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ERROR ANALYSIS OF SPLITTING METHODS FOR SEMILINEAR EVOLUTION EQUATIONS

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Abstract. We consider a Strang-type splitting method for an abstract semilinear evolution equation

$$\partial_t u = Au + F(u).$$

Roughly speaking, the splitting method is a time-discretization approximation based on the decomposition of the operators A and F. Particularly, the Strang method is a popular splitting method and is known to be convergent at a second order rate for some particular ODEs and PDEs. Moreover, such estimates usually address the case of splitting the operator into two parts. In this paper, we consider the splitting method which is split into three parts and prove that our proposed method is convergent at a second order rate.

Keywords: splitting method; semilinear evolution equations; error analysis

MSC 2010: 34B16, 34C25

1. Introduction

Let X be a Hilbert space equipped with a scalar product $(\cdot,\cdot)_X$ and a norm $\|\cdot\|_X$, let A be an m-dissipative linear operator in X with dense domain $D(A) \subset X$.

As is well-known, the operator A generates a contraction semigroup $\Phi_A(t) = \mathrm{e}^{tA}$ if and only if A is m-dissipative with dense domain. We consider the Cauchy problem for semilinear evolution equation

$$\begin{cases} \partial_t u = Au + F(u), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

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where $F\colon D(A) \to D(A)$ is a nonlinear operator. Typical examples of (1.1) are complex Ginzburg-Landau equations in $\Omega \subset \mathbb{R}^d$

$$\partial_t u = (\mathrm{i} + \gamma)\Delta u + \alpha u|u|^2, \tag{1.2}$$

$$\partial_t u = (\mathrm{i} + \gamma)\Delta u + \alpha u|u|^2 + \beta u|u|^4, \tag{1.3}$$

where $\gamma \geqslant 0$ and α and β are complex constants.

The main purpose of this paper is to study the so-called splitting method, which is a semi-discrete approximation of (1.1) with respect to time variable t. The idea behind the splitting method is as follows. We denote the (nonlinear) solution operator (1.1) by $S(t)$. That is, the solution of (1.1) is given as $u(t) = S(t)u_0$; see (1.7) below. Then, we consider the time-discrete approximation to (1.1) at $t = n\Delta t$ as

$$u_n = \Psi(\Delta t)^n u_0,$$

where $\Delta t > 0$ denotes a time increment and n a positive integer. Typical choices of Ψ are, for example,

$$\Psi(t) = \Phi_A(t)\Phi_F(t) \quad (\text{or } \Psi(t) = \Phi_F(t)\Phi_A(t)), \tag{1.4}$$

$$\Psi(t) = \Phi_A(t/2)\Phi_F(t)\Phi_A(t/2), \tag{1.5}$$

where $\Phi_F(t)$ denotes the solution operator of $\partial_t w = F(w)$. Particularly, (1.4) and (1.5) are called the Lie and Strang methods, respectively. For some ordinary differential equations (ODEs) and partial differential equations (PDEs), it is well known that Lie-type splitting methods are first order convergent numerically or rigorously. On the other hand, Strang-type splitting methods are second order convergent. That is, if the time increment Δt is sufficiently small, then we have

$$\|S(n\Delta t)u_0 - \Psi(\Delta t)^n u_0\| \leqslant C\Delta t^2. \tag{1.6}$$

Splitting methods are useful when $S(t)u_0$ is difficult to compute, while $\Phi_A(t)u_0$ and $\Phi_F(t)u_0$ are easy to compute. In addition, if (1.1) has conservation properties, then splitting methods basically preserve its discrete version.

Analysis of splitting methods for ODEs has been presented in many studies. For example, see Hairer et al. [5]. Some results on error analysis are also presented for PDEs. For example, results of error analysis for nonlinear Schrödinger equations can be found in, e.g., Besse et al. [1] and Lubich [7].

However, to our best knowledge, little is known for the abstract Cauchy problem of the form (1.1). Decombes and Thalhammer [4] and Jahnke and Lubich [6] presented an error analysis for the case in which F is a linear operator. For nonlinear abstract Cauchy problems, Borgna et al. [2] demonstrated that various splitting methods involving the Strang method have first order accuracy. However, they did not demonstrate that the Strang-type splitting method is a second order scheme. It should be kept in mind that (1.6) is established for the Strang method applied to particular PDEs; see Besse et al. [1] and Lubich [7]. Therefore, it is worth studying the Strang method for the abstract Cauchy problem of the form (1.1) and deriving the second order error estimate.

On the other hand, the majority of previous studies have considered schemes that are split into two parts; $\partial_t v = Av$ and $\partial_t w = F(w)$. As a matter of fact, if such two-parts splitting is applied to (1.2), then the explicit solution formula for the ordinary differential equation $\partial_t w = \alpha w|w|^2$ is available. However, if the two-parts splitting is applied to (1.3), then we have to solve the ordinary differential equation $\partial_t w = \alpha w|w|^2 + \beta w|w|^4$ by a numerical method, since the exact solution is not available in this case. Therefore, some researchers have proposed schemes that are split into more than two parts. However, the convergence properties of such schemes are not guaranteed in the case of PDEs.

In this paper, we propose a Strang-type splitting method for (1.1) that is split into three parts. Moreover, we show that it is actually convergent at a second order rate.

Let us formulate our problem. For given nonlinear operators $F_1, F_2\colon D(A) \to D(A)$, we set

$$F(v) = F_1(v) + F_2(v), \quad v \in D(A).$$

For $u_0 \in D(A)$, we consider the Cauchy problem (1.1) and the corresponding integral equation

$$u(t) = \Phi_A(t)u_0 + \int_0^t \Phi_A(t-s)F(u(s))\,\mathrm{d}s, \quad t \in [0,T]. \tag{1.7}$$

For $i = 1, 2$, we assume that $F_i\colon D(A) \to D(A)$ satisfies the following conditions.

(F0) $F_i(0) = 0$.

(F1) $\|F_i'(v)w\|_{D(A)} \leqslant L(\|v\|_{D(A)})\|w\|_{D(A)}$ for $v, w \in D(A)$.

(F2) $F_i(v) \in D(A^2)$ and $\|F_i(v)\|_{D(A^2)} \leqslant L_2(\|v\|_{D(A)})\|v\|_{D(A^2)}$ for $v \in D(A^2)$.

(F3) $\|F_i(v) - F_i(w)\|_{D(A^2)} \leqslant L_3(\max\{\|v\|_{D(A^2)}, \|w\|_{D(A^2)}\})\|v - w\|_{D(A^2)}$ for $v, w \in D(A^2)$.

(F4) $\|F_i'(v)w\|_X \leqslant L_4(\|v\|_{D(A)})\|w\|_X$ for $v, w \in D(A)$.

(F5) $\|F_i''(v)(w,w)\|_X \leqslant L_5(\|v\|_{D(A)})\|w\|_X\|w\|_{D(A)}$ for $v, w \in D(A)$.

Herein, F_i' and F_i'' denote the first and second Fréchet derivatives, $L, L_2, \ldots, L_5\colon [0,\infty) \to [0,\infty)$ are nondecreasing functions.

We note that it follows from (F1) and (F0) that

(F6) $\|F_i(v)-F_i(w)\|_{D(A)} \leqslant L(\max\{\|v\|_{D(A)}, \|w\|_{D(A)}\})\|v-w\|_{D(A)}$ for $v, w \in D(A)$;

(F7) $\|F_i(v)\|_{D(A)} \leqslant L(\|v\|_{D(A)})\|v\|_{D(A)}$ for $v \in D(A)$.

Moreover, it follows from (F4) that

(F8) $\|F_i(v) - F_i(w)\|_X \leqslant L_4(\max\{\|v\|_{D(A)}, \|w\|_{D(A)}\})\|v - w\|_X$ for $v, w \in D(A)$.

For simplicity, we write $F''(v)(w, w) = F''(v)w^2$ for $v, w \in D(A)$. Before stating the schemes and main results, we recall a general result for (1.7).

Proposition 1.1. *Assume* (F0)–(F1). *Then, for any* $u_0 \in D(A)$, *there exist* $T_{\max}(u_0) \in (0, \infty]$ *and a unique solution*

$$u \in C([0, T_{\max}(u_0)), D(A)) \cap C^1([0, T_{\max}(u_0)), X)$$

of (1.7) *such that either* (i) *or* (ii) *holds, where*

$$\text{(i) } T_{\max}(u_0) = \infty, \quad \text{(ii) } T_{\max}(u_0) < \infty \quad \text{and} \quad \lim_{t \uparrow T_{\max}(u_0)} \|u(t)\|_{D(A)} = \infty.$$

Moreover, if $u_0 \in D(A^2)$, *then*

$$u \in C([0, T_{\max}(u_0)), D(A^2)) \cap C^1([0, T_{\max}(u_0)), D(A)).$$

For the proof of Proposition 1.1, see e.g., Section 4.3 of [3].

In order to state our scheme, we consider an auxiliary Cauchy problem

$$\begin{cases} \partial_t w_i = F_i(w_i), & t \in [0, T], \\ w_i(0) = w_{i,0}, \end{cases} \qquad i = 1, 2, \tag{1.8}$$

and the corresponding integral equation

$$w_i(t) = w_{i,0} + \int_0^t F_i(w_i(s))\, \mathrm{d}s, \quad t \in [0, T],\ i = 1, 2. \tag{1.9}$$

We denote the solution of (1.9) by $w_i(t) = \Phi_{F_i}(t) w_{i,0}$.

Then, our scheme to find $\Psi(t)u_0 \approx S(t)u_0$ reads

$$\Psi(t)u_0 = \Phi_A(t/2)\Phi_{F_1}(t/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)u_0. \tag{1.10}$$

We are now in a position to state the main results.

Theorem 1.2. *Assume* (F0)–(F5). *Let* $u_0 \in D(A^2)$, $T \in (0, T_{\max}(u_0))$, *and set*

$$m_0 = 8 \max_{t \in [0,T]} \|S(t)u_0\|_{D(A)}.$$

Then there exists a positive constant h_0 which depends only on T, m_0, and $\|u_0\|_{D(A^2)}$ such that

$$\|\Psi(h)^n u_0\|_{D(A)} \leqslant m_0, \quad \|\Psi(h)^n u_0\|_{D(A^2)} \leqslant \mathrm{e}^{\gamma_1 nh} \|u_0\|_{D(A^2)}, \tag{1.11}$$

$$\|S(nh)u_0 - \Psi(h)^n u_0\|_{D(A)} \leqslant \kappa_1 h \|u_0\|_{D(A^2)}, \tag{1.12}$$

$$\|S(nh)u_0 - \Psi(h)^n u_0\|_X \leqslant \kappa_2 h^2 \|u_0\|_{D(A^2)} \tag{1.13}$$

for all $h \in (0, h_0]$ and $n \in \mathbb{N}$ satisfying $nh \leqslant T$, where γ_1 is a positive constant depending only on m_0, and κ_1, κ_2 are positive constants depending only on T and m_0.

The rest of this paper is organized as follows. In Section 2, we collect some lemmas that are needed to prove Theorem 1.2. In Section 3, we give local estimates for the error between $S(h)u_0$ and $\Psi(h)u_0$ in $D(A)$. In Section 4, we give local estimates for the error between $S(h)u_0$ and $\Psi(h)u_0$ in X. In Section 5, we complete the proof of Theorem 1.2. Finally, in Section 6, we present a numerical experiment that illustrate the convergence rate of the scheme numerically.

2. Preliminaries

2.1. Estimates on the contraction semigroup $\Phi_A(t)$.

Lemma 2.1. *Let $k = 0, 1$. Then,*

$$\|\Phi_A(t)v_0 - \Phi_A(s)v_0\|_{D(A^k)} \leqslant (t - s)\|v_0\|_{D(A^{k+1})}$$

for $v_0 \in D(A^{k+1})$ and $0 \leqslant s \leqslant t$.

Proof. Set $v(t) = \Phi_A(t)v_0$. Then we have

$$\Phi_A(t)v_0 - \Phi_A(s)v_0 = v(t) - v(s) = \int_s^t v'(\tau)\,\mathrm{d}\tau = \int_s^t Av(\tau)\,\mathrm{d}\tau.$$

Since

$$\|Av(\tau)\|_{D(A^k)} = \|\Phi_A(\tau)Av_0\|_{D(A^k)} \leqslant \|Av_0\|_{D(A^k)}$$

for $\tau > 0$, we have

$$\begin{aligned}\|\Phi_A(t)v_0 - \Phi_A(s)v_0\|_{D(A^k)} &\leqslant \int_s^t \|Av(\tau)\|_{D(A^k)}\,\mathrm{d}\tau \\ &\leqslant (t - s)\|Av_0\|_{D(A^k)} \leqslant (t - s)\|v_0\|_{D(A^{k+1})}.\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let* $w \in C^1([0,T],D(A)) \cap C([0,T],D(A^2))$. *Then we have*

$$\left\| \int_0^t [\Phi_A(t-s)w(s) - \Phi_A(t/2)w(s)]\,\mathrm{d}s \right\|_X \leqslant t^3(\|w\|_{C^1([0,T],D(A))} + \|w\|_{C([0,T],D(A^2))}) \quad \textit{for } t \in [0,T]. \tag{2.1}$$

Proof. For $0 \leqslant s \leqslant t \leqslant T$, by Taylor's formula, we obtain

$$\Phi_A(t-s)w(s) - \Phi_A(t/2)w(s) = (t/2-s)\Phi_A(t/2)Aw(s) + (t/2-s)^2 \int_0^1 (1-\theta)\Phi_A(\theta(t-s)+(1-\theta)t/2)A^2w(s)\,\mathrm{d}\theta.$$

Let $v(s) = \Phi_A(t/2)Aw(s)$. Then we have

$$\|v'(s)\|_X \leqslant \|Aw'(s)\|_X \leqslant \|w'(s)\|_{D(A)},$$
$$\int_0^t (t/2-s)v(s)\,\mathrm{d}s = \int_0^{t/2} (t/2-s)[v(s)-v(t-s)]\,\mathrm{d}s.$$

Moreover, for $0 \leqslant s \leqslant t/2$, since

$$\begin{aligned}\|v(s)-v(t-s)\|_X &= \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} v(\theta s + (1-\theta)(t-s))\,\mathrm{d}\theta \right\|_X \\ &\leqslant (t-2s)\int_0^1 \|v'(\theta s+(1-\theta)(t-s))\|_X\,\mathrm{d}\theta \\ &\leqslant 2(t/2-s)\|v'\|_{C([0,T],X)},\end{aligned}$$

we get

$$\begin{aligned}\left\| \int_0^t (t/2-s)\Phi_A(t/2)Aw(s)\,\mathrm{d}s \right\|_X &= \left\| \int_0^{t/2} (t/2-s)[v(s)-v(t-s)]\,\mathrm{d}s \right\|_X \\ &\leqslant 2\int_0^{t/2} (t/2-s)^2\,\mathrm{d}s \|v'\|_{C([0,T],X)} \leqslant t^3\|w\|_{C^1([0,T],D(A))}.\end{aligned} \tag{2.2}$$

Furthermore, since

$$\left\| \int_0^1 (1-\theta)\Phi_A(\theta(t-s)+(1-\theta)t/2)A^2w(s)\,\mathrm{d}\theta \right\|_X \leqslant \int_0^1 (1-\theta)\|A^2w(s)\|_X\,\mathrm{d}\theta \leqslant \|w\|_{C([0,T],D(A^2))},$$

we have

$$\left\|\int_0^t (t/2-s)^2\int_0^1(1-\theta)\Phi_A(\theta(t-s)+(1-\theta)t/2)A^2w(s)\,\mathrm{d}\theta\,\mathrm{d}s\right\|_X \tag{2.3}$$
$$\leqslant \|w\|_{C([0,T],D(A^2))}\int_0^t (t/2-s)^2\,\mathrm{d}s \leqslant t^3\|w\|_{C([0,T],D(A^2))}.$$

Thus, by (2.2) and (2.3), we obtain (2.1). □

2.2. Estimates on the nonlinear flows Φ_{F_i}.

Lemma 2.3. *Assume* (F0)–(F1). *For any $M>0$, there exists a positive constant $\tau(M)$ such that if $\|v_0\|_{D(A)}=M$, then*

$$\|\Phi_{F_i}(t)v_0\|_{D(A)}\leqslant 2M,\quad \|S(t)v_0\|_{D(A)}\leqslant 2M \quad \textit{for}\quad t\in[0,\tau(M)],\quad i=1,2.$$

Moreover, if $v_1,v_2\in D(A)$ satisfy $\max\{\|v_1\|_{D(A)},\|v_2\|_{D(A)}\}\leqslant M$, *then*

$$\left.\begin{aligned}\|\Phi_{F_i}(t)v_1-\Phi_{F_i}(t)v_2\|_{D(A)}&\leqslant \mathrm{e}^{L(2M)t}\|v_1-v_2\|_{D(A)},\quad i=1,2,\\ \|S(t)v_1-S(t)v_2\|_{D(A)}&\leqslant \mathrm{e}^{2L(2M)t}\|v_1-v_2\|_{D(A)}\end{aligned}\right\}\ \textit{for } t\in[0,\tau(M)].$$

Proof. See Proposition 4.3.3 of [3]. □

Remark 2.4. We can assume that $\tau\colon (0,\infty)\to(0,\infty)$ is a nonincreasing function.

Lemma 2.5. *Assume* (F0)–(F3). *Let $v_0\in D(A^2)$ and set $M=\|v_0\|_{D(A)}$. Then*

$$\|\Phi_{F_i}(t)v_0\|_{D(A^2)}\leqslant \mathrm{e}^{L_2(2M)t}\|v_0\|_{D(A^2)}\quad \textit{for } t\in[0,\tau(M)],\ i=1,2, \tag{2.4}$$

where $\tau(M)$ is defined above in Lemma 2.3. *Moreover, we have*

$$\|\Psi(t)v_0\|_{D(A^2)}\leqslant \mathrm{e}^{2L_2(8M)t}\|v_0\|_{D(A^2)}\quad \textit{for } t\in[0,\tau(4M)]. \tag{2.5}$$

Proof. First, we note that it follows from (F0)–(F3) that (1.7) is locally well-posed in $D(A^2)$. For $i=1,2$ we set $v_i(t)=\Phi_{F_i}(t)v_0$.

By (1.9) and (F2), we have

$$\begin{aligned}\|v_i(t)\|_{D(A^2)}&\leqslant \|v_0\|_{D(A^2)}+\int_0^t\|F_i(v_i(\tau))\|_{D(A^2)}\,\mathrm{d}\tau\\ &\leqslant \|v_0\|_{D(A^2)}+\int_0^t L_2(\|v_i(\tau)\|_{D(A)})\|v_i(\tau)\|_{D(A^2)}\,\mathrm{d}\tau,\quad i=1,2.\end{aligned}$$

It follows from Lemma 2.3 that

$$\|v_i(t)\|_{D(A^2)} \leqslant \|v_0\|_{D(A^2)} + L_2(2M)\int_0^t \|v_i(\tau)\|_{D(A^2)}\,\mathrm{d}\tau, \quad i=1,2,$$

for $t \in [0,\tau(M)]$. Thus, Gronwall's lemma implies (2.4) for $t\in[0,\tau(M)]$.

Next, since $\|\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A)} \leqslant 2M$ for $t\in[0,\tau(M)]$ and

$$\|\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A)} \leqslant 4M \quad \text{for } t\in[0,\tau(2M)], \tag{2.6}$$

it follows from (2.4) that

$$\begin{aligned}\|\Psi(t)v_0\|_{D(A^2)} &\leqslant \|\Phi_{F1}(t/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A^2)}\\ &\leqslant \mathrm{e}^{L_2(8M)t/2}\|\Phi_{F_2}(t)\Phi_{F1}(t/2)\Phi_A(t/2)v_0\|_{D(A^2)}\end{aligned}$$

for $t\in[0,\tau(4M)]$. Similarly, we have

$$\|\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)v_0\|_{D(A^2)} \leqslant \mathrm{e}^{L_2(4M)t+L_2(2M)t/2}\|v_0\|_{D(A^2)}$$

for $t\in[0,\tau(2M)]$. Therefore, we obtain

$$\|\Psi(t)v_0\|_{D(A^2)} \leqslant \mathrm{e}^{L_2(8M)t/2+L_2(4M)t+L_2(2M)t/2}\|v_0\|_{D(A^2)} \leqslant \mathrm{e}^{2L_2(8M)t}\|v_0\|_{D(A^2)}$$

for $t\in[0,\tau(4M)]$. This completes the proof. □

2.3. Lipschitz property of $S(t)$.

Lemma 2.6. *Assume* (F0)–(F4). *Let* $u_0\in D(A)$, $T\in(0,T_{\max}(u_0))$ *and set*

$$m_1 = 2\max_{t\in[0,T]}\|S(t)u_0\|_{D(A)}, \quad \delta_0=\min\Big\{\frac{m_1}{2},\ m_1\mathrm{e}^{-2L(2m_1)T}\Big\}.$$

If $\|v_0-S(t_0)u_0\|_{D(A)}\leqslant\delta_0$, *then*

$$\|S(t)v_0\|_{D(A)}\leqslant 2m_1 \quad \text{for } t\in[0,T-t_0]. \tag{2.7}$$

Moreover, if $\|v_1-S(t_0)u_0\|_{D(A)}\leqslant\delta_0$ *and* $\|v_2-S(t_0)u_0\|_{D(A)}\leqslant\delta_0$, *then*

$$\left.\begin{aligned}\|S(t)v_1-S(t)v_2\|_{D(A)} &\leqslant \mathrm{e}^{2L(2m_1)t}\|v_1-v_2\|_{D(A)}\\ \|S(t)v_1-S(t)v_2\|_X &\leqslant \mathrm{e}^{2L_4(2m_1)t}\|v_1-v_2\|_X\end{aligned}\right\} \quad \text{for } t\in[0,T-t_0]. \tag{2.8}$$

P r o o f. First, we show (2.7). Since

$$\|v_0\|_{D(A)} \leqslant \|v_0 - S(t_0)u_0\|_{D(A)} + \|S(t_0)u_0\|_{D(A)} \leqslant \delta_0 + \frac{m_1}{2} \leqslant m_1,$$

it follows from Lemma 2.3 that $\|S(t)v_0\|_{D(A)} \leqslant 2m_1$ for $t \in [0, \tau(m_1)]$. Here, we define

$$\widetilde{T} = \sup\{\tau \in (0, T_{\max}(v_0));\ \|S(t)v_0\|_{D(A)} \leqslant 2m_1 \quad \text{for } t \in [0,\tau]\}$$

and suppose $\widetilde{T} < T - t_0$. Then we have

$$S(t)v_0 = \Phi_A(t)v_0 + \int_0^t \Phi_A(t-\tau)F(S(\tau)v_0)\,\mathrm{d}\tau \quad \text{for } t \in [0, \widetilde{T}].$$

Since $0 \leqslant \tau \leqslant \widetilde{T}$ and $\tau + t_0 \leqslant T$ for $\tau \in [0, \widetilde{T}]$, we have

$$\|S(\tau)v_0\|_{D(A)} \leqslant 2m_1, \quad \|S(\tau)(S(t_0)u_0)\|_{D(A)} = \|S(\tau + t_0)u_0\|_{D(A)} \leqslant m_1.$$

Thus, by (F6), we have

$$\begin{aligned}
&\|S(t)v_0 - S(t)(S(t_0)u_0)\|_{D(A)} \\
&\leqslant \|v_0 - S(t_0)u_0\|_{D(A)} + \int_0^t \|F(S(\tau)v_0) - F(S(\tau)S(t_0)u_0)\|_{D(A)}\,\mathrm{d}\tau \\
&\leqslant \delta_0 + 2L(2m_1)\int_0^t \|S(\tau)v_0 - S(\tau)S(t_0)u_0\|_{D(A)}\,\mathrm{d}\tau
\end{aligned}$$

for $t \in [0, \widetilde{T}]$. By Gronwall's lemma, we have

$$\|S(t)v_0 - S(t)S(t_0)u_0\|_{D(A)} \leqslant \delta_0 \mathrm{e}^{2L(2m_1)t} \leqslant \delta_0 \mathrm{e}^{2L(2m_1)T} \leqslant m_1$$

and

$$\begin{aligned}
\|S(t)v_0\|_{D(A)} &\leqslant \|S(t)v_0 - S(t)S(t_0)u_0\|_{D(A)} + \|S(t)S(t_0)u_0\|_{D(A)} \\
&\leqslant m_1 + \frac{1}{2}m_1 < 2m_1 \quad \text{for } t \in [0, \widetilde{T}].
\end{aligned}$$

This contradicts the definition of $\widetilde{T}$. Thus, we conclude $T - t_0 \leqslant \widetilde{T}$, which implies (2.7).

Next, we show (2.8). By (2.7), we have

$$\|S(t)v_1\|_{D(A)} \leqslant 2m_1, \quad \|S(t)v_2\|_{D(A)} \leqslant 2m_1 \quad \text{for } t \in [0, T - t_0]. \tag{2.9}$$

Thus, by (F6), we have

$$\begin{aligned}
&\|S(t)v_1 - S(t)v_2\|_{D(A)} \\
&\qquad \leqslant \|v_1 - v_2\|_{D(A)} + \int_0^t \|F(S(\tau)v_1) - F(S(\tau)v_2)\|_{D(A)}\,\mathrm{d}\tau \\
&\qquad \leqslant \|v_1 - v_2\|_{D(A)} + 2L(2m_1)\int_0^t \|S(\tau)v_1 - S(\tau)v_2\|_{D(A)}\,\mathrm{d}\tau
\end{aligned}$$

for $t \in [0, T - t_0]$. By Gronwall's lemma, we have

$$\|S(t)v_1 - S(t)v_2\|_{D(A)} \leqslant \mathrm{e}^{2L(2m_1)t}\|v_1 - v_2\|_{D(A)} \quad \text{for } t \in [0, T - t_0].$$

Moreover, by (2.9) and (F8), we have

$$\begin{aligned}
&\|S(t)v_1 - S(t)v_2\|_X \\
&\qquad \leqslant \|v_1 - v_2\|_X + \int_0^t \|F(S(\tau)v_1) - F(S(\tau)v_2)\|_X\,\mathrm{d}\tau \\
&\qquad \leqslant \|v_1 - v_2\|_X + 2L_4(2m_1)\int_0^t \|S(\tau)v_1 - S(\tau)v_2\|_X\,\mathrm{d}\tau \quad \text{for } t \in [0, T - t_0].
\end{aligned}$$

Hence, we obtain

$$\|S(t)v_1 - S(t)v_2\|_X \leqslant \mathrm{e}^{2L_4(2m_1)t}\|v_1 - v_2\|_X \quad \text{for } t \in [0, T - t_0].$$

This completes the proof. □

3. Local error estimates in $D(A)$

In this section, we estimate local errors in $D(A)$ between the solution $u(t)$ of (1.7) and $\Psi(t)u_0$ which is defined by (1.10).

Proposition 3.1. *Assume* (F0)–(F3). *Let $u_0 \in D(A^2)$ and set $M = \|u_0\|_{D(A)}$. Then there exists a positive constant $K_1(M)$ depending only on M such that*

$$\|S(t)u_0 - \Psi(t)u_0\|_{D(A)} \leqslant K_1(M)\|u_0\|_{D(A^2)}t^2 \quad \textit{for } t \in [0, \tau(4M)].$$

In what follows, we put

$$u(t) = S(t)u_0, \quad v(t) = \Psi(t)u_0. \tag{3.1}$$

We define $w_1(s,t), w_2(s,t)$ and $w_3(s,t)$ by

$$w_1(s,t) = \Phi_{F_1}(s/2)\Phi_A(t/2)u_0, \quad w_2(s,t) = \Phi_{F_2}(s)\Phi_{F_1}(t/2)\Phi_A(t/2)u_0,$$
$$w_3(s,t) = \Phi_{F_1}(s/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)u_0.$$

Then we have

$$\begin{aligned} w_1(s,t) &= \Phi_A(t/2)u_0 + \frac{1}{2}\int_0^s F_1(w_1(\tau,t))\,\mathrm{d}\tau, \\ w_2(s,t) &= w_1(t,t) + \int_0^s F_2(w_2(\tau,t))\,\mathrm{d}\tau, \\ w_3(s,t) &= w_2(t,t) + \frac{1}{2}\int_0^s F_1(w_3(\tau,t))\,\mathrm{d}s. \end{aligned} \tag{3.2}$$

Therefore, $v(t)$ can be written as

$$\begin{aligned} v(t) &= \Phi_A(t/2)\Phi_{F_1}(t/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)u_0 \\ &= \Phi_A(t/2)w_3(t,t) = \Phi_A(t)u_0 + \boldsymbol{G}_1(t) + \boldsymbol{G}_2(t) + \boldsymbol{G}_3(t), \end{aligned}$$

where

$$\boldsymbol{G}_1(t) = \frac{1}{2}\int_0^t \Phi_A(t/2)F_1(w_1(s,t))\,\mathrm{d}s, \quad \boldsymbol{G}_2(t) = \int_0^t \Phi_A(t/2)F_2(w_2(s,t))\,\mathrm{d}s,$$
$$\boldsymbol{G}_3(t) = \frac{1}{2}\int_0^t \Phi_A(t/2)F_1(w_3(s,t))\,\mathrm{d}s.$$

By using the expression (1.7), we have

$$u(t) - v(t) = \int_0^t \Phi_A(t-s)[F(u(s)) - F(v(s))]\,\mathrm{d}s + R(t), \tag{3.3}$$

where

$$R(t) = \int_0^t \Phi_A(t-s)F(v(s))\,\mathrm{d}s - [\boldsymbol{G}_1(t) + \boldsymbol{G}_2(t) + \boldsymbol{G}_3(t)].$$

We divide $R(t)$ as $R(t) = R_1(t) + R_2(t)$, where

$$\begin{aligned} R_1(t) &= \int_0^t \Phi_A(t-s)F_1(v(s))\,\mathrm{d}s - (\boldsymbol{G}_1(t) + \boldsymbol{G}_3(t)), \\ R_2(t) &= \int_0^t \Phi_A(t-s)F_2(v(s))\,\mathrm{d}s - \boldsymbol{G}_2(t). \end{aligned}$$

Moreover, we split $R_1(t)$ and $R_2(t)$ as $R_1(t) = R_{1a}(t) + R_{1b}(t)$ and $R_2(t) = R_{2a}(t) + R_{2b}(t)$, respectively. Here,

$$R_{1a}(t) = \int_0^t \Phi_A(t-s)\Big[F_1(v(s)) - \frac{1}{2}F_1(w_1(s,t)) - \frac{1}{2}F_1(w_3(s,t))\Big]\,\mathrm{d}s, \tag{3.4}$$

$$R_{1b}(t) = \frac{1}{2}\int_0^t (\Phi_A(t-s) - \Phi_A(t/2))[F_1(w_1(s,t)) + F_1(w_3(s,t))]\,\mathrm{d}s, \tag{3.5}$$

$$R_{2a}(t) = \int_0^t \Phi_A(t-s)[F_2(v(s)) - F_2(w_2(s,t))]\,\mathrm{d}s, \tag{3.6}$$

$$R_{2b}(t) = \int_0^t (\Phi_A(t-s) - \Phi_A(t/2))F_2(w_2(s,t))\,\mathrm{d}s. \tag{3.7}$$

First, we prove the following lemma.

Lemma 3.2. *Assume* (F0)–(F3). *Let* $u_0 \in D(A^2)$ *and set* $M = \|u_0\|_{D(A)}$. *Then there exists a positive constant* C_{12} *depending only on* M *such that*

$$\|R_2(t)\|_{D(A)} \leqslant C_{12}\|u_0\|_{D(A^2)}t^2 \quad \text{for } t \in [0, \tau(4M)]. \tag{3.8}$$

Proof. First, we show that there exists a positive constant C_{12a} depending only on M such that

$$\|R_{2a}(t)\|_{D(A)} \leqslant C_{12a}\|u_0\|_{D(A^2)}t^2 \quad \text{for } t \in [0, \tau(4M)]. \tag{3.9}$$

By Lemma 2.3, we have

$$\|w_2(s,t)\|_{D(A)} \leqslant 4M, \quad \|v(s)\|_{D(A)} \leqslant 8M \quad \text{for } s,t \in [0, \tau(4M)]. \tag{3.10}$$

Thus, by (F6), we have

$$\|R_{2a}(t)\|_{D(A)} \leqslant L(8M)\int_0^t \|v(s) - w_2(s,t)\|_{D(A)}\,\mathrm{d}s \quad \text{for } t \in [0, \tau(4M)]. \tag{3.11}$$

Since

$$\begin{aligned} v(s) &= \Phi_A(s/2)\Phi_{F_1}(s/2)w_2(s,s) \\ &= \Phi_A(s/2)\Big\{w_2(s,s) + \int_0^{s/2} F_1(\Phi_{F_1}(\tau)w_2(s,s))\,\mathrm{d}\tau\Big\}, \end{aligned}$$

we have

$$\begin{aligned} \|v(s) - w_2(s,s)\|_{D(A)} \leqslant{}& \|\Phi_A(s/2)w_2(s,s) - w_2(s,s)\|_{D(A)} \\ &+ \int_0^{s/2} \|F_1(\Phi_{F_1}(\tau)w_2(s,s))\|_{D(A)}\,\mathrm{d}\tau. \end{aligned}$$

By Lemmas 2.1 and 2.5, we have

$$
\begin{aligned}
(3.12)\qquad \|\Phi_A(s/2)w_2(s,s) - w_2(s,s)\|_{D(A)} &\leqslant \frac{s}{2}\|w_2(s,s)\|_{D(A^2)} \\
&\leqslant \frac{s}{2}\mathrm{e}^{L_2(4M)s+L_2(2M)s/2}\|u_0\|_{D(A^2)} \leqslant \frac{s}{2}\mathrm{e}^{2L_2(4M)\tau(2M)}\|u_0\|_{D(A^2)}
\end{aligned}
$$

for $s \in [0, \tau(2M)]$. Moreover, by (F7) and Lemma 2.3, we have

$$
\begin{aligned}
\|F_1(\Phi_{F_1}(\tau)w_2(s,s))\|_{D(A)} &\leqslant L(\|\Phi_{F_1}(\tau)w_2(s,s)\|_{D(A)})\|\Phi_{F_1}(\tau)w_2(s,s)\|_{D(A)} \\
&\leqslant 8L(8M)M \leqslant 8L(8M)\|u_0\|_{D(A^2)}
\end{aligned}
$$

for $\tau, s \in [0, \tau(4M)]$. Thus, we have

$$
\|v(s) - w_2(s,s)\|_{D(A)} \leqslant \frac{s}{2}(\mathrm{e}^{2L_2(4M)\tau(2M)} + 8L(8M))\|u_0\|_{D(A^2)} \quad \text{for } s \in [0, \tau(4M)],
$$

which implies (3.9).

Next, we show that there exists a positive constant C_{12b} depending only on M such that

$$
(3.13)\qquad \|R_{2b}(t)\|_{D(A)} \leqslant C_{12b}\|u_0\|_{D(A^2)}t^2 \quad \text{for } t \in [0, \tau(2M)].
$$

By (F2) and Lemmas 2.1 and 2.5, we have

$$
\begin{aligned}
\|R_{2b}(t)\|_{D(A)} &\leqslant \int_0^t \left|\frac{t}{2} - s\right| \|F_2(w_2(s,t))\|_{D(A^2)}\,\mathrm{d}s \\
&\leqslant \int_0^t \left|\frac{t}{2} - s\right| L_2(4M)\mathrm{e}^{L_2(4M)s+L_2(2M)t/2}\|u_0\|_{D(A^2)}\,\mathrm{d}s \\
&\leqslant \int_0^t \left|\frac{t}{2} - s\right| \mathrm{d}s L_2(4M)\mathrm{e}^{2L_2(4M)\tau(2M)}\|u_0\|_{D(A^2)} \quad \text{for } t \in [0, \tau(2M)],
\end{aligned}
$$

which implies (3.13).

Finally, (3.8) follows from (3.9) and (3.13). □

Lemma 3.3. *Assume* (F0)–(F3). *Let $u_0 \in D(A^2)$ and set $M = \|u_0\|_{D(A)}$. Then there exists a positive constant C_{11} depending only on M such that*

$$
(3.14)\qquad \|R_1(t)\|_{D(A)} \leqslant C_{11}\|u_0\|_{D(A^2)}t^2 \quad \textit{for } t \in [0, \tau(4M)].
$$

Lemma 3.3 can be proved in the same way as Lemma 3.2, so we omit the details. By Lemmas 3.2 and 3.3, we obtain the following lemma.

Lemma 3.4. *Assume* (F0)–(F3). *Let* $u_0 \in D(A^2)$ *and set* $M = \|u_0\|_{D(A)}$. *Then there exists a positive constant* C_1 *depending only on* M *such that*

$$\|R(t)\|_{D(A)} \leqslant C_1 \|u_0\|_{D(A^2)} t^2 \quad \text{for } t \in [0, \tau(4M)]. \tag{3.15}$$

Now, we give the proof of Proposition 3.1.

P r o o f of Proposition 3.1. It follows from (3.3), (F6), and Lemma 3.4 that

$$\begin{aligned}\|u(t) - v(t)\|_{D(A)} &\leqslant \int_0^t \|F(u(s)) - F(v(s))\|_{D(A)} \,\mathrm{d}s + \|R(t)\|_{D(A)} \\ &\leqslant \int_0^t 2L(\max\{\|u(s)\|_{D(A)}, \|v(s)\|_{D(A)}\})\|u(s) - v(s)\|_{D(A)} \,\mathrm{d}s \\ &\quad + C_1 \|u_0\|_{D(A^2)} t^2 \quad \text{for } t \in [0, \tau(4M)].\end{aligned}$$

Moreover, by Lemma 2.3 we have $\|u(s)\|_{D(A)} \leqslant 2M$ and $\|v(s)\|_{D(A)} \leqslant 8M$ for $s \in [0, \tau(4M)]$. Thus, we have

$$\|u(t) - v(t)\|_{D(A)} \leqslant 2L(8M) \int_0^t \|u(s) - v(s)\|_{D(A)} \,\mathrm{d}s + C_1 \|u_0\|_{D(A^2)} t^2$$

for $t \in [0, \tau(4M)]$. By Gronwall's lemma, we obtain

$$\|u(t) - v(t)\|_{D(A)} \leqslant \mathrm{e}^{2L(8M)t} C_1 \|u_0\|_{D(A^2)} t^2 \leqslant \mathrm{e}^{2L(8M)\tau(4M)} C_1 \|u_0\|_{D(A^2)} t^2$$

for $t \in [0, \tau(4M)]$. This completes the proof. □

4. Local error estimates in X

In this section, we prove the following local error estimates in X.

Proposition 4.1. *Assume* (F0)–(F5). *Let* $u_0 \in D(A^2)$ *and set* $M = \|u_0\|_{D(A)}$. *Then there exists a positive constant* $K_2(M)$ *depending only on* M *such that*

$$\|S(t)u_0 - \Psi(t)u_0\|_X \leqslant K_2(M) \|u_0\|_{D(A^2)} t^3 \quad \text{for } t \in [0, \tau(4M)].$$

This proposition is a readily obtainable consequence of

$$\|R_{1a}(t)\|_X \leqslant C_{21a}\|u_0\|_{D(A^2)}t^3, \tag{4.1}$$

$$\|R_{1b}(t)\|_X \leqslant C_{21b}\|u_0\|_{D(A^2)}t^3, \tag{4.2}$$

$$\|R_{2a}(t)\|_X \leqslant C_{22a}\|u_0\|_{D(A^2)}t^3, \tag{4.3}$$

$$\|R_{2b}(t)\|_X \leqslant C_{22b}\|u_0\|_{D(A^2)}t^3, \tag{4.4}$$

for $t \in [0, \tau(4M)]$. Here $C_{21a}, C_{21b}, C_{22a}, C_{22b}$ are positive constants depending only on M and $R_{1a}(t), R_{1b}(t), R_{2a}(t), R_{2b}(t)$ are defined by (3.4)–(3.7).

The proofs of these estimates are given below.

4.1. Proofs of (4.4) and (4.2). We only consider the case (4.4). It follows from Lemmas 2.3, and 2.5 that

$$\|w_2(s,t)\|_{D(A)} \leqslant 4M, \quad \|w_2(s,t)\|_{D(A^2)} \leqslant \mathrm{e}^{2L_2(4M)\tau(2M)}\|u_0\|_{D(A^2)} \tag{4.5}$$

for $s, t \in [0, \tau(2M)]$. Moreover, by Lemma 2.2 we have

$$\begin{aligned}&\left\|\int_0^t [\Phi_A(t-s) - \Phi_A(t/2)]F_2(w_2(s,t))\,\mathrm{d}s\right\|_X\\&\leqslant t^3(\|F_2(w_2)\|_{C^1([0,\tau(2M)],D(A))} + \|F_2(w_2)\|_{C([0,\tau(2M)],D(A^2))}) \quad \text{for } t \in [0, \tau(2M)].\end{aligned}$$

It follows from (F7) and (4.5) that

$$\begin{aligned}\|F_2(w_2(s,t))\|_{D(A)} &\leqslant L(\|w_2(s,t)\|_{D(A)})\|w_2(s,t)\|_{D(A)}\\&\leqslant 4L(4M)M \leqslant 4L(4M)\|u_0\|_{D(A^2)}\end{aligned} \tag{4.6}$$

for $s, t \in [0, \tau(2M)]$. Moreover, by (F2) and (4.5),

$$\begin{aligned}\|F_2(w_2(s,t))\|_{D(A^2)} &\leqslant L_2(\|w_2(s,t)\|_{D(A)})\|w_2(s,t)\|_{D(A^2)}\\&\leqslant L_2(4M)\mathrm{e}^{2L_2(4M)\tau(2M)}\|u_0\|_{D(A^2)} \quad \text{for } s, t \in [0, \tau(2M)].\end{aligned}$$

Thus, there exists a positive constant C_{w_2} depending only on M such that

$$\|F_2(w_2)\|_{C([0,\tau(2M)],D(A^2))} \leqslant C_{w_2}\|u_0\|_{D(A^2)}.$$

Next, since $\partial_s(F_2(w_2(s,t))) = F_2'(w_2(s,t))\partial_s(w_2(s,t)) = F_2'(w_2(s,t))F_2(w_2(s,t))$, it follows from (F1), (4.5), and (4.6) that

$$\begin{aligned}\|\partial_s(F_2(w_2(s,t)))\|_{D(A)} &\leqslant L(\|w_2(s,t)\|_{D(A)})\|F_2(w_2(s,t))\|_{D(A)}\\&\leqslant 4L(4M)^2M \leqslant 4L(4M)^2\|u_0\|_{D(A^2)} \quad \text{for } s, t \in [0, \tau(2M)].\end{aligned}$$

Thus, there exists a positive constant C'_{w_2} depending only on M such that

$$\|F_2(w_2)\|_{C^1([0,\tau(2M)],D(A))} \leqslant C'_{w_2}\|u_0\|_{D(A^2)}.$$

This completes the proof of (4.4).

4.2. Proof of (4.3)**.** In order to prove (4.3), we divide $R_{2a}(t)$ into several parts. By Taylor's formula, we can express $R_{2a}(t)$ as

$$R_{2a}(t) = \sum_{j=0}^{6} Q_j(t),$$

where

$$Q_j(t) = \begin{cases} \displaystyle\int_0^t \Phi_A(t-s)F_2'(u_0)[\Phi_A(s)u_0 - \Phi_A(t/2)u_0]\,\mathrm{d}s, & j=0, \\ \displaystyle(-1)^{j+1}\int_0^t \Phi_A(t-s)J_j(s,t)\,\mathrm{d}s, & j=1,2, \\ \displaystyle\int_0^t \Phi_A(t-s)F_2'(u_0)J_j(s,t)\,\mathrm{d}s, & j=3,4,5,6. \end{cases}$$

Here

$$\begin{aligned} J_1(s,t) &= \int_0^1 (1-\theta)F_2''(\theta\Psi(s)u_0 + (1-\theta)u_0)[\Psi(s)u_0 - u_0]^2\,\mathrm{d}\theta, \\ J_2(s,t) &= \int_0^1 (1-\theta)F_2''(\theta w_2(s,t) + (1-\theta)u_0)[w_2(s,t) - u_0]^2\,\mathrm{d}\theta, \\ J_3(s,t) &= \int_0^s \Phi_A(s/2)F_2(w_2(\tau,s))\,\mathrm{d}\tau - \int_0^s F_2(w_2(\tau,t))\,\mathrm{d}\tau, \\ J_4(s,t) &= \frac{1}{2}\int_0^s \Phi_A(s/2)F_1(w_3(\tau,s))\,\mathrm{d}\tau - \frac{1}{2}\int_0^s \Phi_A(s/2)F_1(w_1(\tau,s))\,\mathrm{d}\tau, \\ J_5(s,t) &= \int_0^s \Phi_A(s/2)F_1(w_1(\tau,s))\,\mathrm{d}\tau - \int_0^s F_1(w_1(\tau,t))\,\mathrm{d}\tau, \\ J_6(s,t) &= \int_0^s F_1(w_1(\tau,t))\,\mathrm{d}\tau - \frac{1}{2}\int_0^t F_1(w_1(\tau,t))\,\mathrm{d}\tau. \end{aligned}$$

Estimation for $Q_0(t)$*.*

In the same way as in the proof of (4.4), we can show

$$\|Q_0(t)\|_X \leqslant (L(M) + L_4(M))\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \geqslant 0.$$

Estimations for $Q_1(t)$ *and* $Q_2(t)$*.*

We only consider the case of $Q_1(t)$. We notice the following holds by Lemma 2.3:

$$\|w_1(s,t)\|_{D(A)} \leqslant 2M, \quad \|w_3(s,t)\|_{D(A)} \leqslant 8M \quad \text{for } s,t \in [0,\tau(4M)]. \tag{4.7}$$

Since $\|\Psi(s)u_0\|_{D(A)} \leqslant 8M$ for $s \in [0, \tau(4M)]$, it follows from (F5) that

$$
\begin{aligned}
\text{(4.8)} \qquad \|Q_1(t)\|_X &\leqslant \int_0^t \|J_1(s)\|_X \,\mathrm{d}s \\
&\leqslant \int_0^t \int_0^1 \|F_2''(\theta\Psi(s)u_0 + (1-\theta)u_0)[\Psi(s)u_0 - u_0]^2\|_X \,\mathrm{d}\theta\,\mathrm{d}s \\
&\leqslant \int_0^t L_5(8M) \cdot \|\Psi(s)u_0 - u_0\|_X \cdot \|\Psi(s)u_0 - u_0\|_{D(A)} \,\mathrm{d}s
\end{aligned}
$$

for $t \in [0, \tau(4M)]$. Moreover, by (3.3) and Lemma 2.1, we have

$$
\begin{aligned}
\|\Psi(s)u_0 - u_0\|_{D(A)} &\leqslant \|\Phi_A(s)u_0 - u_0\|_{D(A)} \\
&\qquad + \|\boldsymbol{G}_1(s)\|_{D(A)} + \|\boldsymbol{G}_2(s)\|_{D(A)} + \|\boldsymbol{G}_3(s)\|_{D(A)} \\
&\leqslant s\|u_0\|_{D(A^2)} + \|\boldsymbol{G}_1(s)\|_{D(A)} + \|\boldsymbol{G}_2(s)\|_{D(A)} + \|\boldsymbol{G}_3(s)\|_{D(A)}.
\end{aligned}
$$

for $s \geqslant 0$. By (F7) and (4.7),

$$
\begin{aligned}
\text{(4.9)} \qquad \|\boldsymbol{G}_1(s)\|_{D(A)} &\leqslant \int_0^s \|F_1(w_1(\tau, s))\|_{D(A)} \,\mathrm{d}\tau \\
&\leqslant 2L(2M)Ms \leqslant 2L(2M)\|u_0\|_{D(A^2)}s
\end{aligned}
$$

for $s \in [0, \tau(M)]$. Similarly, for $s \in [0, \tau(4M)]$, we have

$$
\|\boldsymbol{G}_2(s)\|_{D(A)} \leqslant 4L(4M)\|u_0\|_{D(A^2)}s, \quad \|\boldsymbol{G}_3(s)\|_{D(A)} \leqslant 8L(8M)\|u_0\|_{D(A^2)}s.
$$

Thus, there exists a positive constant C'_{J_1} depending only on M such that

$$
\|\Psi(s)u_0 - u_0\|_{D(A)} \leqslant C'_{J_1}\|u_0\|_{D(A^2)}s \quad \text{for } s \in [0, \tau(4M)].
$$

Similarly, there exists a positive constant C''_{J_1} depending only on M such that

$$
\|\Psi(s)u_0 - u_0\|_X \leqslant C''_{J_1}s \quad \text{for } s \in [0, \tau(4M)].
$$

Therefore, we have

$$
\begin{aligned}
\text{(4.10)} \qquad \|Q_1(t)\|_X &\leqslant L_5(8M)\int_0^t C'_{J_1}C''_{J_1}\|u_0\|_{D(A^2)}s^2 \,\mathrm{d}s \\
&\leqslant L_5(8M)C'_{J_1}C''_{J_1}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \in [0, \tau(4M)].
\end{aligned}
$$

Similarly, we can prove that there exists a positive constant C_{Q_2} depending only on M such that

$$
\text{(4.11)} \qquad \|Q_2(t)\|_X \leqslant C_{Q_2}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \in [0, \tau(4M)].
$$

Estimations for $Q_3(t), Q_4(t),$ and $Q_5(t)$.

We only consider the case $Q_3(t)$. By (F4), we have

$$\|Q_3(t)\|_X \leqslant \int_0^t L_4(\|u_0\|_{D(A)})\|J_3(s,t)\|_X \,\mathrm{d}s.$$

We see that

$$\begin{aligned} J_3(s,t) = &\int_0^s \big[\Phi_A(s/2)F_2(w_2(\tau,s)) - F_2(w_2(\tau,s))\big]\,\mathrm{d}\tau \\ &+ \int_0^s \big[F_2(w_2(\tau,s)) - F_2(w_2(\tau,t))\big]\,\mathrm{d}\tau. \end{aligned}$$

Moreover, by Lemma 2.1 and (4.6), we have

$$\begin{aligned} &\left\|\int_0^s [\Phi_A(s/2)F_2(w_2(\tau,s)) - F_2(w_2(\tau,s))]\,\mathrm{d}\tau\right\|_X \leqslant \int_0^s \frac{s}{2}\|F_2(w_2(\tau,s))\|_{D(A)}\,\mathrm{d}\tau \\ &\quad \leqslant 2L(4M)Ms^2 \leqslant 2L(4M)\|u_0\|_{D(A^2)}s^2 \quad \text{for } s \in [0,\tau(2M)]. \end{aligned}$$

Furthermore, by (F8) and (4.5), we have

$$\begin{aligned} &\left\|\int_0^s [F_2(w_2(\tau,s)) - F_2(w_2(\tau,t))]\,\mathrm{d}\tau\right\|_X \leqslant \int_0^s L_4(4M)\|w_2(\tau,s) - w_2(\tau,t)\|_X\,\mathrm{d}\tau \\ &\quad \leqslant \int_0^s L_4(4M)\|w_2(\tau,s) - w_2(\tau,t)\|_{D(A)}\,\mathrm{d}\tau \quad \text{for } s,t \in [0,\tau(2M)]. \end{aligned}$$

By (3.2), we see that

$$\begin{aligned} w_2(\tau,s) - w_2(\tau,t) = {}&\Phi_A(s/2)u_0 - \Phi_A(t/2)u_0 + \frac{1}{2}\int_0^s F_1(w_1(\tau,s))\,\mathrm{d}\tau \\ &+ \int_0^\tau F_2(w_2(\widetilde{\tau},s))\,\mathrm{d}\widetilde{\tau} - \frac{1}{2}\int_0^t F_1(w_1(\tau,t))\,\mathrm{d}\tau - \int_0^\tau F_2(w_2(\widetilde{\tau},t))\,\mathrm{d}\widetilde{\tau}. \end{aligned}$$

By Lemma 2.1,

$$\|\Phi_A(s/2)u_0 - \Phi_A(t/2)u_0\|_{D(A)} \leqslant \frac{1}{2}|s-t|\|u_0\|_{D(A^2)}.$$

By (F7), (4.5) and (4.7),

$$\|F_j(w_j(\tau,s))\|_{D(A)} \leqslant 4L(4M)M \leqslant 4L(4M)\|u_0\|_{D(A^2)}, \quad j = 1,2,$$

for $\tau, s \in [0,\tau(2M)]$. Thus, there exists a positive constant C_{J_3} depending only on M such that

$$\|J_3(s,t)\|_{D(A)} \leqslant C_{J_3}\|u_0\|_{D(A^2)}(s^2 + st) \quad \text{for } s,t \in [0,\tau(2M)].$$

Therefore, we have

$$\begin{aligned}\|Q_3(t)\|_X &\leqslant \int_0^t L_4(M)C_{J_3}\|u_0\|_{D(A^2)}(s^2+ts)\,\mathrm{d}s\\ &\leqslant L_4(M)C_{J_3}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t\in[0,\tau(2M)].\end{aligned}$$

Similarly, we can prove that there exist positive constants C_{Q_4} and C_{Q_5} such that

$$\|Q_4(t)\|_X \leqslant C_{Q_4}\|u_0\|_{D(A^2)}t^3, \quad \|Q_5(t)\|_X \leqslant C_{Q_5}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t\in[0,\tau(4M)].$$

Estimation for $Q_6(t)$.

We split $J_6(s,t)$ into

$$J_6(s,t) = J_{61}(s,t) + J_{62}(t),$$

where

$$J_{61}(s,t) = \int_{t/2}^s F_1(w_1(\tau,t))\,\mathrm{d}\tau, \quad J_{62}(t) = \int_0^{t/2} F_1(w_1(\tau,t))\,\mathrm{d}\tau - \frac{1}{2}\int_0^t F_1(w_1(\tau,t))\,\mathrm{d}\tau.$$

First, we estimate $\int_0^t \Phi_A(t-s)F_2'(u_0)J_{61}(s,t)\,\mathrm{d}s$. By Taylor's formula, we obtain

$$J_{61}(s,t) = (s-t/2)F_1(w_1(t/2,t)) + (s-t/2)^2 J_{61b}(s,t),$$

where

$$J_{61b}(s,t) = \frac{1}{2}\int_0^1 (1-\theta)F_1'\big(w_1(\theta s+(1-\theta)t/2,t)\big)\cdot F_1\big(w_1((\theta s+(1-\theta)t/2),t)\big)\,\mathrm{d}\theta.$$

In the same way as in the estimate of $Q_0(t)$, we can show

$$\left\|\int_0^t \Phi_A(t-s)F_2'(u_0)(s-t/2)F_1(w_1(t/2))\,\mathrm{d}s\right\|_X \leqslant L(2M)^2\|u_0\|_{D(A^2)}t^3$$

for $t\in[0,\tau(M)]$. By (F1), (F7), and (4.7), we have

$$\|J_{61b}(s,t)\|_X \leqslant \|J_{61b}(s,t)\|_{D(A)} \leqslant L(2M)^2 M \leqslant L(2M)^2\|u_0\|_{D(A^2)}$$

for $s,t\in[0,\tau(M)]$. Thus, by (F4), we obtain

$$\begin{aligned}&\left\|\int_0^t (s-t/2)^2\Phi_A(t-s)F_2'(u_0)J_{61b}(s,t)\,\mathrm{d}s\right\|_X\\ &\quad\leqslant \int_0^t (s-t/2)^2 L_4(M)\|J_{61b}(s,t)\|_X\,\mathrm{d}s\\ &\quad\leqslant L(2M)^2L_4(M)\|u_0\|_{D(A^2)}t^3 \quad \text{for } t\in[0,\tau(M)].\end{aligned} \tag{4.12}$$

Therefore, there exists a positive constant C_{61} depending only on M such that

$$\left\|\int_0^t \Phi_A(t-s)F_2'(u_0)J_{61}(s,t)\,\mathrm{d}s\right\|_X \leqslant C_{61}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t\in[0,\tau(M)]. \tag{4.13}$$

Next, we estimate $\int_0^t \Phi_A(t-s)F_2'(u_0)J_{62}(s,t)\,\mathrm{d}s$. We rewrite $J_{62}(t)$ as

$$J_{62}(t) = \int_0^{t/2} F_1(w_1(\tau,t))\,\mathrm{d}\tau - \int_0^{t/2} F_1(w_1(2\tau,t))\,\mathrm{d}\tau.$$

Hence, by (F6) and (4.7), we have

$$\begin{aligned}
\|J_{62}(t)\|_{D(A)} &\leqslant \int_0^{t/2} \|F_1(w_1(\tau,t)) - F_1(w_1(2\tau,t))\|_{D(A)}\,\mathrm{d}\tau \\
&\leqslant \int_0^{t/2} L(2M)\|w_1(\tau,t) - w_1(2\tau,t)\|_{D(A)}\,\mathrm{d}\tau \\
&= \int_0^{t/2} L(2M)\|\Phi_{F_1}(\tau/2)\Phi_A(t/2)u_0 - \Phi_{F_1}(\tau)\Phi_A(t/2)u_0\|_{D(A)}\,\mathrm{d}\tau \\
&\leqslant \int_0^{t/2} L(2M)\int_{\tau/2}^{\tau} 2L(2M)M\,\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau \leqslant L(2M)^2t^2M \\
&\leqslant L(2M)^2\|u_0\|_{D(A^2)}t^2
\end{aligned}$$

for $t\in[0,\tau(M)]$. Thus, it follows from (F1) that

$$\begin{aligned}
\left\|\int_0^t \Phi_A(t-s)F_2'(u_0)J_{62}(t)\,\mathrm{d}s\right\|_X &\leqslant \left\|\int_0^t \Phi_A(t-s)F_2'(u_0)J_{62}(t)\,\mathrm{d}s\right\|_{D(A)} \\
&\leqslant \int_0^t L(M)\|J_{62}(t)\|_{D(A)}\,\mathrm{d}s \leqslant L(2M)^3\|u_0\|_{D(A^2)}t^3 \quad \text{for } t\in[0,\tau(M)].
\end{aligned} \tag{4.14}$$

Summing up these estimates, we obtain that there exists a positive constant C_{Q_6} depending only on M such that

$$\|Q_6(t)\|_X \leqslant C_{Q_6}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t\in[0,\tau(M)]. \tag{4.15}$$

4.3. Proof of (4.1)**.** To derive an estimation for $R_{1a}(t)$, we divide $R_{1a}(t)$ into

$$R_{1a}(t) = R_{11a}(t) + R_{12a}(t),$$

where

$$\begin{aligned}
R_{11a}(t) &= \int_0^t \Phi_A(t-s)\{F_1(\Psi(s)u_0) - F_1(w_2(s,t))\}\,\mathrm{d}s, \\
R_{12a}(t) &= \frac{1}{2}\int_0^t \Phi_A(t-s)[2F_1(w_2(s,t)) - \{F_1(w_3(s,t)) + F_1(w_1(s,t))\}]\,\mathrm{d}s.
\end{aligned}$$

Then, in exactly the same way as in the proof of (4.3), we can prove that there exists a positive constant C_{211a} depending only on M such that

$$\|R_{11a}(t)\|_X \leqslant C_{211a}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \in [0, \tau(4M)].$$

We proceed to show that there exists a positive constant C_{212a} depending only on M such that

$$\|R_{12a}(t)\|_X \leqslant C_{212a}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \in [0, \tau(M)]. \tag{4.16}$$

To do this, we divide $R_{12a}(t)$ into several parts. By Taylor's formula, we have

$$\begin{aligned} R_{12a}(t) &= \frac{1}{2}\int_0^t \Phi_A(t-s)[2F_1(w_2(s,t)) - F_1(w_3(s,t)) - F_1(w_1(s,t))]\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \Phi_A(t-s)[Q^*(s,t) + 2J_2^*(s,t) - J_3^*(s,t) - J_1^*(s,t)]\,\mathrm{d}s, \end{aligned}$$

where

$$\begin{aligned} Q^*(s,t) &= F_1'(u_0)[2w_2(s,t) - w_3(s,t) - w_1(s,t)], \\ J_j^*(s,t) &= \int_0^1 (1-\theta)F_1''(\theta w_j(s,t) + (1-\theta)u_0)[w_j(s,t) - u_0]^2\,\mathrm{d}\theta, \quad j = 1,2,3. \end{aligned}$$

In exactly the same way as in the estimates of $Q_1(t)$ and $Q_2(t)$, we can prove that there exists a positive constant $C_{J_j^*}$ $(j = 1,2,3)$ depending only on M such that

$$\left\|\int_0^t \Phi_A(t-s)J_j^*(s)\,\mathrm{d}s\right\|_X \leqslant C_{J_j^*}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \in [0, \tau(4M)],\ j = 1,2,3.$$

Hence, it remains to derive the following estimate. There exists a positive constant C_{Q^*} depending only on M such that

$$\left\|\int_0^t \Phi_A(t-s)Q^*(s)\,\mathrm{d}s\right\|_X \leqslant C_{Q^*}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \in [0, \tau(4M)]. \tag{4.17}$$

We rewrite $\int_0^t \Phi_A(t-s)Q^*(s)\,\mathrm{d}s$ as

$$\int_0^t \Phi_A(t-s)Q^*(s)\,\mathrm{d}s = \sum_{j=1}^3 W_j(t)\,\mathrm{d}s,$$

where

$$W_j(t) = \int_0^t \Phi_A(t-s)F_1'(u_0)\widetilde{W}_j(t)\,\mathrm{d}s,$$
$$\widetilde{W}_j(t) = (-1)^j[2I_j(s,t) - I_j(t,t)] \quad (j=1,2), \quad \widetilde{W}_3(t) = [I_1(s,t) - I_3(s,t)].$$

Here,

$$I_2(s,t) = \int_0^s F_2(w_2(\tau,t))\,\mathrm{d}\tau, \quad I_j(s,t) = \int_0^{s/2} F_1(w_j(2\tau,t))\,\mathrm{d}\tau, \quad j=1,3.$$

First, in the same way as in the proof of (4.15), we can prove that there exist positive constants C_{W_1}, C_{W_2} depending only on M such that

$$\|W_j(t)\|_X \leqslant C_{W_j}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \in [0,\tau(2M)],\ j=1,2. \tag{4.18}$$

In the following, we show that there exists a positive constant C_{W_3} depending only on M such that

$$\|W_3(t)\|_X \leqslant C_{W_3}\|u_0\|_{D(A^2)}t^3 \quad \text{for } t \in [0,\tau(4M)]. \tag{4.19}$$

By (F4),

$$\begin{aligned}\|W_3(t)\|_X &= \left\|\int_0^t \Phi_A(t-s)F_1'(u_0)[I_1(s,t) - I_3(s,t)]\,\mathrm{d}s\right\|_X \\ &\leqslant \int_0^t L_4(M)\|I_1(s,t) - I_3(s,t)\|_X.\end{aligned}$$

By (4.7), we have

$$\|w_j(s,t)\|_{D(A)} \leqslant 8M \quad \text{for } s,t \in [0,\tau(4M)],\ j=1,3. \tag{4.20}$$

In view of (4.20) and (F6), we obtain

$$\begin{aligned}\|I_1(s,t) - I_3(s,t)\|_X &\leqslant \|I_1(s,t) - I_3(s,t)\|_{D(A)} \\ &= \left\|\frac{1}{2}\int_0^s (F_1(w_1(\tau,t)) - F_1(w_3(\tau,t)))\,\mathrm{d}\tau\right\|_{D(A)} \\ &\leqslant \int_0^s \frac{1}{2}L(8M)\|w_1(\tau,t) - w_3(\tau,t)\|_{D(A)}\,\mathrm{d}\tau \quad \text{for } s,t \in [0,\tau(4M)].\end{aligned}$$

By (3.2),

$$\begin{aligned}&\|w_1(\tau,t)-w_3(\tau,t)\|_{D(A)}\\&\quad=\left\|w_1(\tau,t)-w_1(t,t)-\int_0^t F_2(w_2(\widetilde{\tau},t))\,\mathrm{d}\widetilde{\tau}-\frac{1}{2}\int_0^\tau F_1(w_3(\widetilde{\tau},t))\,\mathrm{d}\widetilde{\tau}\right\|_{D(A)}.\end{aligned}$$

In the same way as in the estimates of $J_{62}(t)$, we have

$$\|w_1(\tau,t)-w_1(t,t)\|_{D(A)}\leqslant L(M)M|t-\tau|\quad\text{for }\tau,t\in[0,\tau(M)].$$

By (F7), (4.5), and (4.20), we have

$$\|F_2(w_2(\widetilde{\tau},t))\|_{D(A)}\leqslant 4L(4M)M,\quad\|F_1(w_3(\widetilde{\tau},t))\|_{D(A)}\leqslant 8L(8M)M$$

for $\widetilde{\tau},t\in[0,\tau(4M)]$.

Thus, there exists a positive constant $C_{I_{13}}$ depending only on M such that

$$\|I_1(s,t)-I_3(s,t)\|_X\leqslant C_{I_{13}}M(s+t)s\leqslant C_{I_{13}}\|u_0\|_{D(A^2)}(s+t)s$$

for $s,t\in[0,\tau(4M)]$. Therefore, we obtain (4.19).

Summing up these estimates, we get (4.16) and, therefore, (4.1).

By (4.1)–(4.4), we obtain the following lemma.

Lemma 4.2. *Assume* (F0)–(F5)*. Let* $u_0\in D(A^2)$ *and set* $M=\|u_0\|_{D(A)}$*. Then there exists a positive constant* C_2 *depending only on* M *such that*

$$\|R(t)\|_{D(A)}\leqslant C_2\|u_0\|_{D(A^2)}t^3\quad\textit{for }t\in[0,\tau(4M)].\tag{4.21}$$

Now, we can proceed with the proof of Proposition 4.1 in the same way as in the proof of Proposition 3.1 by using Lemma 4.2.

5. Proof of Theorem 1.2

This section is devoted to the proof of the main result, Theorem 1.2. We set

$$\gamma_1 = 2L_2(8m_0), \quad \kappa_1 = \mathrm{e}^{\{2L(2m_0)+\gamma_1\}T} K_1(m_0)T, \quad \kappa_3 = \kappa_1 \|u_0\|_{D(A^2)}, \\ \kappa_2 = \mathrm{e}^{\{2L_4(2m_0)+\gamma_1\}T} K_2(m_0)T. \tag{5.1}$$

We assume that $h_0 > 0$ satisfies

$$h_0 \leqslant \tau(4m_0), \quad \mathrm{e}^{2L(2m_0)h_0}\kappa_3 h_0 \leqslant \delta_0, \quad \kappa_3 h_0 \leqslant \frac{7m_0}{8}, \tag{5.2}$$

where $m_0 = 8 \max_{t\in[0,T]} \|S(t)u_0\|_{D(A)}$ and δ_0 was defined in Lemma 2.6. We note that $\kappa_3 h \leqslant \mathrm{e}^{2L(2m_0)h}\kappa_3 h \leqslant \delta_0$ for $h \in (0, h_0]$.

In what follows, we assume $h \in (0, h_0]$. By induction, we will show

$$\|\Psi(h)^j u_0\|_{D(A^2)} \leqslant \mathrm{e}^{\gamma_1 jh} \|u_0\|_{D(A^2)}, \tag{5.3}$$

$$\|\Psi(h)^j u_0\|_{D(A)} \leqslant m_0, \tag{5.4}$$

$$\|S(jh)u_0 - \Psi(h)^j u_0\|_{D(A)} \leqslant \kappa_3 h, \tag{5.5}$$

$$\|S(jh)u_0 - \Psi(h)^j u_0\|_X \leqslant \kappa_2 \|u_0\|_{D(A^2)} h^2 \tag{5.6}$$

for $j \in \mathbb{N} \cup \{0\}$ satisfying $jh \leqslant T$.

In the case $j = 0$, it is clear that (5.3)–(5.6) hold. We assume $nh \leqslant T$ and (5.3)–(5.6) hold for $j = 0, 1, \ldots, n-1$.

First, it follows from Lemma 2.5 and (5.3) that

$$\begin{aligned}\|\Psi(h)^n u_0\|_{D(A^2)} &= \|\Psi(h)\Psi(h)^{n-1} u_0\|_{D(A^2)} \leqslant \mathrm{e}^{2L_2(8m_0)h} \|\Psi(h)^{n-1}u_0\|_{D(A^2)} \\ &\leqslant \mathrm{e}^{\gamma_1 h}\mathrm{e}^{\gamma_1(n-1)h} \|u_0\|_{D(A^2)} = \mathrm{e}^{\gamma_1 nh}\|u_0\|_{D(A^2)}.\end{aligned}$$

By the triangle inequality, we obtain

$$\begin{aligned}\|S(nh)u_0 - \Psi(h)^n u_0\|_{D(A)} \leqslant \sum_{j=0}^{n-1} &\|S((n-j-1)h)S(h)\Psi(h)^j u_0 \\ &- S((n-j-1)h)\Psi(h)\Psi(h)^j u_0\|_{D(A)}.\end{aligned}$$

Moreover,

$$\|\Psi(h)^j u_0 - S(jh)u_0\|_{D(A)} \leqslant \kappa_3 h \leqslant \delta_0$$

for $j = 0, 1, \ldots, n-2$. Thus, it follows from Lemma 2.6 that

$$\begin{aligned}\|S(h)\Psi(h)^j u_0 - S((j+1)h)u_0\|_{D(A)} &= \|S(h)\Psi(h)^j u_0 - S(h)S(jh)u_0\|_{D(A)} \\ &\leqslant \mathrm{e}^{2L(2m_0)h}\|\Psi(h)^j u_0 - S(jh)u_0\|_{D(A)} \leqslant \mathrm{e}^{2L(2m_0)h}\kappa_3 h \leqslant \delta_0\end{aligned}$$

for $j = 0, 1, \ldots, n-2$. Moreover,

$$\|\Psi(h)\Psi(h)^j u_0 - S((j+1)h)u_0\|_{D(A)} = \|\Psi(h)^{j+1}u_0 - S((j+1)h)u_0\|_{D(A)} \leqslant \kappa_3 h \leqslant \delta_0$$

for $j = 0, 1, \ldots, n-2$. Hence, it follows from Lemma 2.6 that

$$\begin{aligned}
&\|S((n-j-1)h)S(h)\Psi(h)^j u_0 - S((n-j-1)h)\Psi(h)\Psi(h)^j u_0\|_{D(A)} \\
&\qquad \leqslant \mathrm{e}^{2L(2m_0)(n-j-1)h}\|S(h)\Psi(h)^j u_0 - \Psi(h)\Psi(h)^j u_0\|_{D(A)} \\
&\qquad \leqslant \mathrm{e}^{2L(2m_0)T}\|S(h)\Psi(h)^j u_0 - \Psi(h)\Psi(h)^j u_0\|_{D(A)}.
\end{aligned}$$

Hence, we have

$$\|S(nh)u_0 - \Psi(h)^n u_0\|_{D(A)} \leqslant \mathrm{e}^{2L(2m_0)T}\sum_{j=0}^{n-1}\|S(h)\Psi(h)^j u_0 - \Psi(h)\Psi(h)^j u_0\|_{D(A)}.$$

Moreover, it follows from (5.4) that $\|\Psi(h)^j u_0\|_{D(A)} \leqslant m_0$ for $j = 0, 1, \ldots, n-1$. By Proposition 3.1 we obtain

$$\begin{aligned}
\|S(h)\Psi(h)^j u_0 - \Psi(h)\Psi(h)^j u_0\|_{D(A)} &\leqslant K_1(m_0)\|\Psi(h)^j u_0\|_{D(A^2)}h^2 \\
&\leqslant K_1(m_0)\mathrm{e}^{\gamma_1 T}\|u_0\|_{D(A^2)}h^2
\end{aligned}$$

for $j = 0, 1, \ldots, n-1$. Therefore, we have

$$\begin{aligned}
\|S(nh)u_0 - \Psi(h)^n u_0\|_{D(A)} &\leqslant \mathrm{e}^{2L(2m_0)T}\sum_{j=0}^{n-1}K_1(m_0)\mathrm{e}^{\gamma_1 T}\|u_0\|_{D(A^2)}h^2 \\
&\leqslant \mathrm{e}^{\{2L(2m_0)+\gamma_1\}T}K_1(m_0)\|u_0\|_{D(A^2)}nh^2 \leqslant \kappa_3 h.
\end{aligned}$$

Finally, it follows from (5.2) that

$$\|\Psi(h)^n u_0\|_{D(A)} \leqslant \|\Psi(h)^n u_0 - S(nh)u_0\|_{D(A)} + \|S(nh)u_0\|_{D(A)} \leqslant \kappa_3 h + m_0/8 \leqslant m_0.$$

We can also prove (5.6) in the same way as in the proof of (5.5).

Therefore, we showed (5.4) holds for $j = n$. This completes the proof. □

6. Numerical examples

In this section, we present numerical examples. Let $I=(0,1)$. We consider

$$\begin{cases} \partial_t u = \mathrm{i}\partial_x^2 u - \mathrm{i}|u|^2u - 2|u|^4u, & t\in[0,T],\ x\in I,\\ u(t,0)=u(t,1)=0, & t\in[0,T],\\ u(0,x)=u_0(x), & x\in I. \end{cases} \tag{6.1}$$

By letting

$$A=\mathrm{i}\partial_x^2,\quad X=L^2(I),\quad D(A)=H^2(I)\cap H_0^1(I),$$
$$D(A^2)=\{v\in H^4(I);\ v=\partial_x^2 v=0 \text{ on } \partial I\},\quad F(v)=-\mathrm{i}|v|^2v-2|v|^4v,$$

the equation (6.1) fits into the framework of Proposition 1.1. Hence, we can directly apply Theorem 1.2 to (6.1) if $u_0\in D(A^2)$. It is difficult to obtain the exact solution of (6.1). Therefore, instead of (1.13), we numerically investigate the quantity

$$\boldsymbol{e}_Y=\sup_{0\leqslant t_n\leqslant T}\|\Psi(h)^n u_0-\Psi(h/2)^{2n}u_0\|_Y,$$

where $Y=L^\infty(I)$, $L^2(I)$ or $H^1(I)$. In Figures 1 and 3, we present $(\log h,\log \boldsymbol{e}_{L^\infty})$, $(\log h,\log \boldsymbol{e}_{L^2})$ and $(\log h,\log \boldsymbol{e}_{H^1})$. Figure 1 shows that the second-order convergence occurs with the initial value $u_0(x)=\sin(\pi x)$ which is a $D(A^2)$ function. On the other hand, Figure 3 shows that the first-order convergence occurs with the initial value $U_0(x)=|x-\frac12|^3\sin(\pi x)$ for $x\in I$ which is a $D(A)$ function but not a $D(A^2)$ function. Hence, we see that (1.12) and (1.13) are optimal numerically. Moreover, (1.1) has the following dissipative property

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_I |u|^2\,\mathrm{d}x\biggr)^{1/2}\leqslant 0. \tag{6.2}$$

In Figures 2 and 4, we demonstrate that the scheme (1.10) preserves the property (6.2).

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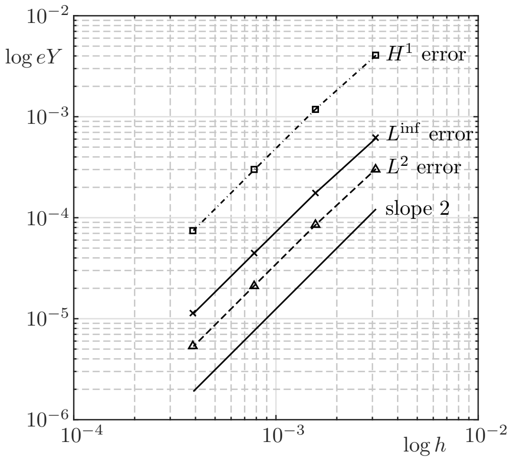

Figure 1. Convergence rate of the scheme (1.10) for (6.1), $u_0(x) = \sin(\pi x)$.

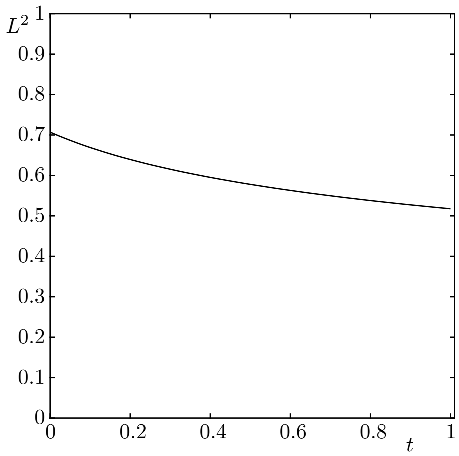

Figure 2. The dissipation property of the scheme (1.10) for (6.1), $u_0(x) = \sin(\pi x)$.

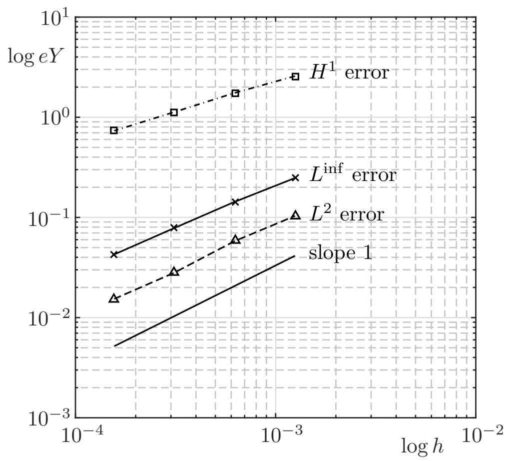

Figure 3. Convergence rate of the scheme (1.10) for (6.1) with the initial value $U_0(x)$.

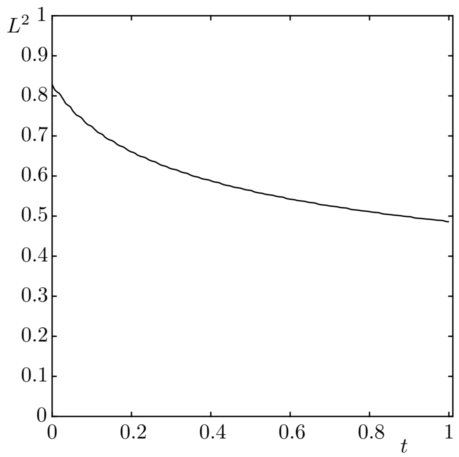

Figure 4. The dissipation property of the scheme (1.10) for (6.1) with the initial value $U_0(x)$.

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