Kybernetika

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Kybernetika, Vol. 53 (2017), No. 1, 129–136

Persistent URL: http://dml.cz/dmlcz/146712

KYBERNETIKA — VOLUME *53* (2017), NUMBER 1, PAGES 129–136

A NOTE ON THE SUPER-ADDITIVE AND SUB-ADDITIVE TRANSFORMATIONS OF AGGREGATION FUNCTIONS: THE MULTI-DIMENSIONAL CASE

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For an aggregation function A we know that it is bounded by A^* and A_* which are its super-additive and sub-additive transformations, respectively. Also, it is known that if A^* is directionally convex, then $A = A^*$ and A_* is linear; similarly, if A_* is directionally concave, then $A = A_*$ and A^* is linear. We generalize these results replacing the directional convexity and concavity conditions by the weaker assumptions of overrunning a super-additive function and underrunning a sub-additive function, respectively.

Keywords: aggregation function, overrunning and underrunning property, sub-additive and super-additive transformation

Classification: 47H04, 47S40

1. INTRODUCTION

Aggregation functions are indispensable in real-world applications where quantitative evaluation data are required to be fused into a single numerical entry. Examples abound and include decision making with the help of aggregating scores or preferences with respect to certain alternatives, or compressing information by merging multiple origin inputs to simplify recognition and classification, and so on, all with applications in artificial intelligence, risk management, statistical inference and many other areas.

Literature on aggregation functions is abundant and we just refer to [2, 4] for basic facts and for a survey of different types of such functions (and their transformations) that have been considered. For the purpose of this paper, an *aggregation function* is any mapping $A: [0,\infty]^n \to [0,\infty]$ which is increasing in every coordinate and its value at the origin equals to 0. The super-additive transformation $A^* : [0,\infty]^n \to [0,\infty]$ of A is defined by

$$A^*(\mathbf{x}) = \sup \left\{ \sum_{j=1}^{k} A(\mathbf{x}^{(j)}) | \sum_{j=1}^{k} \mathbf{x}^{(j)} \leq \mathbf{x} \right\}.$$

DOI: 10.14736/kyb-2017-1-0129

Dually, the sub-additive transformation $A_* : [0,\infty]^n \to [0,\infty]$ of A is defined as

$$A_*(\mathbf{x}) = \inf\left\{\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) | \sum_{j=1}^{k} \mathbf{x}^{(j)} \geq \mathbf{x}\right\}.$$

It easy to show [5] that the functions A^* and A_* are indeed, as their names suggest, super-additive and sub-additive, respectively, that is, $A^*(\mathbf{u}+\mathbf{v}) \geq A^*(\mathbf{u}) + A^*(\mathbf{v})$ and $A_*(\mathbf{u}+\mathbf{v}) \leq A_*(\mathbf{u})+A_*(\mathbf{v})$ for every $\mathbf{u},\ \mathbf{v} \in [0,\infty]^n$, where addition is defined coordinate-wise in the usual manner.

This suggests the question of whether or not for every pair $F, G:\ [0,\infty]^n \to [0,\infty]$ such that $F(\mathbf{0}) = G(\mathbf{0}) = 0$, $F(\mathbf{x}) \leq G(\mathbf{x})$ for every $\mathbf{x} \in [0,\infty]^n$, with F sub-additive and G super-additive, there exists an aggregation function A on $[0,\infty]^n$ such that $A_* = F$ and $A^* = G$.

In [12] (and also in [11] in the one-dimensional case) the authors showed that the answer to this question is negative if relatively mild extra conditions are imposed on F and G. Their results said that if an aggregation function A is such that A^* is directionally convex, then necessarily $A = A^*$ and A_* is linear; dually, if A_* is directionally concave, then $A = A_*$ and A^* is linear.

In [6] the authors studied the one-dimensional results of [11] and [12] under more relaxed conditions, assuming that A^* overruns a super-additive function and A_* underruns a sub-additive function. To explain these new concepts, we say that a function $F:\ [0,\infty] \to [0,\infty]$ *overruns* a super-additive function $H : [0,\infty] \to [0,\infty]$ if the function $F(x)/H(x)$ is strictly increasing on $]0,\infty]$; similarly, we say that such a one-dimensional function F *underruns* a sub-additive one-dimensional function H if $F(x)/H(x)$ is strictly decreasing on $]0,\infty]$. In this paper, we generalize the results of [6] in the multi-dimensional case.

2. RESULTS

Here, points in $[0,\infty]^n$ will be denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, and so on; in particular, $\mathbf{0}$ and $\mathbf{1}$ will stand for the points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. Note that every $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,\infty]^n$ can be written in the form $\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith unit vector. We will use the notation $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in [0,\infty]^n$, and $\mathbf{x} < \mathbf{y}$ if $\mathbf{x} \leq \mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$.

Definition 2.1. Let $F:\ [0,\infty]^n \to [0,\infty]$ be an aggregation function and let $H : [0,\infty]^n \to [0,\infty]$ be a *super-additive* aggregation function. We will say that F *overruns* H if $F(\mathbf{x})/H(\mathbf{x})$ is a strictly increasing function in every coordinate ($\mathbf{x} > \mathbf{0}$). Equivalently, F overruns H if

$$F(\mathbf{x})H(\mathbf{y}) < F(\mathbf{y})H(\mathbf{x}) \quad \text{whenever} \quad \mathbf{0} < \mathbf{x} < \mathbf{y}\ . \tag{1}$$

It is easy to see that if F and H are as above and F overruns H, then F is strictly super-additive. Indeed, by (1), for any $\mathbf{u}, \mathbf{v} \in [0,\infty]^n$ not both equal to $\mathbf{0}$ we have

$F(\mathbf{u})H(\mathbf{u}+\mathbf{v}) < F(\mathbf{u}+\mathbf{v})H(\mathbf{u})$ and $F(\mathbf{v})H(\mathbf{u}+\mathbf{v}) < F(\mathbf{u}+\mathbf{v})H(\mathbf{v})$. Adding the two inequalities together and using super-additivity of H gives

$$(F(\mathbf{u})+F(\mathbf{v}))H(\mathbf{u}+\mathbf{v}) < F(\mathbf{u}+\mathbf{v})(H(\mathbf{u})+H(\mathbf{v})) \le F(\mathbf{u}+\mathbf{v})H(\mathbf{u}+\mathbf{v})$$

and strong super-additivity of F follows by canceling the non-zero term $H(\mathbf{u}+\mathbf{v})$ above.

Example 2.2. Consider aggregation functions $F(x,y) = (x+1)(y+1)(x+y+2)-2$ and $H(x,y) = x+y$. Observe that H is additive and consequently super-additive aggregation function. Now, it can be easily seen that $\varphi(x,y) = \frac{(x+1)(y+1)(x+y+2)-2}{x+y}$ is a strictly increasing function in every coordinate. So, by Definition 2.1, F overruns H.

For an aggregation function $F:\ [0,\infty]^n \to [0,\infty]$ we let ∇_F be the n-dimensional vector whose ith component $(\nabla_F)_i$ is equal to $\liminf_{t\to 0^+} F(t\mathbf{e}_i)/t$ for $i \in \{1,2,\ldots,n\}$. Note that in the ith coordinate, if the first partial derivative of F in $\mathbf{0}$ exist, then it equals $(\nabla_F)_i$.

Theorem 2.3. Let $A:\ [0,\infty]^n \to [0,\infty]$ be an aggregation function. If A^* is continuous and overruns some super-additive aggregation function on $[0,\infty]^n$, then $A^*(\mathbf{x}) = A(\mathbf{x})$ and $A_*(\mathbf{x}) = \nabla_A \cdot \mathbf{x}$ for every $\mathbf{x} \in [0,\infty]^n$.

Proof. Assume that $A^*:\ [0,\infty]^n \to [0,\infty]$ overruns a super-additive aggregation function $H:\ [0,\infty]^n \to [0,\infty]$. That is, let us assume that

$$A^*(\mathbf{x})H(\mathbf{y}) < A^*(\mathbf{y})H(\mathbf{x}) \ \text{ for every } \mathbf{x},\mathbf{y} \in [0,\infty]^n \ \text{ such that } \ \mathbf{0} < \mathbf{x} < \mathbf{y}\ . \qquad (2)$$

Obviously, $A(\mathbf{x}) \le A^*(\mathbf{x})$ for every $\mathbf{x} \in [0,\infty]^n$.

We begin by showing that $A = A^*$. Assume the contrary and let $\overline{\mathbf{x}} \neq \mathbf{0}$ be such that $A(\overline{\mathbf{x}}) < A^*(\overline{\mathbf{x}})$. Recall that

$$A^*(\overline{\mathbf{x}}) = \sup\left\{ \sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \mid \mathbf{0} \neq \mathbf{x}^{(j)} \in [0,\infty]^n \ (1 \le j \le k),\ \sum_{j=1}^{k} \mathbf{x}^{(j)} = \overline{\mathbf{x}} \right\}\ ; \qquad (3)$$

assuming equality in the second summation means no loss of generality. If both sums consist of the single elements $A(\overline{\mathbf{x}})$ and $\overline{\mathbf{x}}$, then automatically $A(\overline{\mathbf{x}}) < A^*(\overline{\mathbf{x}})$, and so it is sufficient to assume that $k \ge 2$ in our arguments. We may also assume that all coordinates of $\overline{\mathbf{x}}$ are positive. Namely, if, say, $\bar{x}_n = 0$, then we would have $x_n^{(j)} = 0$ for all $j \in \{1,2,\ldots,k\}$ in the points entering (3), which just means a reduction in the dimension from n to $n-1$.

We continue by introducing a number of parameters. Let $\delta_1 = \frac{1}{2}(A^*(\overline{\mathbf{x}}) - A(\overline{\mathbf{x}}))$ and let $\xi > 0$ be the smallest coordinate of the point $\overline{\mathbf{x}}$. Further, for every *proper* subset $I \subset \{1,2,\ldots,n\}$ and every $\eta \ge 0$ let $\overline{\mathbf{x}}_{I,\eta}$ be the point whose ith coordinate is equal to η for every $i \in I$ and to $\bar{x}_i$ for every $i \in J = \{1,\ldots,n\}\backslash I$. For every such *non-trivial* partition $\{I,J\}$ of $\{1,2,\ldots,n\}$, strict super-additivity of A^* (as a consequence of its overrunning H) implies that $A^*(\overline{\mathbf{x}}_{I,0}) + A^*(\overline{\mathbf{x}}_{J,0}) < A^*(\overline{\mathbf{x}})$. By continuity and monotonicity of A^*, there exist $\delta_2 > 0$ and $\mu > 0$ such that $A^*(\overline{\mathbf{x}}_{I,\mu}) + A^*(\overline{\mathbf{x}}_{J,\mu}) = A^*(\overline{\mathbf{x}}) - \delta_2$ for every non-trivial partition (I, J) of $\{1, 2, \ldots, n\}$. Applying monotonicity of A^* to the last equation again we conclude that for every $\nu \in [0, \mu]$ and every non-trivial partition $\{I, J\}$ of $\{1, 2, \ldots, n\}$ we have

$$A^*(\overline{\mathbf{x}}_{I,\nu}) + A^*(\overline{\mathbf{x}}_{J,\nu}) \leq A^*(\overline{\mathbf{x}}) - \delta_2 \ . \tag{4}$$

Next, let m be the smallest positive integer satisfying $m \geq \max\{\mu^{-1}, \xi^{-1}, A^*(\mathbf{1})/\delta_1\}$. We will often use the reciprocal value $\varepsilon = 1/m$ of m, so that

$$0 < \varepsilon = m^{-1} \leq \min \{ \ \mu, \ \xi, \ \delta_1/A^*(\mathbf{1}) \ \} \ . \tag{5}$$

Finally, let us apply the 'overrunning' inequality (2) to the pair of points $\overline{\mathbf{x}} - \varepsilon\mathbf{e}_i < \overline{\mathbf{x}}$. As the result we obtain $H(\overline{\mathbf{x}})A^*(\overline{\mathbf{x}} - \varepsilon\mathbf{e}_i) < H(\overline{\mathbf{x}} - \varepsilon\mathbf{e}_i)A^*(\overline{\mathbf{x}})$ for every $i \in \{1, 2, \ldots, n\}$. This means that there exists a $\delta_3 > 0$ such that

$$H(\overline{\mathbf{x}})A^*(\overline{\mathbf{x}} - \varepsilon\mathbf{e}_i) \leq H(\overline{\mathbf{x}} - \varepsilon\mathbf{e}_i)(A^*(\overline{\mathbf{x}}) - \delta_3) \ , \quad 1 \leq i \leq n. \tag{6}$$

From this point on we will distinguish three cases, depending on the distribution of 'large' coordinates in the k-tuple

$$\mathbf{x}^{(j)} \ (1 \leq j \leq k); \quad \sum_{j=1}^{k} \mathbf{x}^{(j)} = \overline{\mathbf{x}} \tag{7}$$

appearing in (3). The three cases will depend on assumptions (A1) – (A3) stated below.

Case 1: Assume that

(A1) *in the k-tuple (7) there exists a $j \in \{1, 2, \ldots, k\}$ such that $\mathbf{x}^{(j)} \geq \overline{\mathbf{x}} - \varepsilon\mathbf{1}$.*

We may let $j = 1$, and then $\sum_{j=2}^{k} \mathbf{x}^{(j)} \leq \varepsilon\mathbf{1}$. To estimate the sum $\sum_{j=1}^{k} A(\mathbf{x}^{(j)})$ in (3), observe first that $A(\mathbf{x}^{(1)}) \leq A(\overline{\mathbf{x}}) = A^*(\overline{\mathbf{x}}) - 2\delta_1$ by our choice of δ_1. Super-additivity of A^* applied to $\sum_{j=2}^{k} \mathbf{x}^{(j)} \leq \varepsilon\mathbf{1}$ further results in the chain of inequalities $\sum_{j=2}^{k} A(\mathbf{x}^{(j)}) \leq \sum_{j=2}^{k} A^*(\mathbf{x}^{(j)}) \leq A^*(\sum_{j=2}^{k} \mathbf{x}^{(j)}) \leq A^*(\varepsilon\mathbf{1})$. Recalling that $\varepsilon = m^{-1}$ and invoking super-additivity of A^* again we obtain $mA^*(m^{-1}\mathbf{1}) \leq A^*(\mathbf{1})$, which means that $A^*(\varepsilon\mathbf{1}) \leq \varepsilon A^*(\mathbf{1})$. But by (5) we have $\varepsilon A^*(\mathbf{1}) \leq \delta_1$, which in combination with the previous inequalities leads to our first partial conclusion: If the sum (7) satisfies the assumption (A1), then

$$\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) = A(\mathbf{x}^{(1)}) + \sum_{j=2}^{k} A(\mathbf{x}^{(j)}) \leq A^*(\overline{\mathbf{x}}) - 2\delta_1 + A^*(\varepsilon\mathbf{1}) \leq A^*(\overline{\mathbf{x}}) - \delta_1 \ . \tag{8}$$

Case 2: Suppose that

(A2) *the k-tuple in (7) is not as in (A1) but has the property that for every $i \in \{1, 2, \ldots, n\}$ there exists a $j = j_i \in \{1, 2, \ldots, k\}$ such that $x_i^{(j)} \geq \bar{x}_i - \varepsilon$.*

Without loss of generality we may assume that there is an $r \in \{1, \ldots, n-1\}$ such that $j_1 = \ldots = j_r = 1$ but $j_i \geq 2$ for all i such that $r + 1 \leq i \leq n$. Let $\mathbf{y} = \mathbf{x}^{(1)}$ and

$\mathbf{z} = \sum_{j=2}^{k} \mathbf{x}^{(j)}$. Super-additivity of A^* implies that

$$\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \le A^*(\mathbf{x}^{(1)}) + \sum_{j=2}^{k} A^*(\mathbf{x}^{(j)}) \le A^*(\mathbf{x}^{(1)}) + A^*\left(\sum_{j=2}^{k} \mathbf{x}^{(j)}\right) = A^*(\mathbf{y}) + A^*(\mathbf{z}) \ . \tag{9}$$

For the partition $\{I, J\}$ of $\{1, 2, \ldots, n\}$ given by $I = \{1, \ldots, r\}$ and $J = \{r+1, \ldots, n\}$, let $\overline{\mathbf{x}}_{I,\varepsilon}$ and $\overline{\mathbf{x}}_{J,\varepsilon}$ be points as introduced earlier when defining the values of δ_2 and ε. Observe that for every $i \in J$ we have $z_i \ge \overline{x}_i - \varepsilon$ and hence $y_i \le \varepsilon$, so that $\mathbf{y} \le \overline{\mathbf{x}}_{J,\varepsilon}$. Similarly, for every $i \in I$ we have $y_i \ge \overline{x}_i - \varepsilon$ and so $z_i \le \varepsilon$, that is, $\mathbf{z} \le \overline{\mathbf{x}}_{I,\varepsilon}$. Applying (4), (5) and monotonicity of A^* it follows that $A^*(\overline{\mathbf{x}}_{J,\varepsilon}) + A^*(\overline{\mathbf{x}}_{I,\varepsilon}) \le A^*(\overline{\mathbf{x}}) - \delta_2$. Combining these inequalities with (9) gives our second partial conclusion: If $\sum_{j=1}^{k} \mathbf{x}^{(j)}$ satisfies the assumption (A2), then

$$\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \le A^*(\mathbf{y}) + A^*(\mathbf{z}) \le A^*(\overline{\mathbf{x}}_{J,\varepsilon}) + A^*(\overline{\mathbf{x}}_{I,\varepsilon}) \le A^*(\overline{\mathbf{x}}) - \delta_2 \ . \tag{10}$$

Case 3: Assume that

(A3) *the k-tuple (7) is such that there is an $i \in \{1, \ldots, n\}$ such that for each $j \in \{1, \ldots, k\}$ one has $x_i^{(j)} \le \bar{x}_i - \varepsilon$, that is, $\mathbf{x}^{(j)} \le \overline{\mathbf{x}} - \varepsilon \mathbf{e}_i$.*

Applying the 'overrunning' inequality (2) to this pair of points gives $H(\overline{\mathbf{x}} - \varepsilon \mathbf{e}_i) A^*(\mathbf{x}^{(j)}) \le H(\mathbf{x}^{(j)}) A^*(\overline{\mathbf{x}} - \varepsilon \mathbf{e}_i)$ for every $j \in \{1, 2, \ldots, k\}$. Summation over j with the help of super-additivity of H yields

$$H(\overline{\mathbf{x}} - \varepsilon \mathbf{e}_i) \sum_{j=1}^{k} A^*(\mathbf{x}^{(j)}) \le \sum_{j=1}^{k} H(\mathbf{x}^{(j)}) A^*(\overline{\mathbf{x}} - \varepsilon \mathbf{e}_i) \le H(\overline{\mathbf{x}}) A^*(\overline{\mathbf{x}} - \varepsilon \mathbf{e}_i) \ . \tag{11}$$

To develop the chain of inequalities (11) further we apply (6), by which, for $\delta_3 > 0$, we have $H(\overline{\mathbf{x}}) A^*(\overline{\mathbf{x}} - \varepsilon \mathbf{e}_i) \le H(\overline{\mathbf{x}} - \varepsilon \mathbf{e}_i)(A^*(\overline{\mathbf{x}}) - \delta_3)$. Canceling then the common term $H(\overline{\mathbf{x}} - \varepsilon \mathbf{e}_i)$ in the last inequality and in (11) results in $\sum_{j=1}^{k} A^*(\mathbf{x}^{(j)}) \le A^*(\overline{\mathbf{x}}) - \delta_3$. Our third partial conclusion now is: For a sum (7) satisfying the assumption (A3) one has

$$\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \le \sum_{j=1}^{k} A^*(\mathbf{x}^{(j)}) \le A^*(\overline{\mathbf{x}}) - \delta_3 \ . \tag{12}$$

It is now easy to draw a conclusion regarding A^*. Observe that for every $k \ge 2$ a k-tuple as in (7) falls under one of the three cases considered above. Letting $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$ it is clear that (8), (10) and (12) imply the inequality $\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \le A^*(\overline{\mathbf{x}}) - \delta$ whenever $k \ge 2$, and we know that $A(\overline{\mathbf{x}}) < A^*(\overline{\mathbf{x}})$. By (3) we thus have $A^*(\overline{\mathbf{x}}) < A^*(\overline{\mathbf{x}})$, a contradiction. This proves that $A(\mathbf{x}) = A^*(\mathbf{x})$ for every $\mathbf{x} \in [0, \infty]^n$.

To finish the proof it remains to show validity of the statement about A_*. Applying Theorem 1 of [10] to the function $x_i \mapsto A(x_i \mathbf{e}_i)$ of one variable $x_i \in [0, \infty]$, $1 \le i \le n$, we obtain the inequality $A_*(x_i \mathbf{e}_i) \le (\nabla_A)_i x_i$ for every $x_i \in [0, \infty]$. We know by [5] that

A_* is sub-additive, which, for every $\mathbf{x} \in [0,\infty]^n$, implies that

$$A_*(\mathbf{x}) = A_*\left(\sum_{i=1}^n x_i\mathbf{e}_i\right) \le \sum_{i=1}^n A_*(x_i\mathbf{e}_i) \le \nabla_A{\cdot}\mathbf{x}\ . \tag{13}$$

To prove the reverse inequality we apply super-additivity of $A^* = A$ together with the inequality $A^*(x_i\mathbf{e}_i) \ge (\nabla_A)_i x_i$, which again follows from Theorem 1 of [10] when applied to the function $x_i \mapsto A(x_i\mathbf{e}_i)$. This results in the chain of inequalities

$$A(\mathbf{x}) = A^*(\mathbf{x}) = A^*\left(\sum_{i=1}^n x_i\mathbf{e}_i\right) \ge \sum_{i=1}^n A^*(x_i\mathbf{e}_i) \ge \sum_{i=1}^n (\nabla_A)_i x_i = \nabla_A{\cdot}\mathbf{x} \tag{14}$$

for every $\mathbf{x} \in [0,\infty]^n$. From (14) we deduce that $A_*(\mathbf{x}) \ge (\nabla_A{\cdot}\mathbf{x})_*$. Since $\nabla_A{\cdot}\mathbf{x}$ is a linear function, we have $(\nabla_A{\cdot}\mathbf{x})_* = \nabla_A{\cdot}\mathbf{x}$, and so $A_*(\mathbf{x}) \ge \nabla_A{\cdot}\mathbf{x}$ for every $\mathbf{x} \in [0,\infty]^n$. In conjunction with (13) this implies that $A_*(\mathbf{x}) = \nabla_A{\cdot}\mathbf{x}$ for every $\mathbf{x} \in [0,\infty]^n$. □

Note that in the above proof, the assumption of continuity of A^* was used exactly once (in Case 2) and, likewise, the assumption of A^* overrunning H was also used just once (in Case 3); in all the remaining places we have only used super-additivity of A^*.

The reader may have noticed that it is the case 2 of the above proof which covers a situation that does not appear in the one-dimensional case, while handling the cases 1 and 3 is an extension of the way the corresponding instances have been treated previously.

Example 2.4. Let $A^*(x,y) = (x+1)(y+1)(x+y+2) - 2$. We know that A^* is a continuous aggregation function and, by Example 2.2, A^* overruns super-additive aggregation function $H(x,y) = x+y$. Then, by Theorem 2.3, we have $A^* = A$ and $A_*(\mathbf{x}) = \nabla_A.\mathbf{x} = 3x + 3y$.

Remark 2.5. Note that $A = A^*$ for any super-additive aggregation function A. Moreover, for any A, as it is mentioned earlier, A^* is super-additive. Theorem 2.3 states the reverse problem, i. e. which properties of A^* ensure $A^* = A$.

In an entirely similar way one can prove a 'dual' statement regarding aggregation functions A for which $A_* = A$.

Definition 2.6. Given an aggregation function $G:\ [0,\infty]^n \to [0,\infty]$ and a *sub-additive* aggregation function $H:\ [0,\infty]^n \to [0,\infty]$, we will say that G *underruns* H if $G(\mathbf{x})/H(\mathbf{x})$ is strictly decreasing in every coordinate ($\mathbf{x} > \mathbf{0}$). Equivalently, G underruns H if

$$G(\mathbf{x})H(\mathbf{y}) > G(\mathbf{y})H(\mathbf{x}) \quad \text{whenever} \quad \mathbf{0} < \mathbf{x} < \mathbf{y}\ .$$

Again, it can be easily seen that if G and H are as above and G underruns H, then G is strictly sub-additive.

Further, for an aggregation function G as above we let ∇^G be the n-dimensional vector with ith component $(\nabla^G)_i$ equal to $\limsup_{t\to 0^+} G(t\mathbf{e_i})/t$ for $i \in \{1,2,\ldots,n\}$. Note that in the ith coordinate, if the first partial derivative of G in $\mathbf{0}$ exist, then it equals $(\nabla^G)_i$. A straightforward modification of the proof of Theorem 2.3 by reversing chains of inequalities appropriately yields the following result.

Theorem 2.7. Let $A : [0,\infty]^n \to [0,\infty]$ be an aggregation function. If A_* is continuous and underruns some sub-additive aggregation function on $[0,\infty]^n$, then $A_*(\mathbf{x}) = A(\mathbf{x})$ and $A^*(\mathbf{x}) = \nabla^A \cdot \mathbf{x}$ for every $\mathbf{x} \in [0,\infty]^n$.

Interesting sufficient conditions for the non-existence of an aggregation function with given super- and sub-additive transformation are given below.

Theorem 2.8. Let $F, G : [0,\infty]^n \to [0,\infty]$ be continuous super-additive and sub-additive functions, respectively, such that $F(\mathbf{x}) \geq G(\mathbf{x})$ for every $\mathbf{x} \in [0,\infty]^n$. If

(a) F overruns some super-additive aggregation function and G is not linear, or

(b) G underruns some sub-additive aggregation function and F is not linear,

then there is no aggregation function $A : [0,\infty]^n \to [0,\infty]$ with $A^* = F$ and $A_* = G$.

Corollary 2.9. Let $F, G : [0,\infty]^n \to [0,\infty]$ be aggregation functions such that $F(\mathbf{x}) \leq G(\mathbf{x})$ for all $\mathbf{x} \in [0,\infty]^n$. If

(a) G overruns some super-additive aggregation function and F is not linear, or

(b) F underruns some sub-additive aggregation function and G is not linear, then there is no aggregation function A such that $A_* = F$ and $A^* = G$.

3. CONCLUSION

Our aim was to relax the assumptions of strict directional convexity or concavity in the results of [11, 12], and in [6] in the one-dimensional case. We have shown that replacing strict directional convexity (concavity) by the weaker condition of overrunning (underrunning) a super-additive (sub-additive) function leads to the same conclusion in the general case.

ACKNOWLEDGEMENT

The first author acknowledges the financial support received from the Ministry of Science, Research and Technology of the Islamic Republic of Iran. The work of the second author was supported by the APVV-14-0013 and the VEGA 1/0420/15 research grants.

(Received June 18, 2016)

REFERENCES

[1] A. Arlotto and M. Scarsini: Hessian orders and multinormal distributions. J. Multivariate Anal. *100* (2009), 2324–2330. DOI:10.1016/j.jmva.2009.03.009

[2] G. Beliakov, A. Pradera, and T. Calvo: Aggregation Functions: A Guide for Practitioners. Springer–Verlag, Berlin 2007. DOI:10.1007/978-3-540-73721-6_5

[3] F. Bernstein and G. Doetsch: Zur Theorie der konvexen Functionen. Math. Annalen *76* (1915), 514–526. DOI:10.1007/bf01458222

[4] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap: Aggregation Functions (Encyklopedia of Mathematics and its Applications). Cambridge University Press, 2009. DOI:10.1017/cbo9781139644150

[5] S. Greco, R. Mesiar, F. Rindone, and L. Šipeky: The superadditive and the subadditive transformations of integrals and aggregation functions. Fuzzy Sets and Systems *291* (2016), 40–53. DOI:10.1016/j.fss.2015.08.006

[6] F. Kouchakinejad and A. Šipošová: A note on the super-additive and sub-additive transformations of aggregation functions: The one-dimensional case. In: Mathematics, Geometry and their Applications, STU Bratislava 2016, pp. 15–19.

[7] M. Kuczma: An Introduction to the Theory of Functional Equations and Inequalities. Second edition. Birkhäuser, 2009. DOI:10.1007/978-3-7643-8749-5

[8] M. Marinacci and L. Montrucchio: Ultramodular functions. Math. Oper. Res. *30* (2005), 311–332. DOI:10.1287/moor.1040.0143

[9] A. Murenko: A generalization of Bernstein–Doetsch theorem. Demonstration Math. XLV *1* (2012), 35–38.

[10] A. Šipošová: A note on the superadditive and the subadditive transformations of aggregation functions. Fuzzy Sets and Systems *299* (2016), 98–104. DOI:10.1016/j.fss.2015.10.008

[11] A. Šipošová and L. Šipeky: On aggregation functions with given superadditive and subadditive transformations. In: Congress on Information Technology, Computational and Experimental Physics, Krakow (Poland) 2015, pp. 199–202.

[12] A. Šipošová, L. Šipeky, and J. Širáň: On the existence of aggregation functions with given super-additive and sub-additive transformations. Fuzzy Sets and Systems (in press). DOI:10.1016/j.fss.2016.11.009

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