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(STRONGLY) GORENSTEIN INJECTIVE MODULES OVER UPPER TRIANGULAR MATRIX ARTIN ALGEBRAS

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Abstract. Let $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ be an Artin algebra. In view of the characterization of finitely generated Gorenstein injective Λ-modules under the condition that M is a cocompatible (A,B)-bimodule, we establish a recollement of the stable category $\overline{\mathrm{Ginj}(\Lambda)}$. We also determine all strongly complete injective resolutions and all strongly Gorenstein injective modules over Λ.

Keywords: (strongly) Gorenstein injective module; upper triangular matrix Artin algebra; triangulated category; recollement

MSC 2010: 18G25, 16E65, 18E30

1. Introduction and preliminaries

The notions of Gorenstein projective and Gorenstein injective modules over an arbitrary ring were introduced by Enochs and Jenda in [5], who formed the basis of the Gorenstein homological algebra. Let Λ be an Artin algebra. A complete injective resolution is an exact sequence

$$I^\bullet\colon \ \ldots \longrightarrow I^{-1} \longrightarrow I^0 \xrightarrow{d^0} I^1 \longrightarrow \ldots$$

of injective Λ-modules, such that $\mathrm{Hom}_\Lambda(J, I^\bullet)$ is also exact for any injective Λ-module J. A Λ-module M is Gorenstein injective if there exists a complete Λ-injective resolution $I^\bullet$ such that $M \cong \mathrm{Im}\, d^0$. Dually, the notions of a complete Λ-projective resolution and a Gorenstein projective Λ-module are defined. Let Λ-mod denote the

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finitely generated category of left modules of the Artin algebra Λ. Let $\mathrm{inj}(\Lambda)$ be the full subcategory of Λ-mod of finitely generated injective modules and $\mathrm{Ginj}(\Lambda)$ the full subcategory of finitely generated Gorenstein injective modules, and let $\mathrm{Proj}(\Lambda)$ be the full subcategory of Λ-mod of finitely generated projective modules and $\mathrm{Gproj}(\Lambda)$ the full subcategory of finitely generated Gorenstein projective modules. By [3], Remark 4.3, the adjoint pair

$$(N^+ = - \otimes_\Lambda D(\Lambda_\Lambda),\ N^- = \mathrm{Hom}_\Lambda(D(\Lambda_\Lambda), -))\colon\ \Lambda\text{-Mod} \to \Lambda\text{-Mod}$$

induces an equivalence

$$(N^+, N^-)\colon\ \mathrm{Gproj}(\Lambda) \xrightarrow{\approx} \mathrm{Ginj}(\Lambda),$$

where D is the duality of Λ.

Many important applications need explicitly constructing all the Gorenstein projective and Gorenstein injective modules of a given algebra. Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be the triangular matrix Artin algebra. Zhang in [12] explicitly described the category of finitely generated Gorenstein projective Λ-modules in view of a compatible A-B-bimodule M, and projective left A-module ${}_AM$. Moreover, he established a left recollement of the stable category $\underline{\mathrm{Gproj}(\Lambda)}$. Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an upper triangulated matrix Artin algebra. It follows from [2], Proposition 2.7, that $\mathrm{proj.dim}_A M \neq \mathrm{proj.dim}\, M_B$ in general. There also exist three different equivalences:

$$\begin{aligned}
(N^+, N^-)_\Lambda\colon\ \mathrm{Gproj}(\Lambda) &\xrightarrow{\approx} \mathrm{Ginj}(\Lambda),\\
(N^+, N^-)_A\colon\ \mathrm{Gproj}(A) &\xrightarrow{\approx} \mathrm{Ginj}(A),\\
(N^+, N^-)_B\colon\ \mathrm{Gproj}(B) &\xrightarrow{\approx} \mathrm{Ginj}(B).
\end{aligned}$$

Thus [12], Theorem 1.4, and [12], Proposition 2.5, are not characterizations for finitely generated Gorenstein injective Λ-modules. In this paper, we introduce the notion of a cocompatible A-B-bimodule which is not a dual of a compatible A-B-bimodule. We describe the finitely generated Gorenstein injective Λ-modules, in view of the projective right B-module M_B. Further, we establish the following recollement of the stable category $\overline{\mathrm{Ginj}(\Lambda)}$.

Theorem A. *Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be a Gorenstein algebra and M_B a projective module. Then we have the following recollement:*

$$\begin{tikzcd}
\overline{\mathrm{Ginj}(B)} \arrow[r, "i_*"] & \overline{\mathrm{Ginj}(\Lambda)} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \overline{\mathrm{Ginj}(A)}. \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

Recently, Bennis and Mahdou [4] investigated a particular case of Gorenstein injective modules which are called strongly Gorenstein injective modules, and proved that a module is Gorenstein injective if and only if it is a direct summand of a strongly Gorenstein injective module, see [4], Theorem 2.7. Note that Gorenstein injective modules are not necessarily strongly Gorenstein injective. Injective modules are strongly Gorenstein injective, and the converse is not true in general. However, in general, not much is known about concrete construction of (finitely generated) noninjective strongly Gorenstein injective modules, even if the algebras are Gorenstein. The current paper determines all strongly complete injective resolutions and all strongly Gorenstein injective modules over an upper triangular matrix Artin algebra Λ. We obtain the following result.

Theorem B. *Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an Artin algebra and N be a Λ-module. Then N is a SGinjective Λ-module if and only if one of the following holds:*

(1) $N \in \begin{pmatrix} 0 \\ D({}_AM)^{\perp} \cap \mathrm{SGinj}(B) \end{pmatrix}$, *where D is the duality of the Artin algebra A;*

(2) $N \cong \begin{pmatrix} L \\ \mathrm{Hom}_A(M,L) \end{pmatrix} \in \begin{pmatrix} \mathrm{SGinj}(A) \\ \mathrm{Hom}_A(M,\mathrm{SGinj}(A)) \end{pmatrix}$, *where L admits a strongly complete A-injective resolution $\ldots \to I \xrightarrow{g} I \xrightarrow{g} I \xrightarrow{g} \ldots$ and $\mathrm{Im}\,\mathrm{Hom}(\mathrm{id}_M, g) = \mathrm{Hom}_A(M, \mathrm{Im}\, g)$;*

(3) $N \cong \begin{pmatrix} \mathrm{Im}\, g \\ \{(h,j) \in \mathrm{Hom}_A(M,I) \oplus J \colon gh=0,\ \alpha(h)+\beta(j)=0\} \end{pmatrix}$, *where I and J are, respectively, an arbitrary nonzero injective A-module and injective B-module, and α, β, g satisfy given conditions.*

Next we recall some notions and definitions which we need in the later sections.

Duality. Let Λ be an Artin algebra. There exists only a finite number of nonisomorphic simple Λ-modules $S_1, \ldots, S_n$. Let $E(S_i)$ be the injective envelope of S_i and let $E = \bigoplus_{i=1}^{n} E(S_i)$. Then E is the injective envelope of $\bigoplus_{i=1}^{n} S_i$ and the contravariant functor $D\colon \Lambda\text{-mod} \to \Lambda^{\mathrm{op}}\text{-mod}$ defined by $D = \mathrm{Hom}_{\Lambda}(-, E)$ is a duality by [2], Theorem 3.3, page 38.

If not otherwise stated, throughout we consider Artin algebras and finitely generated modules. The reason of this assumption is to use the duality of an Artin algebra.

Upper triangular matrix Artin algebras. Let A and B be rings and M an (A,B)-bimodule. Consider the upper triangular matrix ring $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with multiplication given by the one of matrices. We assume that Λ is an Artin algebra: this is exactly the case when there is a commutative Artin ring R such that A and B are Artin R-algebras and M is finitely generated over R which acts centrally on M (see [2], page 72). An algebra C is of this form if and only if there is an idempotent

e such that $(1-e)Ce = 0$. It has an advantage that any Λ-module can be uniquely written as a triple $\binom{X}{Y}_{\varphi}$, where $X \in A$-mod, $Y \in B$-mod and $\varphi\colon M \otimes_B Y \to X$ is an A-map.

Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an Artin algebra. A Λ-module will be identified with a triple $\binom{X}{Y}_{\varphi}$, or simply $\binom{X}{Y}$ if φ is clear, where $X \in A$-mod, $Y \in B$-mod and

$$\varphi\colon M \otimes_B Y \to X$$

is an A-map. A Λ-map $\binom{X}{Y}_{\varphi} \to \binom{X'}{Y'}_{\varphi'}$ will be identified with a pair $\binom{f}{g}$, where $f \in \mathrm{Hom}_A(X, X')$ and $g \in \mathrm{Hom}_B(Y, Y')$, such that the following diagram commutes:

$$\begin{tikzcd} M \otimes_B Y \arrow[r, "\mathrm{id}_M \otimes g"] \arrow[d, "\varphi"] & M \otimes_B Y' \arrow[d, "\varphi'"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \qquad (1.1)$$

A sequence

$$0 \longrightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}_{\varphi_1} \xrightarrow{\binom{f_1}{g_1}} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}_{\varphi_2} \xrightarrow{\binom{f_2}{g_2}} \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}_{\varphi_3} \longrightarrow 0$$

in Λ-mod is exact if and only if $0 \longrightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow 0$ in A-mod is exact and $0 \longrightarrow Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \longrightarrow 0$ in B-mod is exact. Indecomposable injective Λ-modules are objects of the form $\binom{0}{I}$, where I is an indecomposable injective B-module, and objects isomorphic to objects of the form $\binom{J}{\mathrm{Hom}_A(M,J)}_{\varphi}$, where J is an indecomposable injective A-module and $\varphi\colon M \otimes_B \mathrm{Hom}_A(M, J) \to J$ is given by $\varphi(m \otimes f) = f(m)$ for $m \in M$ and $f \in \mathrm{Hom}_A(M, J)$.

Definition 1.1. An (A, B)-bimodule M is cocompatible if the following two conditions hold:

(C1) If $I^{\bullet}$ is an exact sequence of injective A-modules, then $\mathrm{Hom}_A(M, I^{\bullet})$ is exact.

(C2) If $J^{\bullet}$ is a complete B-injective resolution, then $\mathrm{Hom}_B(D(_AM), J^{\bullet})$ is exact, where D is the duality of the Artin algebra A.

Lemma 1.2. *Let M be an (A, B)-bimodule. Then the following are equivalent:*

(1) *M satisfies* (C2)*;*

(2) $\mathrm{Ext}^1_B(\mathrm{Hom}_A(M, I), G) = 0$ *for all* $G \in \mathrm{Ginj}(B)$ *and for all* $I \in \mathrm{inj}(A)$*;*

(3) $\mathrm{Ext}^i_B(\mathrm{Hom}_A(M, I), G) = 0$ *for all* $G \in \mathrm{Ginj}(B)$ *for all* $I \in \mathrm{inj}(A)$ *and for all* $i \geqslant 1$.

Proof. Note that $\mathrm{Hom}_A(M, I) \in \mathrm{add}(D(_AM))$ for all $I \in \mathrm{inj}(A)$, where $\mathrm{add}(D(_AM))$ is the subclass of A-modules consisting of all modules isomorphic to direct summands of finite direct sums of copies of $D(_AM)$. $\square$

Lemma 1.3. *Let M be an (A,B)-bimodule and* $\operatorname{proj.dim}_A M < \infty$.

(1) *If* $\operatorname{proj.dim} M_B < \infty$*, then M is cocompatible.*

(2) *If* $\operatorname{inj.dim} M_B < \infty$*, then M is cocompatible.*

Proof. Let $I^\bullet$ be an exact sequence of injective A-modules. Consider the following projective resolution of ${}_AM$:

$$0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Note that each complex $\operatorname{Hom}_A(P_i, I^\bullet)$ is exact and each term I^i of $I^\bullet$ is injective. We get the following exact sequence of complexes:

$$0 \longrightarrow \operatorname*{Hom}_A(M, I^\bullet) \longrightarrow \operatorname{Hom}_A(P_0, I^\bullet) \longrightarrow \ldots \longrightarrow \operatorname{Hom}_A(P_n, I^\bullet) \longrightarrow 0,$$

which implies that $\operatorname{Hom}_A(M, I^\bullet)$ is exact.

Let $J^\bullet$ be a complete B-injective resolution. If $\operatorname{proj.dim} M_B < \infty$, then [6], Theorem 3.2.9, implies that $\operatorname{inj.dim}_B D({}_AM) < \infty$. Also if $\operatorname{inj.dim} M_B < \infty$, then [6], Theorem 3.2.16, and [1], Corollary 28.8, imply that $\operatorname{proj.dim}_B D({}_AM) < \infty$. By a similar argument as above we see that $\operatorname{Hom}_B(D({}_AM), J^\bullet)$ is exact, as desired. $\square$

Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an Artin algebra. Let ${}_\Lambda\mathscr{C}$ be the category whose objects are triples $\begin{pmatrix} X \\ Y \end{pmatrix}_\psi$, where $X \in A$-mod, $Y \in B$-mod and $\psi\colon Y \to \operatorname{Hom}_A(M, X)$ is a B-map. A morphism from $\begin{pmatrix} X \\ Y \end{pmatrix}_\psi$ to $\begin{pmatrix} X' \\ Y' \end{pmatrix}_{\psi'}$ is a pair $\begin{pmatrix} f \\ g \end{pmatrix}$, where $f \in \operatorname{Hom}_A(X, X')$ and $g \in \operatorname{Hom}_B(Y, Y')$, such that the following diagram commutes:

$$\text{(1.2)} \qquad \begin{CD} Y @>{g}>> Y' \\ @V{\psi}VV @VV{\psi'}V \\ \operatorname{Hom}_A(M,X) @>{\operatorname{Hom}(M,f)}>> \operatorname{Hom}_A(M,X') \end{CD}\ .$$

Define a functor $H\colon \Lambda\text{-mod} \to {}_\Lambda\mathscr{C}$ by $H\begin{pmatrix} X \\ Y \end{pmatrix}_\varphi = \begin{pmatrix} X \\ Y \end{pmatrix}_{\vartheta(\varphi)}$ on objects and $H\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$ on morphisms, where $\vartheta\colon \operatorname{Hom}_A(M \otimes_B Y, X) \to \operatorname{Hom}_B(Y, \operatorname{Hom}_A(M, X))$ is the adjoint isomorphism. Clearly the functor $G\colon {}_\Lambda\mathscr{C} \to \Lambda$-mod given by $G\begin{pmatrix} X \\ Y \end{pmatrix}_\psi = \begin{pmatrix} X \\ Y \end{pmatrix}_{\vartheta^{-1}(\psi)}$ on objects and $G\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$ on morphisms is an inverse of H. Hence H is an isomorphism of categories.

2. Gorenstein injective modules over triangular matrix algebras

Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an Artin algebra. In this section, in view of the characterization of finitely generated Gorenstein injective Λ-modules under the condition that M is a cocompatible (A,B)-bimodule, we establish a recollement of the stable category $\overline{\mathrm{Ginj}(\Lambda)}$ relative to $\overline{\mathrm{Ginj}(A)}$ and $\overline{\mathrm{Ginj}(B)}$.

Lemma 2.1 ([10], Theorem 2.1). *Let M be a cocompatible (A,B)-bimodule and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Then $\begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi} \in \mathrm{Ginj}(\Lambda)$ if and only if $\vartheta(\varphi)\colon Y \to \mathrm{Hom}_A(M,X)$ is a surjective B-map, $\mathrm{Ker}\,\vartheta(\varphi) \in \mathrm{Ginj}(B)$ and $X \in \mathrm{Ginj}(A)$. In this case, $\mathrm{Hom}_A(M,X) \in \mathrm{Ginj}(B)$ if and only if $Y \in \mathrm{Ginj}(B)$.*

Thus, if the Gorenstein symmetric conjecture (i.e., for an Artin algebra A, $\mathrm{inj.dim}\, A_A < \infty$ if and only if $\mathrm{inj.dim}_A\, A < \infty$) holds, then Λ is Gorenstein if and only if A and B are Gorenstein, $\mathrm{proj.dim}_A\, M < \infty$ and $\mathrm{proj.dim}\, M_B < \infty$, where M is an A-B-bimodule and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Here we have the following result.

Proposition 2.2. *Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be a Gorenstein algebra. Then the following are equivalent:*

(1) $\begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi} \in \mathrm{Ginj}(\Lambda)$ *if and only if* $\vartheta(\varphi)\colon Y \to \mathrm{Hom}_A(M,X)$ *is a surjective B-map,* $\mathrm{Ker}\,\vartheta(\varphi) \in \mathrm{Ginj}(B)$ *and* $X \in \mathrm{Ginj}(A)$;

(2) $\begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi} \in \mathrm{Ginj}(\Lambda)$ *implies* $X \in \mathrm{Ginj}(A)$;

(3) M *satisfies* (C1);

(4) M *is cocompatible;*

(5) B *is Gorenstein;*

(6) A *is Gorenstein;*

(7) $\mathrm{proj.dim}_A\, M < \infty$;

(8) $\mathrm{proj.dim}\, M_B < \infty$;

(9) $\mathrm{inj.dim}\, A_A < \infty$;

(10) $\mathrm{inj.dim}_B\, B < \infty$.

P r o o f. The equivalences between (5)–(10) follow from [12], Lemma 2.2. The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (9) Since Λ is Gorenstein and $\mathrm{inj.dim}\, \Lambda_\Lambda = n$, it follows from [6], Theorem 11.2.1, that each Λ-module N has Gorenstein injective dimension at most n, which implies that for any A-module X, we have an exact sequence

$$0 \longrightarrow \begin{pmatrix} X \\ 0 \end{pmatrix} \longrightarrow L^0 \longrightarrow \ldots \longrightarrow L^{n-1} \longrightarrow G \longrightarrow 0,$$

where each $L^i \in \operatorname{inj}(\Lambda)$ and $G = \left(\begin{smallmatrix} X' \\ Y' \end{smallmatrix}\right)_{\varphi'} \in \operatorname{Ginj}(\Lambda)$. Therefore

$$\begin{aligned}\operatorname{Ext}_A^{n+1}(D(A_A), X) &\cong \operatorname{Ext}_\Lambda^{n+1}\left(\left(\begin{smallmatrix} D(A_A) \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right)\right) \\ &\cong \operatorname{Ext}_\Lambda^{1}\left(\left(\begin{smallmatrix} D(A_A) \\ 0 \end{smallmatrix}\right), G\right) \cong \operatorname{Ext}_A^1(D(A_A), X')\end{aligned}$$

(where the first and the last equalities follow from an injective resolution of ${}_\Lambda\left(\begin{smallmatrix} D(A_A) \\ 0 \end{smallmatrix}\right)$, which is exactly induced by an injective resolution of $D(A_A)$; and the second equality follows from dimension shift). By the assumption $X' \in \operatorname{Ginj}(A)$, it follows that $\operatorname{Ext}_A^{n+1}(D(A_A), X) = \operatorname{Ext}_A^1(D(A_A), X') = 0$. This proves that $\operatorname{proj.dim} D(A_A) < \infty$, and so $\operatorname{inj.dim} A_A < \infty$.

(9) $\Rightarrow$ (3) Let $I^\bullet$ be an exact sequence of injective A-modules. Since Λ is a Gorenstein algebra and $\operatorname{inj.dim} A_A < \infty$, it follows from [12], Theorem 2.2, that $\operatorname{proj.dim}_A M < \infty$. This implies that $\operatorname{Hom}_A(M, I^\bullet)$ is exact.

(3) $\Rightarrow$ (4) Let $J^\bullet$ be a complete B-injective resolution. Since Λ is a Gorenstein algebra, $\operatorname{inj.dim}_A M < \infty$ by [12], Lemma 2.1, and so $\operatorname{proj.dim}_B D({}_AM) < \infty$ by [6], Theorem 3.2.16, and [1], Corollary 28.8. This implies that $\operatorname{Hom}_B(D({}_AM), J^\bullet)$ is exact.

The implication (4) $\Rightarrow$ (1) follows from Lemma 2.1. This completes the proof. $\square$

For an Artin algebra Λ, the stable category $\overline{\operatorname{Ginj}(\Lambda)}$ is triangulated, where $\overline{\operatorname{Ginj}(\Lambda)}$ has the same objects as $\operatorname{Ginj}(\Lambda)$, and $\operatorname{Hom}_{\overline{\operatorname{Ginj}(\Lambda)}}(X, Y) = \operatorname{Hom}_\Lambda(X, Y)/\operatorname{Hom}_\Lambda(X, \operatorname{inj}(\Lambda), Y)$, where $\operatorname{Hom}_\Lambda(X, \operatorname{inj}(\Lambda), Y) = \{f \in \operatorname{Hom}_\Lambda(X, Y)\colon f$ factors through an injective module$\}$.

Let $\operatorname{Ind}\operatorname{Ginj}(\Lambda)$ (or, $\operatorname{Ind}\operatorname{inj}(\Lambda)$) denote the set of indecomposable Gorenstein injective (or, injective) Λ-modules. We have the following consequence.

Corollary 2.3. *Let $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ be a Gorenstein algebra.*

(1) *If* $\operatorname{gl.dim} A < \infty$, *then*

$$\operatorname{Ind}\operatorname{Ginj}(\Lambda) = \left\{\left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right) : Y \in \operatorname{Ind}\operatorname{Ginj}(B)\right\} \cup \left\{\left(\begin{smallmatrix} I \\ \operatorname{Hom}_A(M,I) \end{smallmatrix}\right)_\varphi : I \in \operatorname{Ind}\operatorname{inj}(A)\right\}.$$

In particular, we have a triangle equivalence $\overline{\operatorname{Ginj}(\Lambda)} \cong \overline{\operatorname{Ginj}(B)}$.

(2) *If* $\operatorname{gl.dim} B < \infty$, *then*

$$\operatorname{Ind}\operatorname{Ginj}(\Lambda) = \left\{\left(\begin{smallmatrix} X \\ \operatorname{Hom}_A(M,X) \end{smallmatrix}\right) : X \in \operatorname{Ind}\operatorname{Ginj}(A)\right\} \cup \left\{\left(\begin{smallmatrix} 0 \\ J \end{smallmatrix}\right) : J \in \operatorname{Ind}\operatorname{inj}(B)\right\}$$

In particular, we have a triangle equivalence $\overline{\operatorname{Ginj}(\Lambda)} \cong \overline{\operatorname{Ginj}(A)}$.

Proof. (1) Let $\binom{X}{Y}_{\varphi} \in \mathrm{Ginj}(\Lambda)$. By assumption and Proposition 2.2 (1), we have the following exact sequence:

$$0 \longrightarrow \operatorname{Ker}\vartheta(\varphi) \longrightarrow Y \xrightarrow{\vartheta(\varphi)} \mathrm{Hom}_A(M,X) \longrightarrow 0$$

with $X \in \mathrm{Ginj}(A)$ and $\operatorname{Ker}\vartheta(\varphi) \in \mathrm{Ginj}(B)$. Since $\mathrm{gl.dim}\,A < \infty$, $X = I \in \mathrm{inj}(A)$ by the dual of [9], Proposition 2.27. Since $\mathrm{gl.dim}\,A < \infty$, it follows from Proposition 2.2 (8) that $\mathrm{proj.dim}\,M_B < \infty$. Then $\mathrm{inj.dim}\,\mathrm{Hom}_A(M,I) < \infty$ by [6], Theorem 3.2.9, and so $\mathrm{Ext}^1_B(\mathrm{Hom}_A(M,I), \operatorname{Ker}\vartheta(\varphi)) = 0$. Thus the exact sequence above splits, $Y \cong \operatorname{Ker}\vartheta(\varphi) \oplus \mathrm{Hom}_A(M,I)$ and $\binom{X}{Y}_{\vartheta(\varphi)} \cong \binom{0}{\vartheta(\varphi)} \bigoplus \binom{I}{\mathrm{Hom}_A(M,I)}_{\mathrm{id}}$ with $\binom{I}{\mathrm{Hom}_A(M,I)}_{\mathrm{id}} \in \mathrm{inj}({}_{\Lambda}\mathscr{C})$. From this the first assertion follows and $\binom{X}{Y}_{\varphi} \cong \binom{0}{\vartheta(\varphi)}$ in $\overline{\mathrm{Ginj}(\Lambda)}$. Therefore the exact functor

$$F\colon \overline{\mathrm{Ginj}(B)} \longrightarrow \overline{\mathrm{Ginj}(\Lambda)} \quad \text{via} \quad Y \longmapsto \binom{0}{Y}$$

induces a triangle equivalence $\overline{\mathrm{Ginj}(\Lambda)} \cong \overline{\mathrm{Ginj}(B)}$.

(2) Let $\binom{X}{Y}_{\varphi} \in \mathrm{Ginj}(\Lambda)$. By assumption and Proposition 2.2 (1), we have the following exact sequence:

$$0 \longrightarrow \operatorname{Ker}\vartheta(\varphi) \longrightarrow Y \xrightarrow{\vartheta(\varphi)} \mathrm{Hom}_A(M,X) \longrightarrow 0$$

with $X \in \mathrm{Ginj}(A)$ and $\operatorname{Ker}\vartheta(\varphi) \in \mathrm{Ginj}(B)$. Since $\mathrm{gl.dim}\,B < \infty$, it follows that $\operatorname{Ker}\vartheta(\varphi) = J \in \mathrm{inj}(B)$. Thus the exact sequence above splits, $Y \cong J \oplus \mathrm{Hom}_A(M,X)$ and $\binom{X}{Y}_{\varphi} \cong \binom{0}{J} \bigoplus \binom{X}{\mathrm{Hom}_A(M,X)}_{\varphi'}$ with $\binom{0}{J} \in \mathrm{inj}(\Lambda)$. From this the first assertion follows and $\binom{X}{Y}_{\varphi} \cong \binom{X}{\mathrm{Hom}_A(M,X)}_{\varphi'}$ in $\overline{\mathrm{Ginj}(\Lambda)}$. Consider the following equivalence:

$$F\colon \overline{\mathrm{Ginj}(A)} \longrightarrow \overline{\mathrm{Ginj}(\Lambda)} \quad \text{via} \quad X \longmapsto \begin{pmatrix} X \\ \mathrm{Hom}_A(M,X) \end{pmatrix}_{\vartheta^{-1}(\mathrm{id})}.$$

Since $\mathrm{gl.dim}\,B < \infty$, it follows from Proposition 2.2 (7) that $\underline{\mathrm{proj.dim}_A\, M} < \infty$. Thus F is an exact functor and F induces a triangle equivalence $\overline{\mathrm{Ginj}(\Lambda)} \cong \overline{\mathrm{Ginj}(A)}$. $\square$

Let $\mathcal{T}'$, $\mathcal{T}$ and $\mathcal{T}''$ be triangulated categories. The diagram of exact functors

$$(*) \qquad \begin{tikzcd} \mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

is a recollement of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$, if the following four conditions are satisfied:

(R1) $(i^*, i_*), (i_*, i^!), (j_!, j^*)$ and (j^*, j_*) are adjoint pairs;

(R2) $i_*, j_!$ and j_* are fully faithful;

(R3) $j^* i_* = 0$;

(R4) for each object $X \in \mathcal{T}$, the counits and units give rise to distinguished triangles:

$$j_! j^* X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} i_* i^* X \rightsquigarrow \quad \text{and} \quad i_* i^! X \xrightarrow{\omega_X} X \xrightarrow{\xi_X} j_* j^* X \rightsquigarrow .$$

For the diagram $(*)$ of exact functors, under the conditions (R1), (R3) and (R4), one has

(R5) $\operatorname{Im} i_* = \operatorname{Ker} j^*$;

(R6) $\operatorname{Im} j_! = \operatorname{Ker} i^*$;

(R7) $\operatorname{Im} j_* = \operatorname{Ker} i^!$.

Theorem 2.4 ([12], Theorem 3.2)**.** *Let $(*)$ be a diagram of exact functors of triangulated categories. Then the following are equivalent:*

(1) *The diagram $(*)$ is a recollement.*

(2) *The conditions* (R1), (R2) *and* (R5) *are satisfied.*

(3) *The conditions* (R1), (R2) *and* (R6) *are satisfied.*

(4) *The conditions* (R1), (R2) *and* (R7) *are satisfied.*

A right recollement of a triangulated category $\mathcal{T}$ relative to triangulated categories $\mathcal{T}'$ and $\mathcal{T}''$ is a diagram of exact functors consisting of the lower two rows of $(*)$, satisfying all the conditions which involve only the functors $i^!, i_*, j^*, j_*$. Note that the corresponding version of Theorem 2.4 also holds for right recollements.

Theorem 2.5. *Let M be a cocompatible (A, B)-bimodule and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Then we have the following right recollement:*

$$\begin{tikzcd} \overline{\mathrm{Ginj}(B)} \arrow[r, shift left, "i_*"] & \overline{\mathrm{Ginj}(\Lambda)} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \overline{\mathrm{Ginj}(A)} \arrow[l, shift left, "j_*"] \end{tikzcd} .$$

P r o o f. We first construct the functors involved. If a morphism $\begin{pmatrix} X \\ Y \end{pmatrix}_{\vartheta(\varphi)} \longrightarrow \begin{pmatrix} X' \\ Y' \end{pmatrix}_{\vartheta(\varphi')}$ in ${}_\Lambda\mathscr{C}$ factors through an injective object $\begin{pmatrix} 0 \\ J \end{pmatrix} \bigoplus \begin{pmatrix} I \\ \operatorname{Hom}_A(M, I) \end{pmatrix}_{\mathrm{id}}$ of ${}_\Lambda\mathscr{C}$, then the induced B-map $\operatorname{Ker}\vartheta(\varphi) \to \operatorname{Ker}\vartheta(\varphi')$ factors through an injective B-module J. Hence Lemma 2.1 implies that the functor $\mathrm{Ginj}(\Lambda) \to \overline{\mathrm{Ginj}(B)}$ given by $\begin{pmatrix} X \\ Y \end{pmatrix}_\varphi \mapsto \operatorname{Ker}\vartheta(\varphi)$ induces a functor $i^!\colon \overline{\mathrm{Ginj}(\Lambda)} \to \overline{\mathrm{Ginj}(B)}$.

By Lemma 2.1 there is a unique functor $i_*\colon \overline{\mathrm{Ginj}(B)} \to \overline{\mathrm{Ginj}(\Lambda)}$ given by $Y \mapsto \binom{0}{Y}$ which is fully faithful. By Lemma 2.1 there is a unique functor $j_*\colon \overline{\mathrm{Ginj}(A)} \to \overline{\mathrm{Ginj}(\Lambda)}$ given by $X \mapsto \binom{X}{\mathrm{Hom}_A(M,X)}_{\vartheta^{-1}(\mathrm{id})}$ which is fully faithful. By Lemma 2.1 there is a unique functor $j^*\colon \overline{\mathrm{Ginj}(\Lambda)} \to \overline{\mathrm{Ginj}(A)}$ given by $\binom{X}{Y}_\varphi \mapsto X$.

By construction of a distinguished triangle of the stable category $\overline{\mathcal{A}}$ of a Frobenius category $\mathcal{A}$ (see [8], Chapter 1, Section 2), it follows that the functors $i^!, i_*, j^*, j_*$ constructed above are exact functors.

By construction we have $\mathrm{Im}\, i_* \subseteq \mathrm{Ker}\, j^*$ and $\mathrm{Ker}\, j^* = \left\{ \binom{I}{Y}_\varphi : {}_AI \text{ is injective} \right\}$. Let $\binom{I}{Y}_\varphi \in \mathrm{Ker}\, j^*$. Then Lemma 2.1 yields an exact sequence $0 \to \mathrm{Ker}\,\vartheta(\varphi) \to Y \overset{\vartheta(\varphi)}{\to} \mathrm{Hom}_A(M,I) \to 0$ with $\mathrm{Ker}\,\vartheta(\varphi) \in \mathrm{Ginj}(B)$. Since M is cocompatible, we have $\mathrm{Ext}^1_B(\mathrm{Hom}_A(M,I), \mathrm{Ker}\,\vartheta(\varphi)) = 0$ by Lemma 1.2. Therefore $Y \cong \mathrm{Hom}_A(M,I) \oplus \mathrm{Ker}\,\vartheta(\varphi)$, and so

$$\begin{pmatrix} I \\ Y \end{pmatrix}_{\vartheta(\varphi)} \cong \begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \end{pmatrix}_{\mathrm{id}} \oplus \begin{pmatrix} 0 \\ \mathrm{Ker}\,\vartheta(\varphi) \end{pmatrix}.$$

This implies that $\binom{I}{Y}_\varphi \cong i_*(\mathrm{Ker}\,\vartheta(\varphi))$ in $\overline{\mathrm{Ginj}(\Lambda)}$, and hence $\mathrm{Im}\, i_* = \mathrm{Ker}\, j_*$.

Let $\binom{X}{Y}_\varphi \in \Lambda$-mod, $X' \in A$-mod and $Y' \in B$-mod. Clearly we have the isomorphism of abelian groups which is natural in both positions

$$\mathrm{Hom}_B(Y', \mathrm{Ker}\,\vartheta(\varphi)) \cong \mathrm{Hom}_{{}_\Lambda\mathscr{C}}\left(\begin{pmatrix} 0 \\ Y' \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix}_{\vartheta(\varphi)} \right) \tag{2.1}$$

given by $g \mapsto \binom{0}{\iota g}$, where $\iota\colon \mathrm{Ker}\,\vartheta(\varphi) \to Y$ is the inclusion. Also we have the isomorphism of abelian groups which is natural in both positions

$$\mathrm{Hom}_{{}_\Lambda\mathscr{C}}\left(\begin{pmatrix} X \\ Y \end{pmatrix}_{\vartheta(\varphi)}, \begin{pmatrix} X' \\ \mathrm{Hom}_A(M,X') \end{pmatrix}_{\mathrm{id}} \right) \cong \mathrm{Hom}_A(X, X') \tag{2.2}$$

given by $\binom{f}{\mathrm{Hom}(\mathrm{id}_M, f)\vartheta(\varphi)} \mapsto f$.

Let $\binom{X}{Y}_\varphi \in \mathrm{Ginj}(\Lambda)$, $X' \in \mathrm{Ginj}(A)$ and $Y' \in \mathrm{Ginj}(B)$. Then a morphism $\binom{0}{g}$: $\binom{0}{Y'} \to \binom{X}{Y}_{\vartheta(\varphi)}$ in ${}_\Lambda\mathscr{C}$ factors through an injective object $\binom{0}{J} \bigoplus \binom{I}{\mathrm{Hom}_A(M,I)}_{\mathrm{id}}$ in ${}_\Lambda\mathscr{C}$ if and only if the induced B-map $Y' \to \mathrm{Ker}\,\vartheta(\varphi)$ factors through an injective B-module J. This implies that the isomorphism (2.1) induces the following isomorphism, which is natural in both positions:

$$\mathrm{Hom}_{\overline{\mathrm{Ginj}(B)}}(Y', \mathrm{Ker}\,\vartheta(\varphi)) \cong \mathrm{Hom}_{\overline{\mathrm{Ginj}(\Lambda)}}\left(\begin{pmatrix} 0 \\ Y' \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi}\right).$$

That is, $(i_*, i^!)$ is an adjoint pair.

It is easy to see that a morphism $\begin{pmatrix} f \\ \mathrm{Hom}(\mathrm{id}_M, f)\vartheta(\varphi) \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix}_{\vartheta(\varphi)} \to \begin{pmatrix} X' \\ \mathrm{Hom}_A(M, X') \end{pmatrix}_{\mathrm{id}}$ in ${}_{\Lambda}\mathscr{C}$ factors through an injective object $\begin{pmatrix} 0 \\ J \end{pmatrix} \bigoplus \begin{pmatrix} I \\ \mathrm{Hom}_A(M, I) \end{pmatrix}_{\mathrm{id}}$ in ${}_{\Lambda}\mathscr{C}$ if and only if $f\colon X \to X'$ factors through an injective A-module I. This implies that the isomorphism (2.2) induces the following isomorphism, which is natural in both positions:

$$\mathrm{Hom}_{\overline{\mathrm{Ginj}(\Lambda)}}\left(\begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi}, \begin{pmatrix} X' \\ \mathrm{Hom}_A(M, X') \end{pmatrix}_{\vartheta^{-1}(\mathrm{id})}\right) \cong \mathrm{Hom}_{\overline{\mathrm{Ginj}(A)}}(X, X').$$

That is, (j^*, j_*) is an adjoint pair.

Now Theorem 2.5 follows from the right version of Theorem 2.4. □

Lemma 2.6. *Let* $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a Gorenstein algebra and* M_B *a projective module. Then* $\mathrm{Hom}_A(M, X) \in \mathrm{Ginj}(B)$ *for* $X \in \mathrm{Ginj}(A)$.

P r o o f. Let $I^\bullet\colon \ldots \to I^{-1} \to I^0 \xrightarrow{d'^0} I^1 \to \ldots$ be a complete A-injective resolution with $X \cong \mathrm{Im}\, d'^0$. Then Proposition 2.2 (8) and (3) imply that $\mathrm{Hom}_A(M, I^\bullet)$ is exact, and hence $\mathrm{Hom}_A(M, X) \cong \mathrm{Im}(\mathrm{Hom}(\mathrm{id}_M, d'^0))$. Since M_B is projective, it follows that $\mathrm{Hom}_A(M, I^\bullet)$ is an exact sequence of injective B-modules. Since Λ is Gorenstein and M_B is projective, it follows from Proposition 2.2 (5) that B is also Gorenstein. Let J be an injective B-module. Then $\mathrm{proj.dim}_B\, J < \infty$, and hence $\mathrm{Hom}_B(J, \mathrm{Hom}_A(M, I^\bullet))$ is exact. This implies that $\mathrm{Hom}_A(M, I^\bullet)$ is a complete injective resolution. Thus $\mathrm{Hom}_A(M, X) \in \mathrm{Ginj}(B)$. □

Theorem 2.7. *Let* $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a Gorenstein algebra and* M_B *a projective module. Then we have the following recollement:*

$$\begin{tikzcd} \overline{\mathrm{Ginj}(B)} \arrow[r, "i_*"] & \overline{\mathrm{Ginj}(\Lambda)} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \overline{\mathrm{Ginj}(A)} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}.$$

P r o o f. It follows from Proposition 2.2 (8) and (4) that M is cocompatible. Therefore Theorem 2.5 implies that there are exact functors $i^!, i_*, j^*, j_*$ such that they make the right recollement.

If a map $\begin{pmatrix} f \\ g \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix}_{\vartheta(\varphi)} \to \begin{pmatrix} X' \\ Y' \end{pmatrix}_{\vartheta(\varphi')}$ factors through an injective object $\begin{pmatrix} 0 \\ J \end{pmatrix} \bigoplus \begin{pmatrix} I \\ \mathrm{Hom}_A(M, I) \end{pmatrix}_{\mathrm{id}}$ in ${}_{\Lambda}\mathscr{C}$, then $g\colon Y \to Y'$ factors through an injective B-module $J \oplus \mathrm{Hom}_A(M, I)$, where $\mathrm{Hom}_A(M, I)$ is an injective B-module since M_B

is projective. By Lemma 2.6 there exists a unique functor $i^*\colon \overline{\mathrm{Ginj}(\Lambda)} \to \overline{\mathrm{Ginj}(B)}$ given by $\begin{pmatrix} X \\ Y \end{pmatrix}_\varphi \mapsto Y$.

We claim that there exists a unique fully faithful functor $j_!\colon \overline{\mathrm{Ginj}(A)} \to \overline{\mathrm{Ginj}(\Lambda)}$ given by $X \mapsto \begin{pmatrix} X \\ J \end{pmatrix}_\sigma$, where $J \in \mathrm{inj}(B)$ such that there is an exact sequence $0 \to \operatorname{Ker}\vartheta(\sigma) \to J \stackrel{\vartheta(\sigma)}{\to} \mathrm{Hom}_A(M,X) \to 0$ with $\operatorname{Ker}\vartheta(\sigma) \in \mathrm{Ginj}(B)$. We justify this claim as follows. Let ${}_AX \in \mathrm{Ginj}(A)$. By Lemma 2.6 $\mathrm{Hom}_A(M,X) \in \mathrm{Ginj}(B)$, hence there is an exact sequence $0 \to \operatorname{Ker}\overline{\sigma} \to J \stackrel{\overline{\sigma}}{\to} \mathrm{Hom}_A(M,X) \to 0$ with $\operatorname{Ker}\overline{\sigma} \in \mathrm{Ginj}(B)$ and $J \in \mathrm{inj}(B)$. Let $f\colon X' \to X$ be an A-map in $\mathrm{Ginj}(A)$ and $J' \in \mathrm{inj}(B)$ such that $0 \to \operatorname{Ker}\overline{\sigma}' \to J' \stackrel{\overline{\sigma}'}{\to} \mathrm{Hom}_A(M,X') \to 0$ is exact with $\operatorname{Ker}\overline{\sigma}' \in \mathrm{Ginj}(B)$. Since $\mathrm{Ext}^1_B(J', \operatorname{Ker}\overline{\sigma}) = 0$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\overline{\sigma}' \arrow[r, "\iota'"] \arrow[d] & J' \arrow[r, "\overline{\sigma}'"] \arrow[d, "g"] & \mathrm{Hom}_A(M,X') \arrow[r] \arrow[d, "\mathrm{Hom}(\mathrm{id}_M, f)"] & 0 \\
0 \arrow[r] & \operatorname{Ker}\overline{\sigma} \arrow[r, "\iota"] & J \arrow[r, "\overline{\sigma}"] & \mathrm{Hom}_A(M,X) \arrow[r] & 0.
\end{tikzcd}$$

Note that one can take $f = \mathrm{id}_X$. If we have another map $g'\colon J' \to J$ such that $\overline{\sigma}g' = \mathrm{Hom}(\mathrm{id}_M, f)\overline{\sigma}'$, then $g - g'$ factors through $\operatorname{Ker}\overline{\sigma}$. Since $\operatorname{Ker}\overline{\sigma} \in \mathrm{Ginj}(B)$, we have an epimorphism $\widetilde{\sigma}\colon \widetilde{J} \to \operatorname{Ker}\overline{\sigma}$ with $\widetilde{J}$ injective. Then we easily see that $\begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} f \\ g' \end{pmatrix}$ factors through an injective Λ-module $\begin{pmatrix} 0 \\ \widetilde{J} \end{pmatrix}$, and hence $\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g' \end{pmatrix}$ in $\overline{\mathrm{Ginj}(\Lambda)}$ and the object $\begin{pmatrix} X \\ J \end{pmatrix}_\sigma \in \overline{\mathrm{Ginj}(\Lambda)}$ $(\sigma = \vartheta^{-1}(\overline{\sigma}))$ is independent of the choice of J. Thus we get a unique functor $j_!\colon \mathrm{Ginj}(A) \to \overline{\mathrm{Ginj}(\Lambda)}$ given by $X \mapsto \begin{pmatrix} X \\ J \end{pmatrix}_\sigma$ and $f \mapsto \begin{pmatrix} f \\ g \end{pmatrix}$. Assume that $f\colon X' \to X$ factors through an injective A-module I with $f = f_2 f_1$. Since $\mathrm{Hom}_A(M,I) \in \mathrm{inj}(B)$, it is a projective object in $\mathrm{Ginj}(B)$. Thus there is $\alpha\colon \mathrm{Hom}_A(M,I) \to J$ such that $\mathrm{Hom}(\mathrm{id}_M, f_2) = \overline{\sigma}\alpha$. Since

$$\overline{\sigma}(g - \alpha\,\mathrm{Hom}(\mathrm{id}_M, f_1)\overline{\sigma}') = 0,$$

there is $\widetilde{g}\colon J' \to \operatorname{Ker}\overline{\sigma}$ such that $\iota\widetilde{g} = g - \alpha\,\mathrm{Hom}(\mathrm{id}_M, g_1)\overline{\sigma}'$. Let $\widetilde{\sigma}\colon \widetilde{J} \to \operatorname{Ker}\overline{\sigma}$ be an epimorphism with $\widetilde{J}$ injective. Then we get $\beta\colon J' \to \widetilde{J}$ such that $\widetilde{\sigma}\beta = \widetilde{g}$. Thus $\begin{pmatrix} f \\ g \end{pmatrix}$ factors through an injective object $\begin{pmatrix} 0 \\ \widetilde{J} \end{pmatrix} \bigoplus \begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \end{pmatrix}_{\mathrm{id}}$ in ${}_\Lambda\mathscr{C}$ with $\begin{pmatrix} f_2 \\ (\iota\widetilde{\sigma}, \alpha) \end{pmatrix} \left(\begin{pmatrix} f_1 \\ \beta \\ \mathrm{Hom}(\mathrm{id}_M, f_1)\overline{\sigma}' \end{pmatrix} \right)$. Therefore $j_!\colon \mathrm{Ginj}(A) \to \overline{\mathrm{Ginj}(\Lambda)}$ induces a functor $\overline{\mathrm{Ginj}(A)} \to \overline{\mathrm{Ginj}(\Lambda)}$, again denoted by $j_!$, which is given by $X \mapsto \begin{pmatrix} X \\ J \end{pmatrix}_\sigma$ and $f \mapsto \begin{pmatrix} f \\ g \end{pmatrix}$. If $\begin{pmatrix} f \\ g \end{pmatrix}$ factors through an injective object $\begin{pmatrix} 0 \\ \widetilde{J} \end{pmatrix} \bigoplus \begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \end{pmatrix}_{\mathrm{id}}$ in ${}_\Lambda\mathscr{C}$, then f factors through an injective A-module I. Thus $j_!$ is fully faithful.

As in the proof of Theorem 2.5, we know that i^* and $j_!$ constructed above are exact functors.

Let $\binom{X}{Y}_\varphi \in \operatorname{Ginj}(\Lambda)$, $X' \in \operatorname{Ginj}(A)$ and $Y' \in \operatorname{Ginj}(B)$. Clearly we have an isomorphism

$$\operatorname{Hom}_{{}_\Lambda\mathscr{C}}\left(\begin{pmatrix} X \\ Y \end{pmatrix}_{\vartheta(\varphi)}, \begin{pmatrix} 0 \\ Y' \end{pmatrix}\right) \cong \operatorname{Hom}_B(Y, Y') \tag{2.3}$$

of abelian groups which is natural in both positions. It is easy to see that a morphism $\binom{0}{g} : \binom{X}{Y}_{\vartheta(\varphi)} \to \binom{0}{Y'}$ in ${}_\Lambda\mathscr{C}$ factors through an injective object $\binom{0}{J} \bigoplus \binom{I}{\operatorname{Hom}_A(M,I)}_{\mathrm{id}}$ in ${}_\Lambda\mathscr{C}$ if and only if $g\colon Y \to Y'$ factors through an injective B-module $J \oplus \operatorname{Hom}_A(M, I)$. Therefore the isomorphism (2.3) induces the following isomorphism which is natural in both positions:

$$\operatorname{Hom}_{\overline{\operatorname{Ginj}(\Lambda)}}\left(\begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi}, \begin{pmatrix} 0 \\ Y' \end{pmatrix}\right) \cong \operatorname{Hom}_{\overline{\operatorname{Ginj}(B)}}(Y, Y').$$

That is, (i^*, i_*) is an adjoint pair.

Let $\binom{f}{g} : \binom{X'}{J'}_{\sigma'} \to \binom{X}{Y}_\varphi$ be a Λ-map,

$$0 \longrightarrow \operatorname{Ker}\sigma' \longrightarrow J' \xrightarrow{\vartheta(\sigma')} \operatorname{Hom}_A(M, X') \longrightarrow 0$$

be an exact sequence with $\operatorname{Ker}\vartheta(\sigma') \in \operatorname{Ginj}(B)$ and J' injective. Then $\binom{f}{g}$ factors through an injective object $\binom{0}{J} \bigoplus \binom{I}{\operatorname{Hom}_A(M,I)}_{\mathrm{id}}$ in ${}_\Lambda\mathscr{C}$ if and only if $f\colon X' \to X$ factors through an injective A-module I. This implies that the map $f \mapsto \binom{f}{g} = j_!(f)$ gives rise to the following isomorphism which is natural in both positions:

$$\operatorname{Hom}_{\overline{\operatorname{Ginj}(\Lambda)}}\left(\begin{pmatrix} X' \\ J' \end{pmatrix}_{\sigma'}, \begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi}\right) \cong \operatorname{Hom}_{\overline{\operatorname{Ginj}(A)}}(X', X).$$

And that is, $(j_!, j^*)$ is an adjoint pair.

Therefore Theorem 2.4 yields the recollement of $\overline{\operatorname{Ginj}(\Lambda)}$ relative to $\overline{\operatorname{Ginj}(B)}$ and $\overline{\operatorname{Ginj}(A)}$ in Theorem 2.7. □

3. Strongly Gorenstein injective modules over triangular matrix algebras

In this section, we determine all strongly complete injective resolutions and all strongly Gorenstein injective modules over upper triangular matrix Artin algebras.

Following Bennis and Mahdou [4], a complete injective resolution of the form

$$I^\bullet\colon \ \ldots \xrightarrow{f} I \xrightarrow{f} I \xrightarrow{f} I \xrightarrow{f} \ldots$$

is called a strongly complete injective resolution and denoted by $(I^\bullet, f)$. A Λ-module X is called strongly Gorenstein injective (SGinjective for short) if $X \cong \operatorname{Im} f$ for some strongly complete injective resolution $(I^\bullet, f)$. Denote by $S\,\mathrm{GInj}(\Lambda)$ (or, $\mathrm{SGinj}(\Lambda)$) the full subcategory of SGinjective modules in Λ-Mod (or, in Λ-mod).

Lemma 3.1. *For an Artin algebra* Λ, *we have* $\mathrm{SGinj}(\Lambda) = \mathrm{SGInj}(\Lambda) \cap \Lambda\text{-mod}$.

Proof. This follows from [3], Remark 4.3, and [7], Proposition 1.1. $\square$

Given a datum (I, J, α, β, g), where I and J are injective A-module, and injective B-module, respectively, $g\colon I \to I$ is an A-map, and $\alpha\colon \operatorname{Hom}_A(M, I) \to J$, $\beta\colon J \to J$ are B-maps, put

$$X = \begin{pmatrix} I \\ \operatorname{Hom}_A(M, I) \oplus J \end{pmatrix}, \quad f = \begin{pmatrix} g \\ \begin{pmatrix} \operatorname{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix} \end{pmatrix}\colon X \longrightarrow X, \tag{3.1}$$

where the Λ-action on X is given by $(\varphi, 0)\colon M \otimes_B (\operatorname{Hom}_A(M, I) \oplus J) \to I$, where $\varphi\colon M \otimes_B \operatorname{Hom}_A(M, I) \to I$ is the natural A-map. Therefore $X = \begin{pmatrix} I \\ \operatorname{Hom}_A(M,I) \end{pmatrix}_\varphi \oplus \begin{pmatrix} 0 \\ J \end{pmatrix}$ is an injective Λ-module. It is clear that $f\colon X \to X$ is a Λ-map. Here I and J could be zero.

Consider the following conditions (i)–(v):

(i) $I^\bullet\colon \ \ldots \longrightarrow I \xrightarrow{g} I \xrightarrow{g} I \xrightarrow{g} \ldots$ is exact;

(ii) $J^\bullet\colon \ \ldots \longrightarrow J \xrightarrow{\beta} J \xrightarrow{\beta} J \xrightarrow{\beta} \ldots$ is a complete B-injective resolution;

(iii) $\beta\alpha + \alpha \operatorname{Hom}(\mathrm{id}_M, g) = 0$;

(iv) if $gh = 0, \alpha(h) + \beta(j) = 0$, then there exists $(h', j') \in \operatorname{Hom}_A(M, I) \oplus J$ such that $h = gh', j = \alpha(h') + \beta(j')$;

(v) for any indecomposable injective A-module I', if

$$(t, s) \in \operatorname{Hom}_B(\operatorname{Hom}_A(M, I'), J) \oplus \operatorname{Hom}_A(I', I)$$

with $gs = 0, \alpha \operatorname{Hom}(\mathrm{id}_M, s) + \beta t = 0$, then there exists $(s', t') \in \operatorname{Hom}_A(I', I) \oplus \operatorname{Hom}_B(\operatorname{Hom}_A(M, I'), J)$ such that $s = gs'$ and $t = \alpha \operatorname{Hom}(\mathrm{id}_M, s') + \beta t'$.

Lemma 3.2. *Given a datum* (I, J, α, β, g) *as above, if the conditions* (i)–(v) *are satisfied, then*

$$X^\bullet\colon \ \dots \longrightarrow X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \dots \tag{3.2}$$

is a strongly complete Λ*-injective resolution; conversely, any strongly complete* Λ*-injective resolution is of the form* (3.2) *with* X *and* f *given in* (3.1), *satisfying the conditions* (i)–(v).

Proof. "$\Rightarrow$" It follows from (i)–(iii) that

$$\begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g^2) & 0 \\ \beta\alpha + \alpha\,\mathrm{Hom}(\mathrm{id}_M, g) & \beta^2 \end{pmatrix} = 0.$$

Note that (iv) implies $\mathrm{Ker}\begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix} \subseteq \mathrm{Im}\begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix}$, it follows that

$$\mathrm{Ker}\begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix} = \mathrm{Im}\begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix},$$

and hence (i) implies that the sequence $X^\bullet$ is exact. Let J' be an indecomposable injective B-module. Then $\mathrm{Hom}_{\Lambda\mathscr{C}}\left(\begin{pmatrix} 0 \\ J' \end{pmatrix}, \begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \end{pmatrix}_{\mathrm{id}}\right) = 0$, and hence

$$\mathrm{Hom}_\Lambda\left(\begin{pmatrix} 0 \\ J' \end{pmatrix}, X^\bullet\right) \cong \mathrm{Hom}_{\Lambda\mathscr{C}}\left(\begin{pmatrix} 0 \\ J' \end{pmatrix}, H(X^\bullet)\right) \cong \mathrm{Hom}_B(J', J^\bullet)$$

is exact by (ii). Let I' be an indecomposable injective A-module. Then

$$\mathrm{Hom}_{\Lambda\mathscr{C}}\left(\begin{pmatrix} I' \\ \mathrm{Hom}_A(M,I') \end{pmatrix}_{\mathrm{id}}, H(X)\right) \cong \mathrm{Hom}_A(I', I) \oplus \mathrm{Hom}_B(\mathrm{Hom}_A(M, I'), J).$$

Put $Y = \mathrm{Hom}_A(I', I) \oplus \mathrm{Hom}_B(\mathrm{Hom}_A(M, I'), J)$. By (i)–(iii), we see that the sequence

$$\dots \longrightarrow Y \xrightarrow{\phi} Y \xrightarrow{\phi} Y \xrightarrow{\phi} \dots$$

is a complex, where $\phi(s,t) = (gs, \alpha\,\mathrm{Hom}(\mathrm{id}_M, s) + \beta t)$ for any $(s,t) \in \mathrm{Hom}_A(I', I) \oplus \mathrm{Hom}_B(\mathrm{Hom}_A(M, I'), J)$, and by (v) it is exact. This means that

$$\mathrm{Hom}_\Lambda\left(\begin{pmatrix} I' \\ \mathrm{Hom}_A(M, I') \end{pmatrix}, X^\bullet\right) \cong \mathrm{Hom}_{\Lambda\mathscr{C}}\left(\begin{pmatrix} I' \\ \mathrm{Hom}_A(M, I') \end{pmatrix}_{\mathrm{id}}, H(X^\bullet)\right)$$

is exact. Therefore $X^\bullet$ is a strongly complete Λ-injective resolution.

"$\Leftarrow$" Let $X^\bullet\colon \ldots \to X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \ldots$ be a strongly complete Λ-injective resolution. Then X is of the form given in (3.1). Write f as

$$\begin{pmatrix} g \\ \begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & \gamma \\ \alpha & \beta \end{pmatrix} \end{pmatrix}\colon \begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \oplus J \end{pmatrix} \to \begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \oplus J \end{pmatrix}.$$

Since $\mathrm{Hom}_{\Lambda\mathscr{C}}\Bigl(\begin{pmatrix} 0 \\ J \end{pmatrix}, \begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \end{pmatrix}_{\mathrm{id}}\Bigr) = 0$, we get $\gamma = 0$, i.e., f is given as in (3.1). By $\mathrm{Ker}\, f = \mathrm{Im}\, f$, we have $\mathrm{Ker}\, g = \mathrm{Im}\, g$ and $\mathrm{Ker}\begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix} = \mathrm{Im}\begin{pmatrix} \mathrm{Hom}(\mathrm{id}_M, g) & 0 \\ \alpha & \beta \end{pmatrix}$. These imply that (i), (iii), (iv) are satisfied and $\beta^2 = 0$. By the exactness of $\mathrm{Hom}_{\Lambda\mathscr{C}}\Bigl(\begin{pmatrix} 0 \\ J' \end{pmatrix}, H(X^\bullet)\Bigr)$ and of $\mathrm{Hom}_{\Lambda\mathscr{C}}\Bigl(\begin{pmatrix} I' \\ \mathrm{Hom}_A(M,I') \end{pmatrix}, H(X^\bullet)\Bigr)$, we see that $\mathrm{Hom}_B(J', J^\bullet)$ is exact and that (v) is satisfied. This completes the proof. $\square$

Keeping the notations as before, we put

$$D({}_AM)^\perp = \{L \in B\text{-mod}\colon\ \mathrm{Ext}^i_B(D({}_AM), L) = 0,\ \forall\, i \geqslant 1\};$$
$$\begin{pmatrix} 0 \\ D({}_AM)^\perp \cap \mathrm{SGinj}(B) \end{pmatrix} = \Bigl\{\begin{pmatrix} 0 \\ J \end{pmatrix} \in \Lambda\text{-mod}\colon\ J \in D({}_AM)^\perp \cap \mathrm{SGinj}(B)\Bigr\};$$
$$\begin{pmatrix} \mathrm{SGinj}(A) \\ \mathrm{Hom}_A(M, \mathrm{SGinj}(A)) \end{pmatrix} = \Bigl\{\begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \end{pmatrix} \in \Lambda\text{-mod}\colon\ I \in \mathrm{SGinj}(A)\Bigr\}.$$

Theorem 3.3. *Let N be a Λ-module and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Then N is a SGinjective Λ-module if and only if one of the following holds:*

(1) $N \in \begin{pmatrix} 0 \\ D({}_AM)^\perp \cap \mathrm{SGinj}(B) \end{pmatrix}$;

(2) $N \cong \begin{pmatrix} L \\ \mathrm{Hom}_A(M,L) \end{pmatrix} \in \begin{pmatrix} \mathrm{SGinj}(A) \\ \mathrm{Hom}_A(M, \mathrm{SGinj}(A)) \end{pmatrix}$, *where L admits a strongly complete A-injective resolution $\ldots \to I \xrightarrow{g} I \xrightarrow{g} I \xrightarrow{g} \ldots$ and* $\mathrm{Im}\,\mathrm{Hom}(\mathrm{id}_M, g) = \mathrm{Hom}_A(M, \mathrm{Im}\, g)$;

(3) $N \cong \mathrm{Im}\, f = \begin{pmatrix} \mathrm{Im}\, g \\ (h,j) \in \mathrm{Hom}_A(M,I) \oplus J \mid gh = 0, \alpha(h) + \beta(j) = 0 \end{pmatrix}$, *where f is given in* (3.1), *I and J are, respectively, an arbitrary nonzero injective A-module and injective B-module, and α, β, g satisfy the conditions* (i)–(v).

Proof. "$\Leftarrow$" If $N \in \begin{pmatrix} 0 \\ D({}_AM)^\perp \cap \mathrm{SGinj}(B) \end{pmatrix}$, then $N \cong \mathrm{Im}\,\beta$ with $J^\bullet\colon \ldots \to J \xrightarrow{\beta} J \xrightarrow{\beta} J \xrightarrow{\beta} \ldots$ a strongly complete B-injective resolution. Note that $N \in D({}_AM)^\perp$ implies that

$$\mathrm{Hom}_\Lambda\Bigl(\begin{pmatrix} I \\ \mathrm{Hom}_A(M,I) \end{pmatrix}, J^\bullet\Bigr) \cong \mathrm{Hom}_B(\mathrm{Hom}_A(M,I), J^\bullet)$$

is exact for any indecomposable injective A-module I. Therefore $J^\bullet$ is also a strongly complete Λ-injective resolution, and N is a SGinjective Λ-module.

Let $N \cong \binom{L}{\mathrm{Hom}_A(M,L)} \in \binom{\mathrm{SGinj}(A)}{\mathrm{Hom}_A(M,\mathrm{SGinj}(A))}$, where $L \cong \mathrm{Im}\, g \in \mathrm{SGinj}(A)$ with a strongly complete A-injective resolution $\ldots \to I \xrightarrow{g} I \xrightarrow{g} I \xrightarrow{g} \ldots$ such that $\mathrm{Im}\,\mathrm{Hom}(\mathrm{id}_M, g) = \mathrm{Hom}_A(M, \mathrm{Im}\, g)$. It is easy to verify that the sequence of injective objects in ${}_\Lambda\mathscr{C}$

$$\ldots \longrightarrow \binom{I}{\mathrm{Hom}_A(M,I)}_{\mathrm{id}} \xrightarrow{f} \binom{I}{\mathrm{Hom}_A(M,I)}_{\mathrm{id}} \xrightarrow{f} \binom{I}{\mathrm{Hom}_A(M,I)}_{\mathrm{id}} \xrightarrow{f} \ldots$$

is a strongly complete resolution, where $f = \binom{g}{\mathrm{Hom}(\mathrm{id}_M,g)}$. Hence [11], Proposition 3.6, implies that $N \cong \mathrm{Im}\, f$ is a SGinjective Λ-module.

Case (3) follows directly from Lemma 3.2.

"$\Rightarrow$" If N is a SGinjective Λ-module, then $N \cong \mathrm{Im}\, f$, where f is the Λ-map which occurred in a strongly complete Λ-injective resolution $X^\bullet$. By Lemma 3.2, $X^\bullet$ is of the form (3.2), where X and f are given in (3.1) satisfying the conditions (i)–(v). If $I = 0$ in (3.1), then $\alpha = 0$, $g = 0$, and hence by (ii) and (iv), we know that $X^\bullet$ is also a strongly complete B-injective resolution; and (v) means that $\mathrm{Hom}_B(D({}_AM), X^\bullet)$ is exact, which implies $N \cong \mathrm{Im}\, f \in D({}_AM)^\perp$. Thus N is of the form (1). If $J = 0$ in (3.1), then $\alpha = 0$, $\beta = 0$. By (i) and (v), we see that $\ldots \longrightarrow I \xrightarrow{g} I \xrightarrow{g} I \xrightarrow{g} \ldots$ is a strongly complete A-injective resolution; and by (iv), we have $\mathrm{Im}\,\mathrm{Hom}(\mathrm{id}_M, g) = \mathrm{Hom}_A(M, \mathrm{Im}\, g)$. It follows that

$$N \cong \mathrm{Im}\, f = \begin{pmatrix} \mathrm{Im}\, g \\ \mathrm{Im}\,\mathrm{Hom}(\mathrm{id}_M, g) \end{pmatrix} = \begin{pmatrix} \mathrm{Im}\, g \\ \mathrm{Hom}(M, \mathrm{Im}\, g) \end{pmatrix} \in \begin{pmatrix} \mathrm{SGinj}(A) \\ \mathrm{Hom}_A(M, \mathrm{SGinj}(A)) \end{pmatrix}.$$

Case (3) follows directly from (1) and (2). This completes the proof. $\square$

Corollary 3.4. *Let M be a cocompatible (A,B)-bimodule and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Then $\binom{X}{Y}_\varphi \in \mathrm{SGinj}(\Lambda)$ if and only if one of the following holds:*

(1) $\binom{X}{Y}_\varphi \in \binom{0}{\mathrm{SGinj}(B)}$;

(2) $\binom{X}{Y}_\varphi \cong \binom{X}{\mathrm{Hom}_A(M,X)} \in \binom{\mathrm{SGinj}(A)}{\mathrm{Hom}_A(M,\mathrm{SGinj}(A))}$;

(3) $\binom{X}{Y}_\varphi \cong \binom{X}{\mathrm{Hom}_A(M,X) \oplus L}$, *where $X \in \mathrm{SGinj}(A)$ and $L \in \mathrm{SGinj}(B)$.*

P r o o f. "$\Rightarrow$" This follows from Theorem 3.3.

"$\Leftarrow$" If $\binom{X}{Y}_\varphi \in \binom{0}{\mathrm{SGinj}(B)}$, then $Y \cong \mathrm{Im}\,\beta$ with $J^\bullet\colon \ldots \to J \xrightarrow{\beta} J \xrightarrow{\beta} J \xrightarrow{\beta} \ldots$ a strongly complete B-injective resolution. But M is cocompatible, it follows that $\mathrm{Hom}_B(D({}_AM), J^\bullet)$ is exact and $Y \in D({}_AM)^\perp$. Therefore $\binom{X}{Y}_\varphi$ is a SGinjective Λ-module by Theorem 3.3.

Let $\begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi} \cong \begin{pmatrix} X \\ \mathrm{Hom}_A(M,X) \end{pmatrix} \in \begin{pmatrix} \mathrm{SGinj}(A) \\ \mathrm{Hom}_A(M,\mathrm{SGinj}(A)) \end{pmatrix}$, where $X \cong \mathrm{Im}\, g \in \mathrm{SGinj}(A)$ with a strongly complete A-injective resolution $I^{\bullet}\colon \ldots \to I \xrightarrow{g} I \xrightarrow{g} I \xrightarrow{g} \ldots$. Then $\mathrm{Hom}_A(M, I^{\bullet})$ is exact by (C1). Hence Theorem 3.3 implies that $\begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi}$ is a SGinjective Λ-module.

Case (3) follows directly from (1) and (2). $\square$

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