \quad z(0) = x, \quad t \in [0,1]. \tag{2.4}$$

If the differential equation

$$z'(t) = -R(z(t))e, \quad z(0) = x, \quad t \in [0,1] \tag{2.5}$$

has a solution $z \in C^1([0,1], X)$, then it satisfies (2.4), which gives

$$F(z(t)) + te = c, \quad t \in [0,1] \tag{2.6}$$

for a constant c. But putting $t = 0$ into (2.6) we derive

$$c = F(z(0)) = F(x) = y + e,$$

which gives (2.3). So we need to solve (2.5). Since $R(x)$ is locally Lipschitz, the Cauchy problem (2.5) has a unique local solution. To prolong it, we note

$$|z(t)| \leqslant |x| + |e| \int_0^t \|R(z(s))\| \, \mathrm{d}s. \tag{2.7}$$

So Theorem 2.1 gives

$$|z(t)| \leqslant G^{-1}(G(|x|) + |e|), \quad t \in [0,1],$$

which by (2.5) implies

$$\begin{aligned} |x_y - x| = |z(1) - z(0)| &\leqslant |e| \int_0^1 \|R(z(s))\| \, \mathrm{d}s \\ &\leqslant |e| \int_0^1 g(|z(s)|) \, \mathrm{d}s \leqslant |e| g(G^{-1}(G(|x|) + |e|)). \end{aligned}$$

So

$$\varphi(r_1, r_2) = r_2 g(G^{-1}(G(r_1) + r_2)).$$

The proof is finished. □

Corollary 2.3. *In addition to Theorem 2.2, if F is locally bounded, i.e.,*

$$F_M := \sup_{|x| \leqslant M} |F(x)| < \infty$$

for any $M > 0$, then we can take $\varphi(r) = c_M r$ with $c_M = g(G^{-1}(G(M) + M + F_M))$ whenever $|x| \leqslant M$ and $|y| \leqslant M$.

R e m a r k 2.4. Taking $x = 0$ in Theorem 2.2, we see that F is surjective and moreover, for any $y \in Y$ there is an $x_y \in X$ such that $F(x_y) = y$ with

$$|x_y| \leqslant (|y| + |F(0)|)g(G^{-1}(G(0) + |y| + |F(0)|)).$$

Hence Theorem 2.2 is an extension of global invertible mapping results, especially Hadamard [4], [8], to surjectivity.

Following the proof of Theorem 2.2, we get the next result.

Theorem 2.5. *Assume* (i) *and* (ii). *Then for any $x \in X$ and $y \in Y$ such that*

$$|F(x) - y| < \int_{|x|}^{\infty} \frac{1}{g(u)}\,\mathrm{d}u, \tag{2.8}$$

there is an $x_y \in X$ such that $F(x_y) = y$ and

$$|x_y - x| \leqslant |F(x) - y|\, g(G^{-1}(G(|x|) + |F(x) - y|)). \tag{2.9}$$

Of course, estimate (2.8) is useful when $G(\infty) < \infty$ for

$$G(\infty) := \int_{1}^{\infty} \frac{1}{g(u)}\,\mathrm{d}u. \tag{2.10}$$

Then Theorem 2.5 gives an error estimate for PUHRS of F. The next simple example shows that (2.8) is optimal in some sense.

E x a m p l e 2.6. Take $X = Y = \mathbb{R}$ and $F(x) = \arctan x$. Then $DF(x) = F'(x) = 1/(1+x^2)$ and $R(x) = 1 + x^2$. So $g(u) = 1 + u^2$ and (2.8) has the form

$$y - \arctan x < \frac{\pi}{2} - \arctan x \Leftrightarrow y < \frac{\pi}{2} \tag{2.11}$$

for $x > 0$ and $y > \arctan x$. But now (2.11) cannot be improved since the range of F is $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$.

Now we present a simple local result.

Theorem 2.7. *Assume there is a locally Lipschitz right inverse* $R\colon B_r \to L(Y,X)$ *of* $DF(x)$ *on a ball* $B_r := \{x \in X\colon |x| < r\}$ *such that* $\|R(x)\| \leqslant M$ *for any* $x \in B_r$ *and a constant* $M > 0$. *Then for any* $x \in B_r$ *and* $y \in Y$ *such that*

$$|F(x) - y| < \frac{r - |x|}{M}, \tag{2.12}$$

there is an $x_y \in B_r$ *such that* $F(x_y) = y$ *and*

$$|x_y - x| \leqslant M|F(x) - y|. \tag{2.13}$$

P r o o f. From (2.7) and (2.12), we derive

$$|z(t)| \leqslant |x| + |F(x) - y|M < r, \quad t \in [0,1].$$

Hence (2.5) has a unique solution $z(t)$ in B_r. Finally, (2.13) follows from

$$|x_y - x| \leqslant |F(x) - y| \int_0^1 \|R(z(s)\| \, \mathrm{d}s \leqslant |F(x) - y|M.$$

The proof is complete. □

Certainly, assumption (i) gives the surjectivity of $DF(x)$ for any $x \in X$. In general, finding a right inverse $R(x)$ is not so easy. We have discussed this for the linear case in [23]. The simplest case is when F is semilinear (see also [23], Theorem 7):

Theorem 2.8. *Let* $F \in C^2(X,Y)$. *If there is a surjective* $A \in L(X,Y)$ *with a right inverse* $R \in L(Y,X)$ *such that*

$$\sup_{x \in X} \|DF(x) - A\| < \frac{1}{\|R\|} \tag{2.14}$$

for any $x \in X$, *then* F *is UHRS.*

P r o o f. Since

$$DF(x)R = (DF(x) - A)R + AR = (DF(x) - A)R + \mathbb{I}_Y$$

and by (2.14)

$$\|(DF(x) - A)R\| \leqslant \|DF(x) - A\|\|R\| < 1,$$

the Neumann theorem [21] gives that $(DF(x) - A)R + \mathbb{I}_Y$ is invertible on Y, i.e., $(DF(x) - A)R + \mathbb{I}_Y \in L(Y)$, so we derive

$$DF(x)R((DF(x) - A)R + \mathbb{I}_Y)^{-1} = \mathbb{I}_Y.$$

Hence we take

$$R(x) := R((DF(x) - A)R + \mathbb{I}_Y)^{-1}$$

for any $x \in X$. From the same Neumann theorem we know that $R \in C^1(X, L(Y, X))$, which of course implies the local Lipschitzness of $R(x)$. Moreover, by (2.14), we derive

$$\begin{aligned}\|R(x)\| &\leqslant \|R\| \|((DF(x) - A)R + \mathbb{I}_Y)^{-1}\| \\ &\leqslant \frac{\|R\|}{1 - \|R\| \sup\limits_{x \in X} \|DF(x) - A\|}.\end{aligned}$$

So we take

$$g(u) = \frac{\|R\|}{1 - \|R\| \sup\limits_{x \in X} \|DF(x) - A\|},$$

and then

$$\varphi(r_1, r_2) = \frac{\|R\|}{1 - \|R\| \sup\limits_{x \in X} \|DF(x) - A\|} r_2.$$

The proof is finished by Theorem 2.2. □

On the other hand, if X and Y are Hilbert spaces, then we can take ([3], page 344, Example 17)

$$R(x) = DF(x)^* [DF(x) DF(x)^*]^{-1}, \tag{2.15}$$

i.e., $R(x)$ is the Moore-Penrose inverse $DF(x)^\dagger$ of $DF(x)$. We also know (see [3], page 344, Example 17)

$$\|R(x)\| = \inf_{0 \neq z \in [\ker DF(x)]^\perp} \frac{|DF(x)z|}{|z|}. \tag{2.16}$$

Now we can extend Theorem 2.8 as follows.

Theorem 2.9. *Let $F \in C^2(X, Y)$ for Hilbert spaces X and Y such that $DF(x)$ is surjective for any $x \in X$. If there is a surjective $A \in L(X, Y)$ such that*

$$\lim_{|x| \to \infty} DF(x) = A \tag{2.17}$$

in $L(X, Y)$ and the Moore-Penrose inverse $DF(x)^\dagger$ of $DF(x)$ is a bounded function from X to $L(Y, X)$, then F is UHRS.

P r o o f. Clearly by (2.15) (see also [25], Theorem 2)

$$\lim_{|x| \to \infty} DF(x)^\dagger = A^\dagger.$$

Since $DF(x)^\dagger$ is a bounded function from X to $L(Y,X)$, we see that

$$\sup_{x\in X}\|R(x)\|<\infty.$$

So we can take a constant nonzero g in assumption (ii), and the proof is finished by Theorem 2.2. $\square$

Corollary 2.10. *If $F\in C^2(\mathbb{R}^m,\mathbb{R}^n)$ for $m\geqslant n$ is such that $DF(x)$ is surjective for any $x\in\mathbb{R}^m$ and there is a surjective $A\in L(\mathbb{R}^m,\mathbb{R}^n)$ such that (2.17) holds, then F is UHRS.*

In general, when we know only that the Moore-Penrose inverse $DF(x)^\dagger$ is a bounded function from X to $L(Y,X)$, then we can apply Theorem 2.5 with

$$g(u):=\sup_{|x|\leqslant u}\|DF(x)^\dagger\|,\quad u\geqslant 0$$

to get that F is PUHRS.

3. Examples

In this section we give two examples to illustrate the above results.

Example 3.1. Let $J=[0,1]$ and $X=C(J,\mathbb{R})$, $Y=\{y\in C^1(J,\mathbb{R})\colon\ y(0)=0\}$, where X is endowed with the norm $\|x\|_0=\max_{t\in J}|x(t)|$ and Y with $\|y\|_1=\|y'\|_0$. Define the operator $F\colon\ X\to Y$ by

$$F(x)=\int_0^t (x^3(s)+x(s))\,\mathrm{d}s.$$

Clearly $F\in C^1(X,Y)$ and

$$DF(x)v=\int_0^t (3x^2(s)+1)v(s)\,\mathrm{d}s.$$

Then

$$R(x)y=\frac{y'}{3x^2(s)+1}.$$

So $\|R(x)\|\leqslant 1$ for any $x\in X$, and thus $g(r)=1$. Consequently, by Theorem 2.2, F is UHRS with $\varphi(r)=r$.

E x a m p l e 3.2. Consider $F \in C^2(\mathbb{R}^m, \mathbb{R})$ such that $\nabla F(x) \neq 0$ for any $x \in \overline{B}_r$ and some $r > 0$. Since $DF(x)v = \nabla F(x)v^*$, we take $R(x)y = y|\nabla F(x)|^{-2}\,\nabla F(x)$. Thus $\|R(x)\| = 1/|\nabla F(x)|$. Setting $M := \max_{x \in \overline{B}_r} 1/|\nabla F(x)|$, Theorem 2.7 can be applied. For $m = 2$, this could express climbing on a hill, when the relief of the hill is given by $F \in C^2(\mathbb{R}^3, (0, \infty))$. So we are at a position $(x_1, x_2, F(x_1, x_2))$, $(x_1, x_2) \in B_r$, and we try to reach the given altitude y within the region B_r. The inequality (2.12) is sufficient to reach the altitude y at the region B_r and the location of this new position is given by (2.13).

References

[1] *E. L. Allgower, K. Georg*: Numerical Continuation Methods. An Introduction. Springer Series in Computational Mathematics 13, Springer, Berlin, 1990. zbl MR doi

[2] *S. András, A. R. Mészáros*: Ulam-Hyers stability of dynamic equations on time scales via Picard operators. Appl. Math. Comput. *219* (2013), 4853–4864. zbl MR doi

[3] *A. Ben-Israel, T. N. E. Greville*: Generalized Inverses. Theory and Applications. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 15, Springer, New York, 2003. zbl MR doi

[4] *M. S. Berger*: Nonlinearity and Functional Analysis. Lectures on Nonlinear Problems in Mathematical Analysis. Pure and Applied Mathematics 74, Academic Press (Harcourt Brace Jovanovich, Publishers), New York, 1977. zbl MR

[5] *I. Bihari*: A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. Acta Math. Acad. Sci. Hung. *7* (1956), 81–94. zbl MR doi

[6] *M. Burger, N. Ozawa, A. Thom*: On Ulam stability. Isr. J. Math. *193* (2013), 109–129. zbl MR doi

[7] *L. Cădariu*: Stabilitatea Ulam-Hyers-Bourgin pentru ecuatii functionale. Univ. Vest Timişoara, Timişoara, 2007.

[8] *S.-N. Chow, J. K. Hale*: Methods of Bifurcation Theory. Grundlehren der Mathematischen Wissenschaften 251. A Series of Comprehensive Studies in Mathematics, Springer, New York, 1982. zbl MR doi

[9] *D. S. Cimpean, D. Popa*: Hyers-Ulam stability of Euler's equation. Appl. Math. Lett. *24* (2011), 1539–1543. zbl MR doi

[10] *B. Hegyi, S.-M. Jung*: On the stability of Laplace's equation. Appl. Math. Lett. *26* (2013), 549–552. zbl MR doi

[11] *D. H. Hyers*: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA *27* (1941), 222–224. zbl MR doi

[12] *D. H. Hyers, G. Isac, T. M. Rassias*: Stability of Functional Equations in Several Variables. Progress in Nonlinear Differential Equations and Their Applications 34, Birkhäuser, Boston, 1998. zbl MR doi

[13] *S.-M. Jung*: Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis. The Hadronic Press Mathematics Series. Hadronic Press, Palm Harbor, 2001. zbl MR

[14] *V. Lakshmikantham, S. Leela, A. A. Martynyuk*: Practical Stability of Nonlinear Systems. World Scientific, Singapore, 1990. zbl MR

[15] *N. Lungu, D. Popa*: Hyers-Ulam stability of a first order partial differential equation. J. Math. Anal. Appl. *385* (2012), 86–91. zbl MR doi

[16] *C. Park*: Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras. Bull. Sci. Math. *132* (2008), 87–96. zbl MR doi

[17] *D. Popa, I. Raşa*: On the Hyers-Ulam stability of the linear differential equation. J. Math. Anal. Appl. *381* (2011), 530–537. zbl MR doi
[18] *T. M. Rassias*: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. *72* (1978), 297–300. zbl MR doi
[19] *I. A. Rus*: Ulam stabilities of ordinary differential equations in a Banach space. Carpathian J. Math. *26* (2010), 103–107. zbl MR
[20] *R. Seydel*: Practical Bifurcation and Stability Analysis. Interdisciplinary Applied Mathematics 5, Springer, New York, 2010. zbl MR doi
[21] *A. E. Taylor, D. C. Lay*: Introduction to Functional Analysis. John Wiley & Sons, New York, 1980. zbl MR
[22] *S. M. Ulam*: A Collection of Mathematical Problems. Interscience Tracts in Pure and Applied Mathematics 8, Interscience Publishers, New York, 1960. zbl MR
[23] *J. Wang, M. Fečkan*: Ulam-Hyers-Rassias stability for semilinear equations. Discontin. Nonlinearity Complex *3* (2014), 379–388. zbl doi
[24] *J. Wang, M. Fečkan, Y. Zhou*: Ulam's type stability of impulsive ordinary differential equations. J. Math. Anal. Appl. *395* (2012), 258–264. zbl MR doi
[25] *Y. Wei, J. Ding*: Representations for Moore-Penrose inverses in Hilbert spaces. Appl. Math. Lett. *14* (2001), 599–604. zbl MR doi

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