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On a Semi-symmetric Metric Connection in an Almost Kenmotsu Manifold with Nullity Distributions

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Abstract

We consider a semisymmetric metric connection in an almost Kenmotsu manifold with its characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution respectively. We first obtain the expressions of the curvature tensor and Ricci tensor with respect to the semisymmetric metric connection in an almost Kenmotsu manifold with ξ belonging to $(k,\mu)'$- and (k,μ)-nullity distribution respectively. Then we characterize an almost Kenmotsu manifold with ξ belonging to $(k,\mu)'$-nullity distribution admitting a semisymmetric metric connection.

Key words: Semisymmetric metric connection, almost Kenmotsu manifold, Einstein manifold, sectional curvature, Ricci tensor, Weyl conformal curvature tensor.

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1 Introduction

In 1924, Friedmann and Schouten [12] introduced the idea of semisymmetric connection on a differentiable manifold. A linear connection $\bar{\nabla}$ on a differentiable manifold M is said to be a semisymmetric connection if the the torsion tensor T of the connection $\bar{\nabla}$ satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form and ξ is a vector field defined by

$$\eta(X) = g(X, \xi)$$

for all vector fields $X \in \chi(M)$, where $\chi(M)$ is the set off all differentiable vector fields on M.
In, 1932, Hayden [14] introduce the idea of semisymmetric metric connection on a Riemannian manifold (M, g). A semisymmetric connection $\bar{\nabla}$ is said to be a semisymmetric metric connection if $\bar{\nabla}g = 0$. K. Yano [30] started to study the systematic study on semi-symmetric metric connection and this was further studied by T. Imai [15], M. Pravanović and N. Pušić [21], N. Pušić [22], Lj. S. Velimirović et al ([25, 26]), R. S. Mishra [18], U. C. De and G. Pathak [8], T. Q. Binh [4], Y. Liang [17], P. Zhao and H. Song [32], Z. I. Szabó [23], Ajit Barman ([5, 6, 7]) and many others.

The notoin of k-nullity distribution was introduced by Gray [13] and Tanno [24] in the study of Riemannian manifolds (M, g), which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}, \tag{1.1}$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type (1,3). Blair, Koufogiorgos and Papantoniou [1] introduced a generalized notion of the k-nullity distribution, named the (k, μ)-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k, \mu) = \{Z \in T_pM : R(X, Y)Z &= k[g(Y, Z)X - g(X, Z)Y] \\ &+ \mu[g(Y, Z)hX - g(X, Z)hY]\}, \end{aligned} \tag{1.2}$$

where $h = \frac{1}{2}\pounds_\xi \phi$ and $\pounds$ denotes the Lie derivative.

In ([9],[10],[19]) Dileo and Pastore introduce the notion of $(k, \mu)'$-nullity distribution, another generalized notion of the k-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k, \mu)' = \{Z \in T_pM : R(X, Y)Z &= k[g(Y, Z)X - g(X, Z)Y] \\ &+ \mu[g(Y, Z)h'X - g(X, Z)h'Y]\}, \end{aligned} \tag{1.3}$$

where $h' = h \circ \phi$.

A Riemannian manifold is said to be Ricci semisymmetric if $R(X, Y).S = 0$, where S denotes the Ricci tensor of type (0,2). This paper is organized in the following way. In Section 2, we give a brief account on an almost Kenmotsu manifold, while Section 3 contains some results on an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution. In Section 4, we first obtain the expressions for the curvature tensor and Ricci tensor with respect to the semisymmetric metric connection and also we prove that in an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution the manifold is an Einstein manifold with respect to the semisymmetric metric connection. In Section 5, we characterize Ricci semisymmetric almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution. In Section 6, we consider an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution satisfying the curvature condition $\bar{C}.\bar{S} = 0$, where $\bar{C}$ denotes the Weyl conformal curvature tensor with respect to the semisymmetric metric connection. Finally, we prove that in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the manifold is Ricci semisymmetric if and only if the manifold is an Einstein manifold with respect to the semisymmetric metric connection.

2 Almost Kenmotsu manifolds

A differentiable $(2n+1)$-dimensional manifold M is said to have a (ϕ,ξ,η)-structure or an almost contact structure, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([2, 3]),

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1) easily.
If a manifold M with a (ϕ,ξ,η)-structure admits a Reimannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y of T_pM^{2n+1}, then M is said to have an almost contact structure (ϕ,ξ,η,g). The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X,\Phi Y)$ for any X, Y of T_pM^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the (1,2)-type torsion tensor N_ϕ, defined by

$$N_\phi = [\phi,\phi] + 2d\eta \otimes \xi,$$

where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ [2]. Recently in ([9, 10, 11, 19, 20]), almost contact metric manifold such that η is closed and $d\Phi = 2\eta\wedge\Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by

$$(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

for any vector fields X, Y. It is well known [16] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an intregrable distribution. Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi\phi$ and $l = R(\cdot,\xi)\xi$

on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [19]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.5}$$

for any vector fields X, Y. The (1,1)-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and $h'\xi = 0$. Also it is clear that ([9, 29])

$$h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k+1)\phi^2 \quad (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.6}$$

Almost Kenmotsu manifold have been studied by several authors such as Dileo and Pastore ([9, 10, 11]), Wang and X. Liu ([27, 28, 29]) and many others.

3 ξ belongs to the $(k,\mu)'$-nullity distribution

This section is devoted to study almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (2.6) it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigen value λ and $-\lambda$ of h', respectively. Before presenting our main theorems we recall some results:

Lemma 3.1. (Prop. 4.1 and Prop. 4.3 of [9]) *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given by the following:*

(a) $K(X,\xi) = k - 2\lambda$ *if* $X \in [\lambda]'$ *and*
$K(X,\xi) = k + 2\lambda$ *if* $X \in [-\lambda]'$,

(b) $K(X,Y) = k - 2\lambda$ *if* $X, Y \in [\lambda]'$;
$K(X,Y) = k + 2\lambda$ *if* $X, Y \in [-\lambda]'$ *and*
$K(X,Y) = -(k+2)$ *if* $X \in [\lambda]'$, $Y \in [-\lambda]'$,

(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k - 2n)$.

Lemma 3.2. (Lemma 3 of [27]) *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{3.1}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Lemma 3.3. (Proposition 4.2 of [9]) *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(k, -2)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies*

$$\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0, \\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0, \\
R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\
R(X_\lambda, Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

From (1.3) we have,

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{3.2}$$

where $k, \mu \in \mathbb{R}$. Also we get from (3.2)

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X]. \tag{3.3}$$

Contracting X in (3.2) we have

$$S(Y, \xi) = 2nk\eta(Y). \tag{3.4}$$

Moreover in an almost Kenmotsu manifold with $(k, \mu)'$-nullity distribution,

$$\nabla_X \xi = X - \eta(X)\xi + h'X \tag{3.5}$$

and

$$(\nabla_X \eta)Y = g(Y, X) - \eta(X)\eta(Y) + g(Y, h'X) \tag{3.6}$$

4 Curvature tensor of an almost Kenmotsu manifold with $(k, \mu)'$-nullity distribution with respect to the semisymmetric metric connection

Let ∇ be the Riemannian connection. Also let R be the curvature tensor of ∇ and $\bar{R}$ be the curvature tensor of $\bar{\nabla}$ in an almost contact metric manifold. In view of [30] R and $\bar{R}$ are related by

$$\bar{R}(X, Y)Z = R(X, Y)Z - \alpha(Y, Z)X + \alpha(X, Z)Y - g(Y, Z)AX + g(X, Z)AY, \tag{4.1}$$

where α is a $(0, 2)$ tensor defined by

$$\alpha(X, Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y) + \frac{1}{2}\eta(\xi)g(X, Y), \tag{4.2}$$

and

$$g(AX, Y) = \alpha(X, Y), \tag{4.3}$$

Using (2.1), (3.5), and (3.6) we get from (4.2) and (4.3),

$$\alpha(X, Y) = \frac{3}{2}g(X, Y) + g(X, h'Y) - 2\eta(X)\eta(Y) \tag{4.4}$$

and

$$AX = \frac{3}{2}X - 2\eta(X)\xi + h'X \tag{4.5}$$

Using (4.4) and (4.5) in (4.1) we get,

$$\begin{aligned}\bar{R}(X, Y)Z = {} & R(X, Y)Z - 3g(Y, Z)X + 3g(X, Z)Y \\ & - g(Z, h'Y)X + g(Z, h'X)Y + 2g(Y, Z)\eta(X)\xi \\ & - 2g(X, Z)\eta(Y)\xi - g(Y, Z)h'X + g(X, Z)h'Y \\ & + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y,\end{aligned} \tag{4.6}$$

Setting $X = \xi$ in (4.6), we get

$$\bar{R}(\xi, Y)Z = R(\xi, Y)Z - g(Y, Z)\xi - g(Z, h'Y)\xi + \eta(Z)h'Y + \eta(Z)Y, \tag{4.7}$$

Using (1.3) in (4.7) we get,

$$\bar{R}(\xi, Y)Z = (k-1)g(Y, Z)\xi - 3g(z, h'Y)\xi - (k-1)\eta(Z)Y + 3\eta(Z)h'Y. \tag{4.8}$$

Setting $Z = \xi$ in (4.8) we have,

$$\bar{R}(\xi, Y)\xi = (k-1)\eta(Y)\xi - (k-1)Y + 3h'Y. \tag{4.9}$$

Taking inner product with W of (4.6),

$$\begin{aligned}\bar{R}(X, Y, Z, W) = {} & g(R(X, Y)Z, W) - 3g(Y, Z)g(X, W) + 3g(X, Z)g(Y, W) \\ & - g(Z, h'Y)g(X, W) + g(Z, h'X)g(Y, W) + 2g(Y, Z)\eta(X)\eta(W) \\ & - 2g(X, Z)\eta(Y)\eta(W) - g(Y, Z)g(h'X, W) + g(X, Z)g(h'Y, W) \\ & + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y,\end{aligned} \tag{4.10}$$

Substituting $X = W = e_i$ in (4.10), where $\{e_1, e_2, \ldots, e_{2n+1}\}$ be a local orthonormal basis of vector fields in M and taking summation over i, $1 \leq i \leq (2n+1)$, then (4.10) takes the form

$$\bar{S}(Y, Z) = S(Y, Z) - 2ng(h'Y, Z) - 2(3n-1)g(Y, Z) + 2(2n-1)\eta(Y)\eta(Z), \tag{4.11}$$

where S and $\bar{S}$ denote the Ricci tensor of M with respect to ∇ and $\bar{\nabla}$ respectively. Hence we have following theorem:

Theorem 4.1. *For an almost Kenmotsu manifold (M, g) with respect to the semisymmetric metric connection $\bar{\nabla}$*
(a) *The curvature tensor is given by* (4.6),
(b) *The Ricci tensor $\bar{S}$ is Symmetric,*
(c) $\bar{R}(X, Y, Z, W) + \bar{R}(Y, X, W, Z) = 0$,
(d) $\bar{R}(X, Y, Z, W) + \bar{R}(X, Y, W, Z) = 0$,
(e) $\bar{R}(X, Y, Z, W) = \bar{R}(Z, W, X, Y)$,
(f) $\bar{S}(\xi, X) = 2n(k-1)\eta(X)$, *where X, Y, Z, W are the vector fields on M^{2n+1}.*

Now from (3.1), it follows that

$$S(Y, Z) = -2ng(Y, Z) + 2n(k+1)\eta(Y)\eta(Z) - 2ng(h'Y, Z). \tag{4.12}$$

Letting $Y, Z \in [\lambda]'$. Then from (4.12) it follows that

$$S(Y, Z) = -2n(1+\lambda)g(Y, Z). \tag{4.13}$$

Now from (4.11), using (4.13) and the fact $Y, Z \in [\lambda]'$ we get,

$$\bar{S}(Y, Z) = 2[1 - 2n(\lambda + 2)]g(Y, Z).$$

This leads to the following:

Theorem 4.2. *In an almost Kenmotsu manifold with ξ belongs to the $(k, \mu)'$-nullity distribution, the manifold is an Einstein manifold with respect to the semisymmetric metric connection.*

5 Ricci semisymmetric almost Kenmotsu manifolds with ξ belongs to $(k, \mu)'$-nullity distribution with respect to semisymmetric metric connection

In this section we characterize Ricci semisymmetric almost Kenmotsu manifolds with respect to the semisymmetric metric connection, that is, $\bar{R}.\bar{S} = 0$.
Now we prove the following:

Theorem 5.1. *Let, M^{2n+1} be an almost Kenmotsu manifold with characterstic vector ξ belonging to $(k, \mu)'$-nullity distribution with $h' \neq 0$. If the manifold is Ricci semisymmetric with respect to the semisymmetric connection then the following cases occur:*
(i) Einstein manifold with respect to the semisymmetric metric connection.
(ii) locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold with constant sectional curvature -4 and a flat n-dimensional manifold.
(iii) locally isometric to the Riemannian product of an $n+1$ dimensional manifold with constant sectional curvature -9 and n-dimensional manifold with constant sectional curvature -1.

Proof. Suppose, $(\bar{R}(X, Y).\bar{S})(Z, W) = 0$ for all vector fields X, Y, Z, W. This implies,

$$\bar{S}(\bar{R}(X, Y)Z, W) + \bar{S}(Z, \bar{R}(X, Y)W) = 0. \tag{5.1}$$

Putting $X = Z = \xi$ in (5.1), we get

$$\bar{S}(\bar{R}(\xi, Y)\xi, W) + \bar{S}(\xi, \bar{R}(\xi, Y)W) = 0. \tag{5.2}$$

Using (4.8) and (4.9) we get from (5.2),

$$-(k-1)\bar{S}(Y,W)+3\bar{S}(h'Y,W)+2n(k-1)^2g(Y,W)-6n(k-1)g(h'Y,w) = 0 \tag{5.3}$$

for any vector fields Y, W on M^{2n+1}.
Substituting $Y = h'Y$ in (5.3) we get

$$\begin{aligned} &-(k-1)\bar{S}(h'Y,W) + 3\bar{S}(h'^2Y,W) + 2n(k-1)^2g(h'Y,W) \\ &\qquad - 6n(k-1)g(h'^2Y,W) = 0. \end{aligned} \tag{5.4}$$

Again substituting $h'^2 = (k+1)\phi^2$ in (5.4) ,

$$\begin{aligned} &-(k-1)\bar{S}(h'Y,W) - 3(k+1)\bar{S}(Y,W) + 2n(k-1)^2g(h'Y,W) \\ &\qquad + 6n(k-1)^2g(Y,W) = 0. \end{aligned} \tag{5.5}$$

From (5.3) and (5.5) we have,

$$(k+2)(k+5)[\bar{S}(Y,W) - 2n(k-1)g(Y,W)] = 0.$$

Then the following cases arise:
Case 1:

$$\bar{S}(Y,W) - 2n(k-1)g(Y,W) = 0,$$

which implies that the manifold is an Einstein manifold with respect to the semisymmetric metric connection.
Case 2: $(k+2) = 0$, that is, $k = -2$.
Without loss of generality we may choose $\lambda = 1$. Then we have from Lemma 3.3,

$$\mathrm{R}(X_\lambda, Y\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$

$$\mathrm{R}(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0,$$

for any vector field $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu = -2$, it follows from Lemma 3.1 that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.1, we see that $K(X,Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]'$, $Y \in [-\lambda]'$. As is shown [9] that the distribution $[\xi] \oplus [\lambda]'$ is intregrable with totally geodesic leaves and the distribution $[-\lambda]'$ is intregrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $H^{2n+1}(-4) \times \mathbb{R}^n$.

Case 3: $k+5=0$, that is, $k=-5$. Without loss of generality we may choose $\lambda=2$. Then we have from Lemma 3.3,

$$R(X_\lambda, Y_\lambda)Z_\lambda = -9[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$
$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = [g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}],$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu=-2$ it follows from Lemma 3.1, $K(X,\xi)=-9$ for any $X \in [\lambda]'$ and $K(X,\xi)=-1$ for any $X \in [-\lambda]'$. Again from Lemma 3.1 we see that $K(X,Y)=-9$ for any $X, Y \in [\lambda]'$; $K(X,Y)=-1$ for any $X,Y \in [-\lambda]'$ and $K(X,Y)=3$ for any $X \in [\lambda]'$, $Y \in [-\lambda]'$. As is shown [9] that the distribution $[\xi] \oplus [\lambda]'$ is intregrable with totally geodesic leaves and the distribution $[-\lambda]'$ is intregrable with totally umbilical leaves by $H=-(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda=2$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $H^{2n+1}(-9) \times \mathbb{R}^n$. □

6 Almost Kenmotsu manifolds with ξ belongs to $(k,\mu)'$-nullity distribution with respect to the semisymmetric metric connection satisfying $\bar{C}.\bar{S}=0$

This section deals with the study of an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ satisfying the curvature condition $\bar{C}.\bar{S}=0$, where $\bar{C}$ is the Weyl conformal curvature tensor with respect to the semisymmetric metric connection. The Weyl conformal curvature tensor C on a $(2n+1)$-dimensional Riemannian manifold is defined by [31],

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z &- \frac{1}{(2n-1)}[S(Y,Z)X \\ &- S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{6.1}$$

where, X, Y, Z are any vector fields, S is the Ricci tensor of type $(0,2)$ and Q is the Ricci operator defined by $S(X,Y)=g(QX,Y)$. Using the results (3.1) and (3.3) one can easily obtain the following:

$$C(\xi, Y)Z = (\mu + \frac{2n}{(2n-1)})[g(h'Y, Z)\xi - \eta(Z)h'Y], \tag{6.2}$$

Putting $Z=\xi$ we get,

$$C(\xi, Y)\xi = -(\mu + \frac{2n}{(2n-1)})h'Y, \tag{6.3}$$

Now we prove the following:

Theorem 6.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution. If $\bar{C}.\bar{S} = 0$, then the manifold is an Einstein one with respect to the semisymmetric metric connection.*

Proof. Now $(\bar{C}(X, Y).\bar{S})(U, V) = 0$ implies

$$\bar{S}(\bar{C}(X, Y)U, V) + \bar{S}(U, \bar{C}(X, Y)V) = 0, \tag{6.4}$$

Setting $X = U = \xi$ in (6.4), we have

$$\bar{S}(\bar{C}(\xi, Y)\xi, V) + \bar{S}(\xi, \bar{C}(\xi, Y)V) = 0. \tag{6.5}$$

Using (6.2) and (6.3) we get from (6.5),

$$\bar{S}(h'Y, V) - g(h'Y, V)\bar{S}(\xi, \xi) = 0. \tag{6.6}$$

By the help of (3.1) and (4.11) we get from (6.6),

$$\bar{S}(h'Y, V) - 2n(k-1)g(h'Y, V) = 0. \tag{6.7}$$

Setting $Y = h'Y$ in (6.6) we get,

$$\bar{S}(h'^2Y, V) - 2n(k-1)g(h'^2Y, V) = 0. \tag{6.8}$$

Putting $h'^2 = (k+1)\phi^2$ in (6.8), we get,

$$\bar{S}((k+1)\phi^2Y, V) - 2n(k-1)g((k+1)\phi^2Y, V) = 0.$$

which implies that,

$$(k+1)[\bar{S}(Y, V) - 2n(k-1)g(Y, V)] = 0,$$

for any vector fields Y, V on M^{2n+1}.
Suppose $(k+1) = 0$, that is $k = -1$. Dileo and Pastore [9] prove that in almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution if $k = -1$, then $h' = 0$ and the manifold M^{2n+1} is locally a wrapped product of an almost Kähler manifold and an open interval. Thus $k + 1 = 0$, contradicts our hypothesis $h' \neq 0$.
Therefore, $\bar{S}(Y, V) = 2n(k-1)g(Y, V)$, for any vector fields V, Y on M^{2n+1}. Thus the manifold is an Einstein manifold with respect to the semisymmetric metric connection. □

7 ξ belongs to the (k, μ)-nullity distribution

In this section we study $\bar{R}.\bar{S} = 0$ on an almost Kenmotsu manifolds with ξ belonging to the (k, μ)-nullity distribution, where $\bar{R}$ and $\bar{S}$ are the Riemannian curvature tensor and Ricci tensor with respect to semisymmetric metric connection $\bar{\nabla}$. From (1.2) we obtain

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{7.1}$$

where $k, \mu \in \mathbb{R}$. Before proving our main results in this section we first state the following:

Lemma 7.1. [9] *Let M^{2n+1} be an almost Kenmotsu manifold of dimension $(2n+1)$. Suppose that the characterstic vector field ξ belonging to the (k,μ)-nullity distribution. Then $k=-1$, $h=0$ and M^{2n+1} is locally a wrapped product of an open interval and an almost Kähler manifold.*

In view of Lemma 7.1 it follows from (7.1),

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{7.2}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{7.3}$$

$$S(X,\xi) = -2n\eta(X), \tag{7.4}$$

for any vector fields X, Y on M^{2n+1}. Let, R and $\bar{R}$ be the curvature tensor of ∇ and $\bar{\nabla}$ respectively. Then R and $\bar{R}$ are related by [30],

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y \\ &\quad -g(Y,Z)AX + g(X,Z)AY,\end{aligned} \tag{7.5}$$

where

$$\alpha(X,Y) = \frac{3}{2}g(X,Y) - 2\eta(X)\eta(Y), \tag{7.6}$$

and

$$AX = \frac{3}{2}X - 2\eta(X)\xi, \tag{7.7}$$

Using (7.6) and (7.7) we get from (7.5),

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - 3g(Y,Z)X + 3g(X,Z)Y \\ &\quad + 2g(Y,Z)\eta(X)\xi - 2g(X,Z)\eta(Y)\xi + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y,\end{aligned} \tag{7.8}$$

Putting $X=\xi$ in (7.8),

$$\bar{R}(\xi,Y)Z = R(\xi,Y)Z - g(Y,Z)\xi + \eta(Z)Y, \tag{7.9}$$

Using (7.3) we get from (7.9),

$$\bar{R}(\xi,Y)Z = -2\eta(Y)\xi + 2Y, \tag{7.10}$$

Substituting $Z=\xi$ in (7.10) we have,

$$\bar{R}(\xi,Y)\xi = -2\eta(Y)\xi + 2Y. \tag{7.11}$$

Contracting X in (7.8) gives

$$\bar{S}(Y,Z) = S(Y,Z) - 3(2n+1)g(Y,Z) + (2n-3)\eta(Y)\eta(Z). \tag{7.12}$$

Setting $Y=\xi$ in (7.12),

$$\bar{S}(\xi,W) = S(\xi,W) - (4n+1)\eta(W). \tag{7.13}$$

By the help of (4.3) we get,

$$\bar{S}(\xi, W) = -(6n+1)\eta(W), \tag{7.14}$$

Putting $W = \xi$ we get,

$$\bar{S}(\xi, \xi) = -(6n+1). \tag{7.15}$$

Now we are in a position to state and proof our main theorem in this section:

Theorem 7.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the (k, μ)-nullity distribution. Then M^{2n+1} is Ricci semisymmetric if and only if the manifold is an Einstein manifold with respect to the semisymmetric metric connection.*

Proof. Now $(\bar{R}(X, Y).\bar{S})(Z, W) = 0$ implies

$$\bar{S}(\bar{R}(X, Y)Z, W) + \bar{S}(Z, \bar{R}(X, Y)W) = 0. \tag{7.16}$$

Putting $X = Z = \xi$ in (7.16) and using (7.10) we have,

$$\bar{S}(-2\eta(Y)\xi + 2Y, W) + \bar{S}(\xi, -2g(Y, W)\xi + 2\eta(W)Y) = 0. \tag{7.17}$$

Using (7.14) and (7.15) we get,

$$\bar{S}(Y, W) + (1 + 6n)g(Y, W) = 0.$$

which implies that the manifold is an Einstein manifold with respect to the semisymmetric metric connection.
Cnversely, let the manifold is an Einstein manifold. Then obviously $\bar{R}.\bar{S} = 0$. This complete the proof. □

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