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# GLOBAL OUTPUT-FEEDBACK FINITE-TIME STABILIZATION FOR A CLASS OF STOCHASTIC NONLINEAR CASCADED SYSTEMS

QIXUN LAN, HUAWEI NIU, YAMEI LIU AND HUAFENG XU

In this paper, the problem of global finite-time stabilization via output-feedback is investigated for a class of stochastic nonlinear cascaded systems (SNCSs). First, based on the adding a power integrator technique and the homogeneous domination approach, a global output-feedback finite-time control law is constructed for the driving subsystem. Then, based on homogeneous systems theory, it is shown that under some mild conditions the global finite-time stability in probability of the driving subsystem implies the global finite-time stability in probability of the whole SNCS. Finally, a simulation example is given to illustrate the effectiveness of the proposed control design approach.

*Keywords:* stochastic nonlinear systems, cascaded systems, output-feedback stabilization, finite-time control

*Classification:* 68M15,39A13

## 1. INTRODUCTION

Nonlinear cascaded systems (NCSs) present an important class of nonlinear systems. In general, a cascaded system can be divided into two or more subsystems with cascaded structure. And many practical systems can be ascribed to cascaded systems, such as, material processing systems, chemical processes systems, robots systems, multi-agent systems, and so on [17]. Moreover, for many highly nonlinear and strong coupling systems, cascaded based control strategy usually has the advantage of reducing complexity of the control law design and the difficulty of the stability analysis. The problems of global stability analysis and stabilization of NCSs have been received a lot of attentions recently [1, 4, 13, 17, 34]. Global output-feedback stabilization (GOFS) of nonlinear systems is a challenging problem in the control field, because the so-called separation principle usually does not hold for nonlinear systems. In fact, counterexamples were given in [23] illustrating that GOFS of nonlinear systems is impossible in general, without introducing extra growth conditions on the unmeasurable states of the system. Since then, the GOFS problems for nonlinear systems have received intensive attentions [11, 30, 34, 38]. At the same time, with the development of stochastic system theory [22, 32, 33], GOFS for stochastic nonlinear systems (SNSs) has been an active

area of research [10, 15, 20, 21] ever since the first result on GOFs for SNS was proposed in [2].

Nonsmooth control method is a new control method developed recently [12, 25, 29]. Compared with smooth control methods, the closed-loop systems under nonsmooth control law usually demonstrates faster convergence rate, higher control precision as well as better disturbance rejection properties [12, 29]. In view of these advantages, nonsmooth control methods have recently received a lot of attentions [7, 11, 12, 18], and have been successfully used in practical systems [3, 5, 6, 7, 8, 9, 24, 26, 27, 28, 31]. Recently, the definition of finite-time stability and the finite-time stability theorem for SNS were proposed in [36]. Based on [36], [14] considered the problem of almost surely finite-time stabilization for a class of SNSs by using adding a power integrator technique. [35] not only improved the definition of finite-time stability in probability for SNSs further, but also proposed the sufficient condition to ensure the existence of a solution for a stochastic system. [37] and [16] considered the global output feedback finite-time stabilization (GOFFTS) problem for a class of SNSs. Currently, it is just the underway step on studying the nonsmooth analysis and synthesis and still needs further investigation.

Consider stochastic nonlinear cascaded systems (SNCSs) described by

$$d\zeta = \eta(\zeta) dt + f_0(\zeta, y) dt + g_0^T(\zeta, y) d\omega \tag{1}$$

$$\begin{cases} dx_i = x_{i+1} dt + f_i(x) dt + g_i^T(x) d\omega, & i = 1, \dots, n - 1, \\ dx_n = u dt + f_n(x) dt + g_n^T(x) d\omega \\ y = x_1 \end{cases} \tag{2}$$

where  $(\zeta^T, x^T) = (\zeta_1, \dots, \zeta_m, x_1, x_2, \dots, x_n)^T \in \mathbb{R}^{m+n}$ ,  $u \in \mathbb{R}$  and  $y \in \mathbb{R}$  are the system states, input and output of the system, respectively.  $\omega$  is an  $m$  – dimensional Brownian motion defined on the complete probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  with  $\Omega$  being a sample space,  $\mathcal{F}$  being a  $\sigma$  – field,  $\{\mathcal{F}_t\}_{t \geq 0}$  being a filtration, and  $P$  being a probability measure. The drift terms  $\eta, f_i$  and the diffusion terms  $g_i$   $i = 0, 1, \dots, n$ , are assumed to be continuous functions, and satisfy  $\eta(0) = 0, f_i(0) = 0$  and  $g_i(0) = 0, i = 0, 1, \dots, n$ . A wide class of SNSs can be described by SNCS (1)–(2). In literature, for cascaded (or interconnected) nonlinear systems, Lyapunov’s method [1, 17], input-to-state stability (ISS) [12], stochastic ISS [21], integral ISS [34], small-gain techniques [13], and homogeneous techniques [4, 7] are the main fundamental approaches. Among the existing results about cascaded (or interconnected) nonlinear systems, there is few results regarding to the finite-time stabilization problems for cascaded nonlinear systems, except [4, 12, 19]. The authors of [19] consider the finite-time stability for a class of cascaded time-varying systems firstly, and proposed a forward completeness condition to guarantee the global finite-time stability of the whole cascaded system. [12] provides a new framework for tackling finite-time control problems, and proposed finite-time ISS (FTISS), finite-time input-to-output stability, and finite-time small-gain theorem for nonlinear cascaded systems. Recently, based on homogeneous technique and the adding a power integrator technique, [4] further develops the results of [19]; The merits of this method is that one does not need to construct proper Lyapunov function to test ISS (or integral ISS) or matching conditions imposed on the driven subsystem, thus it can be used to solve the global stabilization problems for a wide class of nonlinear cascaded systems. It is should

be pointed out that all the mentioned results (i. e., [4, 12, 19]) are focus on the state-feedback stabilization problem of determined nonlinear cascaded systems. However, to the best of the authors' knowledge, the existing results about GOFFTS of nonlinear systems are mainly focused on the deterministic counterpart with  $\zeta \equiv 0$ , such as, [11, 18]. There are few results for the GOFFTS problem of SNSs. Motivated by the finite-time stability theory for SNSs [14, 35, 36], [37] and [16] consider the GOFFTS problem of different classes of SNSs by adding a power integrator technique and homogeneous domination approach. How to design a global output-feedback finite-time control law for SNCS (1)–(2) is still an open problem.

In this paper, we will address GOFFTS problem for SNCS (1)–(2). To solve the GOFFTS problem of SNCS (1)–(2), the main difficulty lies: (i) There is no existing result in literature about the GOFFTS problems for SNCS (1)–(2), even for its deterministic counterpart. (ii) The drift terms  $f, f_i$  and the diffusion terms  $g_i, i = 0, 1, \dots, n$ , may be not satisfy local Lipschitz condition, which means that SNCS (1)–(2) may have more than one weak solution. (iii) Compared with deterministic nonlinear systems, SNCS (1)–(2) involves not only the gradient but also the Hessian terms in the design procedure, so that the control law design for system (1)–(2) is much more difficult and tedious, even impossible [20]. Therefore, how to global stabilize SNCS (1)–(2) in finite time by an output feedback control law is more challenging. Motivated by the finite-time stability theory for SNSs in [14, 35, 36], the sufficient condition for the existence of the solutions of stochastic system [35] and the homogeneous systems theory [4], it is shown that GOFFTS problem for SNCS (1)–(2) is solvable. Specifically, based on adding a power integrator method and homogeneous system theory, we first construct a homogeneous output feedback finite-time control law for the nominal driving subsystem (2). Secondly, for the driving subsystem (2), a scaling gain is introduced into output feedback control law to dominate the drift and diffusion terms. Then according to the conditions imposed on the driven subsystem (1), we show that the driven subsystem (1) is global bounded by using homogeneous properties. Finally, a numerical example is proposed to illustrate the effectiveness of the proposed control method.

## 2. PRELIMINARIES AND PROBLEM DESCRIPTION

In this section, we introduce some useful definitions and lemmas which will used throughout the paper.

Consider the following stochastic nonlinear system

$$dx = h(x) dt + g^T(x) dw, \quad x(0) = x_0 \in \mathbb{R}^n \quad (3)$$

where  $x \in \mathbb{R}^n$  is the state,  $\omega$  is a  $r$  – dimensional Brownian motion,  $h(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g^T(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r}$  are continuous functions, and vanish at the origin, that is,  $h(0) = 0, g(0) = 0$ .

**Definition 2.1.** (Mao [22], Deng and Kristic [2]) For any given  $V(x) \in \mathbb{C}^2$ , associated with stochastic system (3), the differential operator  $\mathcal{L}$  is defined as

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x} h(x) + \frac{1}{2} \text{trace} \left\{ g(x) \frac{\partial^2 V}{\partial x^2} g^T(x) \right\}.$$

**Definition 2.2.** (Khoo et al. [14]) The trivial solution of (3) is said to be finite-time stable in probability if the following statements hold:

- (i) Finite-time attractiveness in probability: The trivial solution of (3) is said to be finite-time attractive in probability, if the stochastic system admits a solution (either in the strong sense or in the weak sense) for any initial data  $x_0 \in \mathbb{R}^n$ , denoted by  $x(t; x_0)$ ; moreover, any initial value  $x_0 \in \mathbb{R}^n \setminus \{0\}$ , the first hitting time  $\tau_{x_0} = \inf\{t; x(t; x_0) = 0\}$ , which is called the stochastic settling time, is finite almost surely, that is,  $P\{\tau_{x_0} < \infty\} = 1$ ;
- (ii) Stable in probability: The trivial solution is said to be stable in probability if for every pair of  $\varepsilon \in (0, 1)$  and  $r > 0$ , there exists a  $\delta = \delta(\varepsilon, r) > 0$  such that  $P\{|x(t; x_0)| < r, \text{ for all } t \geq 0\} \geq 1 - \varepsilon$ , whenever  $|x_0| < \delta$ .

**Lemma 2.3.** (Yin and Khoo [35]) Suppose that there exists a nonnegative function  $V(x) \in \mathbb{C}^2$ , which is radially unbounded, that is,  $\lim_{|x| \rightarrow \infty} V(x) = +\infty$ ; moreover, (3) is autonomous. If  $\mathcal{L}V(x) \leq 0, \forall x \in \mathbb{R}^n$ , then (3) has a solution for any initial data.

**Lemma 2.4.** (Yin et al. [36]) For stochastic system (3), if there exists a  $\mathbb{C}^2$  Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $\mathcal{K}_\infty$  class functions  $\mu_1$  and  $\mu_2$ , positive real numbers  $c > 0$  and  $0 < \gamma < 1$ , such that for all  $x \in \mathbb{R}^n$  and  $t \geq 0$ ,

$$\mu_1(|x|) \leq V(x) \leq \mu_2(|x|), \mathcal{L}V(x) \leq -c(V(x))^\gamma, \tag{4}$$

then the trivial solution of (3) is finite-time stable in probability.

In what follows, we will introduce the notion of weighted homogeneity and some useful properties of homogeneous systems [25].

**Definition 2.5.** For fixed coordinates  $(x_1, \dots, x_n) \in \mathbb{R}^n$  and real numbers  $r_i > 0, i = 1, \dots, n$ :

- (i) the dilation  $\Delta_\varepsilon(x) = (\varepsilon^{r_1}x_1, \dots, \varepsilon^{r_n}x_n), \forall \varepsilon > 0$ , with  $r_i$  being called as the weights of the coordinates( for simplicity of notation, we define dilation weight  $\Delta = (r_1, \dots, r_n)$  );
- (ii) a function  $V \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$  is said to be homogeneous of degree  $\tau$  if there is a real number  $\tau \in \mathbb{R}$  such that  $\forall x \in \mathbb{R} \setminus \{0\}, \varepsilon > 0, V(\Delta_\varepsilon(x)) = \varepsilon^\tau V(x_1, \dots, x_n)$ ;
- (iii) a vector field  $f \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$  is said to be homogeneous of degree  $\tau$  if there is a real number  $\tau \in \mathbb{R}$  such that for  $i = 1, \dots, n \forall x \in \mathbb{R} \setminus \{0\}, \varepsilon_i > 0, f_i(\Delta_\varepsilon(x)) = \varepsilon^{\tau+r_i} f_i(x_1, \dots, x_n)$ ;
- (iv) a homogeneous  $p$ -norm is defined as  $\|x\|_{\Delta, p} = (\sum_{i=1}^n |x_i|^{p/r_i})^{1/p}, \forall x \in \mathbb{R}^n$ , for a constant  $p \geq 1$ .

**Lemma 2.6.** Given a dilation weight  $\Delta = (r_1, \dots, r_n)$ , suppose  $V_1(x)$  and  $V_2(x)$  are homogeneous functions of degree  $\tau_1$  and  $\tau_2$ , respectively. Then  $V_1(x)V_2(x)$  is also homogeneous with respect to the same dilation weight  $\Delta$ . Moreover, the homogeneous degree of  $V_1 \cdot V_2$  is  $\tau_1 + \tau_2$ .

**Lemma 2.7.** Suppose  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a homogeneous function of degree  $\tau$  with respect to the dilation weight  $\Delta$ . Then the following holds:

- (i)  $\partial V/\partial x_i$  is homogeneous of degree  $\tau - r_i$  with  $r_i$  being the homogeneous weight of  $x_i$ ;
- (ii) there is a constant  $c$  such that  $V(x) \leq c\|x\|_{\Delta}^{\tau}$ . Moreover, if  $V(x)$  is positive definite,  $\underline{c}\|x\|_{\Delta}^{\tau} \leq V(x)$  for a constant  $\underline{c} > 0$ .

**Lemma 2.8.** Let  $f(\cdot)$  be a continuous vector function on  $\mathbb{R}^n$  such that the trivial solution  $x = 0$  of system  $\dot{x} = f(x)$  is asymptotically stable. Suppose  $f(\cdot)$  is homogeneous of degree  $\alpha$  with respect to  $(r_1, \dots, r_n)$ . Then for any positive integer  $p$  and any  $\sigma > p \times \max\{r_1, \dots, r_n\}$ , there exists a  $\mathbb{C}^p$  homogeneous Lyapunov function  $V(x)$  of degree  $\sigma$  with respect to  $(r_1, \dots, r_n)$ . As a direct consequence the time-derivative  $\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x)$  is homogeneous of degree  $\sigma + \alpha$  with respect to  $(r_1, \dots, r_n)$ .

The next three lemmas, which were first introduced in [29], play key roles in constructing the control law.

**Lemma 2.9.** Let  $c$  and  $d$  be positive constants. Given any positive number  $\gamma > 0$ , then, for  $x \in \mathbb{R}, y \in \mathbb{R}$

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-c/d}|y|^{c+d}.$$

**Lemma 2.10.** Let  $p$  be an odd real number and  $x, y \in \mathbb{R}$ , then

$$|x^p - y^p| \leq 2^{1-p}|x - y|^p, \text{ for } p \in (0, 1), \quad |x - y|^p \leq 2^{p-1}|x^p - y^p|, \text{ for } p \geq 1.$$

**Lemma 2.11.** Let  $p \in \mathbb{R}_{odd}^{\geq 1}$  and  $x, y$  be real-valued functions, then, for a constant  $c > 0$  the following inequality holds

$$|x^p - y^p| \leq p|x - y|(x^{p-1} + y^{p-1}) \leq c|x - y|(|x - y|^{p-1} + |y|^{p-1}).$$

In this paper, the following assumptions are needed.

**Assumption 2.1.** There are constants  $\tau \in (-\frac{1}{2n}, 0)$ ,  $c_1 > 0$  and  $c_2 > 0$  such that for  $i = 1, \dots, n$

$$|f_i(\cdot)| \leq c_1 \left( |x_1|^{\frac{r_i+\tau}{r_1}} + |x_2|^{\frac{r_i+\tau}{r_2}} + \dots + |x_i|^{\frac{r_i+\tau}{r_i}} \right), \tag{5}$$

$$|g_i(\cdot)| \leq c_2 \left( |x_1|^{\frac{2r_i+\tau}{2r_1}} + |x_2|^{\frac{2r_i+\tau}{2r_2}} + \dots + |x_i|^{\frac{2r_i+\tau}{2r_i}} \right), \tag{6}$$

where  $r_1 \in (0, \frac{1}{2})$  is a ratio of two odd numbers, and  $r_i$  is defined as

$$r_i = r_{i-1} + \tau, \quad i = 2, \dots, n + 1. \tag{7}$$

**Assumption 2.2.** The system

$$d\zeta = \eta(\zeta) dt \tag{8}$$

is homogeneous of degree  $\tau_1 < 0$  with respect to  $s = (s_1, \dots, s_m)$  and is globally finite-time stable. For  $f_0(\zeta, y) = (f_{01}(\zeta, y), \dots, f_{0m}(\zeta, y))^T$ ,  $g_0(\zeta, y) = (g_{01}(\zeta, y), \dots, g_{0m}(\zeta, y))^T$ , there exist continuous functions  $F_{0i}(\zeta, y) \geq 0$ ,  $G_{0i}(\zeta, y) \geq 0$ ,  $i = 1, \dots, m$  with  $F_{0i}(\zeta, 0) = G_{0i}(\zeta, 0) = 0$ , such that for all  $y \in \mathbb{R}$ ,  $|f_{0i}(\zeta, y)| \leq F_{0i}(\zeta, y)$ ,  $|g_{0i}(\zeta, y)| \leq G_{0i}(\zeta, y)$ , and for all  $\varepsilon > 0$

$$\begin{aligned} \lim_{\varepsilon \rightarrow +\infty} \frac{F_{0i}(\varepsilon^{s_1} \zeta_1, \dots, \varepsilon^{s_m} \zeta_m, y)}{\varepsilon^{s_i + \tau_0}} &= M_{1i}(\zeta, y), \\ \lim_{\varepsilon \rightarrow +\infty} \frac{G_{0i}(\varepsilon^{s_1} \zeta_1, \dots, \varepsilon^{s_m} \zeta_m, y)}{\varepsilon^{s_i + \frac{\tau_0}{2}}} &= M_{2i}(\zeta, y) \end{aligned} \tag{9}$$

where  $\tau_0 = 0$  or  $-\min\{s_1, \dots, s_m\} \leq \tau_0 < \tau_1$ ,  $M_{1i}(\zeta, y) \geq 0$ ,  $M_{2i}(\zeta, y) \geq 0$ ,  $i = 1, \dots, m$ , are continuous functions.

For simplicity, we assume that  $\tau = -\frac{p}{q}$  with  $p$  being an even integer and  $q$  being an odd integer. Under this assumption and the definition of  $r_i$ ,  $r_i$  will always be a ratio of odd numbers.

For SNCS (1), (2), under Assumptions 2.1 and 2.2, the control objective is to design a nonsmooth dynamic output-feedback control law

$$\dot{\chi} = \phi(\chi, y), \quad u = \varphi(\chi, y), \tag{10}$$

such that the closed-loop system consisting of (1), (2) and (10) is globally finite-time stable in probability (GFTSiP).

**Remark 2.12.** For Assumption 2.1, there are several points should be pointed out:

- (i) The condition that  $r_1 \in (0, \frac{1}{2})$  is a ratio of two odd numbers plays a key role to avoiding nonsense terms during the design procedure. According to the systems' dimension, the structure of drift and diffusion terms, one can choose appropriate values for  $r_1$  and  $\tau$ .
- (ii) If one chooses  $\tau = 0$  in Assumption 2.1, then Assumption 2.1 reduces to Assumption 1 in [10], and A1 and B2 in [21]. In this case, according to the design methods in this paper, the proposed control law will render the the considered SNCSs globally asymptotically stable in probability.
- (iii) The existing results about GOFs problems [2, 10, 20, 21] of SNS are obtained based on the assumption that  $f_i(x)$  and  $g_i(x)$  are local Lipschitz functions with  $f(0) = g(0) = 0$ . However, in this paper,  $\tau \in (-\frac{1}{2n}, 0)$ , this means that  $f_i(x)$  and  $g_i(x)$  are allowed to be non-Lipschitz continuous functions.

**Remark 2.13.** (Khoo et al. [14]) considered the finite-time stabilization problem of subsystem (2) as  $\tau = 0$ . However, as  $\tau < 0$ , the proposed method in [14] is invalid. Under Assumption 2.1, both the methods proposed in [37] and [16] can be used to solve the GOFs problem for SNS (2), but these two methods do not suitable for SNCS (1)–(2). Under Assumptions 2.1 and 2.2, how to design a global output feedback finite-time

control law for SNCS (1)–(2) by partial states feedback, there is no existing results to refer in the literature. In this case, if we can construct a control law based on  $y = x_1$  and the estimations of the other states of system (2), and design a  $\mathbb{C}^2$  Lyapunov function  $V$  satisfying the condition of Lemma 2.4, then the trivial solution of the closed-loop system of SNCS (1)–(2) is GFTSiP, that is, the global output feedback finite-time stabilization problem of SNCS (1)–(2) is solvable.

**Remark 2.14.** To ensure that the behavior of the nonlinear terms  $f_{0i}(\zeta, y)$  and  $g_{0i}(\zeta, y)$ ,  $i = 1, \dots, m$  are "not bad" when  $\zeta$  gets large, the homogeneous inequalities (9) are introduced in Assumption 2.2 which is different from the famous ISS (or FTISS) condition. To illustrate this point, consider the following SNS

$$d\zeta = -\zeta^{\frac{1}{3}} dt + \zeta y dt + \zeta \sin(y) d\omega \quad (11)$$

where  $y$  is considered as an update law. Obviously, SNS (11) is not ISS with  $y$  as the input. However, it is easy to show that system  $d\zeta = -\zeta^{\frac{1}{3}} dt$  is homogeneous of degree  $\tau_1 = -\frac{1}{3}$  with respect  $s_1 = \frac{1}{2}$ . By choosing  $F_{01}(\zeta, y) = |\zeta y|$  and  $G_{01}(\zeta, y) = |\zeta y|$  and  $\tau_0 = 0$ , through simple calculations, it is not difficult to obtain that  $\lim_{\varepsilon \rightarrow +\infty} \frac{F_{01}(\varepsilon^{s_1} \zeta, y)}{\varepsilon^{s_1 + \tau_0}} = |\zeta y|$  and  $\lim_{\varepsilon \rightarrow +\infty} \frac{G_{01}(\varepsilon^{s_1} \zeta, y)}{\varepsilon^{s_1 + \frac{\tau_0}{2}}} = |\zeta y|$ , which imply SNS (11) satisfies the condition of Assumption 2.2.

### 3. MAIN RESULTS

In this section, we will present a recursive design method to construct global finite-time stabilizing law via output-feedback for SNCS (1)–(2). The design procedure is composed of two steps: the first step is to design a global output-feedback finite-time control law for the driving subsystem (2), and the second step is to show that the states of the driven subsystem (1) are globally bounded.

#### 3.1. Global output-feedback finite-time control law design for driving subsystem (2)

To achieve the objective, we first consider the following nominal system of the driving subsystem (2)

$$\begin{aligned} dz_i &= z_{i+1} dt, \quad i = 1, \dots, n-1, \\ dz_n &= v dt, \quad y = z_1. \end{aligned} \quad (12)$$

##### 3.1.1. Global output-feedback finite-time control law for system (12)

**Theorem 3.1.** For any constant  $\tau \in (-\frac{1}{2n}, 0)$ , there is a homogeneous output feedback control law of degree  $\tau$  rendering system (12) is globally finite-time stable.

*Proof.* The proof of this theorem is divided into three steps: First, a homogeneous finite-time control law is constructed for system (12) by using the adding a power integrator technique. Then, a homogeneous reduced-order observer is constructed whose



gains to be determined later. Finally, we show that the closed-loop system is global finite-time stable in probability by choosing appropriate observer gains.

A. *Finite-time State feedback control law design*

**Initial Step:** For system (12), a Lyapunov function is constructed as  $V_1(z_1) = \frac{r_1}{4} z_1^{\frac{4}{r_1}}$ , where  $r_1$  and  $\tau$  are defined in (7). Clearly, the virtual control law  $z_2^* = -n z_1^{\frac{r_2}{r_1}}$  renders

$$\mathcal{L}V_1(z_1) = z_1^{\frac{4-r_1}{r_1}} \dot{z}_2 = z_1^{\frac{4-r_1}{r_1}} z_2^* + z_1^{\frac{4-r_1}{r_1}} (z_2 - z_2^*) \leq -n \xi_1^{4+\tau} + \xi_1^{4-r_1} (z_2 - z_2^*) \quad (13)$$

where  $\xi_1 = z_1^{\frac{1}{r_1}}$ .

**Inductive Step:** Suppose at step  $i-1$ , there exist a  $\mathbb{C}^2$  Lyapunov function  $V_{i-1}(\bar{z}_{i-1})$ , which is positive definite and proper, and satisfying

$$V_{i-1} \leq 2(\xi_1^4 + \dots + \xi_{i-1}^4), \quad (14)$$

with a set of virtual control laws  $z_1^*, \dots, z_i^*$  defined as

$$\begin{aligned} z_1^* &= 0, & \xi_1 &= z_1^{\frac{1}{r_1}} - z_1^{*\frac{1}{r_1}} \\ z_k^* &= -\beta_{k-1} \xi_{k-1}^{r_k}, & \xi_k &= z_k^{\frac{1}{r_k}} - z_k^{*\frac{1}{r_k}}, \quad k = 2, \dots, i, \end{aligned} \quad (15)$$

with  $\beta_1 > 0, \dots, \beta_{i-1} > 0$ , such that

$$\mathcal{L}V_{i-1}(\bar{z}_{i-1}) \leq -(n-i+2) (\xi_1^{4+\tau} + \xi_2^{4+\tau} + \dots + \xi_{i-1}^{4+\tau}) + \xi_{i-1}^{4-r_{i-1}} (x_i - x_i^*). \quad (16)$$

In what follows, we will show that (16) still holds at step  $i$ . To prove this claim, consider the Lyapunov function

$$V_i(\bar{z}_i) = V_{i-1}(\bar{z}_{i-1}) + W_i(\bar{z}_i), \quad \text{with } W_i(\bar{z}_i) = \int_{z_i^*}^{z_i} (s^{\frac{1}{r_i}} - z_i^{*\frac{1}{r_i}})^{4-r_i} ds. \quad (17)$$

The Lyapunov function  $V_i$  has some useful properties collected in the following propositions:

**Proposition 3.2.**  $V_i(\bar{z}_i)$  is  $\mathbb{C}^2$ , positive definite, proper, and satisfy the inequality  $V_i(\bar{z}_i) \leq 2(\xi_1^4 + \dots + \xi_i^4)$ .

By (17), we have

$$\begin{aligned} \mathcal{L}V_i(\bar{z}_i) &\leq -(n-i+2) (\xi_1^{4+\tau} + \xi_2^{4+\tau} + \dots + \xi_{i-1}^{4+\tau}) + \xi_{i-1}^{4-r_{i-1}} (z_i - z_i^*) \\ &\quad + \frac{\partial W_i}{\partial z_i} z_{i+1} + \sum_{k=1}^{i-1} \frac{\partial W_i}{\partial z_k} z_{k+1} \end{aligned} \quad (18)$$

For the second term in the righthand side of the inequality (18), by using Lemmas 2.10 and 2.11, we can show that there exists a constant  $a_1 > 0$  such that

$$\xi_{i-1}^{4-r_{i-1}}(z_i - z_i^*) \leq |\xi_{i-1}|^{4-r_{i-1}} 2^{1-r_i} |\xi_i|^{r_i} \leq \frac{1}{2} \xi_{i-1}^{4+\tau} + a_1 \xi_i^{4+\tau}. \tag{19}$$

Similar to the proof of the proposition 3.2 in [16], for the last term in the righthand side of the inequality (18), we have the following proposition.

**Proposition 3.3.** There exists a positive constant  $a_2 > 0$  such that

$$\sum_{k=1}^{i-1} \frac{\partial W_i}{\partial z_k} z_{k+1} \leq \frac{1}{2} (\xi_1^{4+\tau} + \dots + \xi_{i-1}^{4+\tau}) + a_2 \xi_i^{4+\tau}.$$

According to the inequalities (18), (19), Propositions 3.2 and 3.3, we have

$$\mathcal{L}V_i(\bar{z}_i) \leq -(n-i+1) (\xi_1^{4+\tau} + \xi_2^{4+\tau} + \dots + \xi_{i-1}^{4+\tau}) + (a_1 + a_2) \xi_i^{4+\tau} + \xi_i^{4-r_i} z_{i+1}. \tag{20}$$

Clearly, if the virtual control law is chosen as  $z_{i+1}^* = -\beta_i \xi_i^{r_{i+1}} = -(n-i+1+a_1+a_2) \xi_i^{r_{i+1}}$ , then it follows from (20) that

$$\mathcal{L}V_i(\bar{z}_i) \leq -(n-i+1) (\xi_1^{4+\tau} + \xi_2^{4+\tau} + \dots + \xi_i^{4+\tau}) + \xi_i^{4-r_i} (z_{i+1} - z_{i+1}^*). \tag{21}$$

This completes the inductive proof.

Similarly, as  $i = n$ , there is a virtual control law  $z_{n+1}^* = -\beta_n \xi_n^{r_{n+1}}$  and a Lyapunov function  $V_n(\bar{z}_n)$  such that

$$\mathcal{L}V_n(\bar{z}_n) \leq -(\xi_1^{4+\tau} + \xi_2^{4+\tau} + \dots + \xi_n^{4+\tau}) + \xi_n^{4-r_n} (v - z_n^*). \tag{22}$$

*B. Homogeneous reduced-order observer design*

Since  $z_2, \dots, z_n$  in system (12) are not measurable, in this section, the following homogeneous observer is constructed for system (12)

$$\dot{\eta}_i = -\ell_{i-1} \hat{z}_i, \quad \hat{z}_i = (\eta_i + \ell_{i-1} \hat{z}_{i-1})^{\frac{r_i}{r_{i-1}}}, \quad i = 2, \dots, n, \tag{23}$$

where  $\hat{z}_1 = z_1$  and gains  $\ell_i > 0$ ,  $i = 1, \dots, n-1$  to be determined later.

Based on the estimated states, we design an output feedback control law of the form

$$v(\hat{z}) = -\beta_n \left( \hat{z}_n^{\frac{1}{r_n}} + \beta_{n-1}^{\frac{1}{r_n}} \left( \hat{z}_{n-1}^{\frac{1}{r_{n-1}}} + \dots + \beta_2^{\frac{1}{r_3}} \left( \hat{z}_2^{\frac{1}{r_2}} + \beta_1^{\frac{1}{r_2}} \hat{z}_1^{\frac{1}{r_1}} \right) \dots \right) \right)^{r_{n+1}} \tag{24}$$

where  $\hat{z} = (z_1, \hat{z}_2, \dots, \hat{z}_n)$ .

For  $i = 2, \dots, n$ , we choose

$$U_i = \int_{(\eta_i + \ell_{i-1} z_{i-1})^{\frac{r_i}{r_{i-1}}}}^{z_i^{\frac{4-r_{i-1}}{r_i}}} \frac{4-r_{i-1}}{r_{i-1}} \left( s^{\frac{r_i-1}{4-r_{i-1}}} - (\eta_i + \ell_{i-1} z_{i-1}) \right) ds, \tag{25}$$

and set  $e_i = (z_i - \hat{z}_i)^{\frac{1}{r_i}}$ ,  $i = 2, \dots, n$ ,  $z_{n+1} = v(\hat{z})$ . It is deduced from (12), (23) and (25) that

$$\begin{aligned} \mathcal{L}U_i &= \frac{\partial U_i}{\partial z_i} z_{i+1} + \frac{\partial U_i}{\partial z_{i-1}} z_i - \frac{\partial U_i}{\partial \eta_i} \ell_{i-1} \hat{z}_i \\ &= \frac{4 - r_{i-1}}{r_i} z_i^{\frac{4-r_{i-1}}{r_i}-1} \left( z_i^{\frac{r_i-1}{r_i}} - (\eta_i + \ell_{i-1} z_{i-1}) \right) z_{i+1} \\ &\quad - \ell_{i-1} e_i^{r_i} \left( z_i^{\frac{4-r_{i-1}}{r_i}} - \hat{z}_i^{\frac{4-r_{i-1}}{r_i}} \right) - \ell_{i-1} e_i^{r_i} \left( \hat{z}_i^{\frac{4-r_{i-1}}{r_i}} - (\eta_i + \ell_{i-1} z_{i-1})^{\frac{4-r_{i-1}}{r_i-1}} \right). \end{aligned} \tag{26}$$

Next, we estimate the nonlinear terms in right-hand side of (26). First, by Lemma 2.10 and  $e_i = (z_i - \hat{z}_i)^{\frac{1}{r_i}}$ , we have

$$\ell_{i-1} 2^{\frac{r_i+r_{i-1}-4}{r_i}} e_i^{4+\tau} \leq \ell_{i-1} e_i^{r_i} \left( z_i^{\frac{4-r_{i-1}}{r_i}} - \hat{z}_i^{\frac{4-r_{i-1}}{r_i}} \right), \quad i = 2, \dots, n. \tag{27}$$

To estimate the remaining terms of (26), we introduce the following propositions, the proof of which are omitted here for the space limitations, the interested reader can refer [16] for more details.

**Proposition 3.4.** There exist a constant  $\alpha_i > 0$  and a  $\mathcal{K}_\infty$  function  $h_i(\cdot)$  such that

$$\frac{4 - r_{i-1}}{r_i} z_i^{\frac{4-r_{i-1}}{r_i}-1} \left( z_i^{\frac{r_i-1}{r_i}} - s_i \right) z_{i+1} \leq \frac{1}{12} (\xi_{i-1}^{4+\tau} + \xi_i^{4+\tau} + \xi_{i+1}^{4+\tau}) + \alpha_i e_i^{4+\tau} + h_i(\ell_{i-1}) e_{i-1}^{4+\tau}.$$

**Proposition 3.5.** There exist a constant  $\bar{\alpha} > 0$  and a  $\mathcal{K}_\infty$  function  $h_n(\cdot)$  such that

$$\frac{4 - r_{n-1}}{r_n} z_n^{\frac{4-r_{n-1}}{r_n}-1} \left( z_n^{\frac{r_n-1}{r_n}} - s_n \right) v(\hat{z}) \leq \frac{1}{8} \sum_{i=1}^n \xi_i^{4+\tau} + \bar{\alpha} \sum_{i=2}^n e_i^{4+\tau} + h_n(\ell_{n-1}) e_{n-1}^{4+\tau}.$$

**Proposition 3.6.** There exists a  $\mathcal{K}_\infty$  function  $\hat{h}_i(\cdot)$  such that

$$-\ell_{i-1} e_i^{r_i} \left( \hat{z}_i^{\frac{4-r_{i-1}}{r_i}} - s_i^{\frac{4-r_{i-1}}{r_i-1}} \right) \leq e_i^{4+\tau} + \frac{1}{16} (\xi_i^{4+\tau} + \xi_{i-1}^{4+\tau}) + \hat{h}_i(\ell_{i-1}) e_{i-1}^{4+\tau}, \quad i = 3, \dots, n.$$

Consider  $U = \sum_{i=2}^n U_i$ , in conjunction with Propositions 3.3–3.6 leads to

$$\begin{aligned} \mathcal{L}U &= \sum_{i=1}^n \frac{1}{2} \xi_i^{4+\tau} - \left( \ell_1 2^{\frac{r_2+r_1-4}{r_2}} - \alpha_2 - \bar{\alpha} - h_3(\ell_2) - \hat{h}_3(\ell_2) \right) e_2^{4+\tau} \\ &\quad - \sum_{i=3}^n \left( \ell_{i-1} 2^{\frac{r_i+r_{i-1}-4}{r_i}} - \alpha_i - 1 - \bar{\alpha} - h_{i+1}(\ell_i) - \hat{h}_{i+1}(\ell_i) \right) e_i^{4+\tau} \\ &\quad - \left( \ell_{n-1} 2^{\frac{r_n+r_{n-1}-4}{r_n}} - 1 - \bar{\alpha} \right) e_n^{4+\tau}. \end{aligned} \tag{28}$$

C. *Determination of the homogeneous observer gains*

Since  $z_2, \dots, z_n$  are not measurable, the control law  $v = v(\hat{z})$  results a reductant term  $\xi_n^{4-r_n}(v(\hat{z}) - v^*(z))$  in (22). For this term, we have the following estimation.

**Proposition 3.7.** There exists a constant  $\tilde{\alpha} \geq 0$  such that

$$\xi_n^{4-r_n}(v(\hat{z}) - v^*(z)) \leq \sum_{i=1}^n \frac{1}{4} \xi_i^{4+\tau} + \sum_{i=1}^n \tilde{\alpha} e_i^{4+\tau}.$$

It follows from (22), (28) and Proposition 3.7 that

$$\begin{aligned} \mathcal{L}W &= - \sum_{i=1}^n \frac{1}{4} \xi_i^{4+\tau} - \left( \ell_1 2^{\frac{r_2+r_1-4}{r_2}} - \alpha_2 - \bar{\alpha} - \tilde{\alpha} - h_3(\ell_2) - \hat{h}_3(\ell_2) \right) e_2^{4+\tau} \\ &\quad - \sum_{i=3}^{n-1} \left( \ell_{i-1} 2^{\frac{r_i+r_{i-1}-4}{r_i}} - \alpha_i - 1 - \bar{\alpha} - \tilde{\alpha} - h_{i+1}(\ell_i) - \hat{h}_{i+1}(\ell_i) \right) e_i^{4+\tau} \\ &\quad - \left( \ell_{n-1} 2^{\frac{r_n+r_{n-1}-4}{r_n}} - 1 - \bar{\alpha} - \tilde{\alpha} \right) e_n^{4+\tau}, \end{aligned} \tag{29}$$

where the Lyapunov function  $W = V_n + U$ .

Obviously, if we choose

$$\begin{aligned} \ell_{n-1} &= 2^{\frac{4-r_n-r_{n-1}}{r_n}} \left( \frac{5}{4} + \bar{\alpha} + \tilde{\alpha} \right) \\ \ell_{i-1} &= 2^{\frac{4-r_i-r_{i-1}}{r_i}} \left( \frac{5}{4} + \alpha_i + \bar{\alpha} + \tilde{\alpha} + h_{i+1}(\ell_i) + \hat{h}_{i+1}(\ell_i) \right), \quad i = n-1, \dots, 3 \\ \ell_1 &= 2^{\frac{4-r_2-r_1}{r_2}} \left( \frac{1}{4} + \alpha_2 + \bar{\alpha} + \tilde{\alpha} + h_3(\ell_2) + \hat{h}_3(\ell_2) \right), \end{aligned} \tag{30}$$

then (30) in conjunction with (29) lead to

$$\mathcal{L}W = -\frac{1}{4} \left( \sum_{i=1}^n \xi_i^{4+\tau} + \sum_{i=2}^n e_i^{4+\tau} \right). \tag{31}$$

From the construction of  $W$ , one can show that  $W$  is positive definite and proper with respect to  $Z = (z_1, \dots, z_n, \eta_2, \dots, \eta_n)^T$ . Moreover, the closed-loop system composed by (12), (23) and (24) can be written as the following compact form

$$dZ = \Xi(Z) dt = (z_2, \dots, z_n, v, f_{n+1}, \dots, f_{2n-1})^T dt \tag{32}$$

with  $f_{n+i} = -\ell_i \hat{z}_{i+1}$ ,  $i = 1, \dots, n-1$ . In fact, by choosing the dilation weight  $\Delta = (\underbrace{r_1, r_2, \dots, r_n}_{\text{for } z_1, \dots, z_n}, \underbrace{r_1, r_2, \dots, r_{n-1}}_{\text{for } \eta_2, \dots, \eta_n})$ , with  $r_i$  defined in (7), it can be shown that system (32),

$W$  and the right hand side of (31) is homogeneous of degree  $\tau$ , 4 and  $4 + \tau$ , respectively,

with respect to  $\Delta$ . By Lemma 2.7, we can show that there exist two constant  $\theta_1 > 0$  and  $\theta_2 > 0$  such that

$$W \leq \theta_1 \|Z\|_{\Delta}^4, \quad \mathcal{L}W = \frac{\partial W}{\partial Z} \Xi(Z) \leq -\theta_2 \|Z\|_{\Delta}^{4+\tau}, \quad (33)$$

thus there exists a constant  $c > 0$  such that  $\mathcal{L}W \leq -cW^\gamma$ , with  $\gamma = \frac{4+\tau}{4} \in (0, 1)$ . Therefore, it can be concluded that the closed-loop system composed by (12), (23) and (24) is globally finite-time stable.  $\square$

**Remark 3.8.** To design and analysis of the control law for deterministic systems, a  $\mathbb{C}^1$  Lyapunov function is enough [18, 29]. However, for stochastic systems, due to the appearance of Hessian term, if one still use a  $\mathbb{C}^1$  Lyapunov function rather than a  $\mathbb{C}^2$  one, there will lead to some nonsense terms. This can be avoid, if one chooses a appropriate  $\mathbb{C}^2$  Lyapunov function, which can be shown by the proof of Proposition 3.2. For GOFTS problem of system (12), both the methods proposed in [37] and [16] can be used. Here, the analysis and design methods are different from [16, 37], by choosing appropriate dilation weight, homogeneous degree, less design parameters are needed, thus the proposed homogeneous Lyapunov function has more simple structure.

### 3.2. Global output-feedback finite-time control law for the driving subsystem (2)

Consider the following coordinate transformation

$$z_i = \frac{x_i}{L^{i-1}}, \quad v = \frac{u}{L^n}, \quad i = 1, \dots, n, \quad (34)$$

where  $L > 1$  is a constant to be determined. Under (34), driving subsystem (2) is transformed into the system

$$\begin{aligned} dz_i &= Lz_{i+1} dt + \frac{f_i(\cdot)}{L^{i-1}} dt + \frac{g_i^T(\cdot)}{L^{i-1}} d\omega, \quad i = 1, \dots, n-1, \\ dz_n &= Lv dt + \frac{f_n(\cdot)}{L^{n-1}} dt + \frac{g_n^T(\cdot)}{L^{n-1}} d\omega. \end{aligned} \quad (35)$$

For system (35), we design the following reduced-order homogeneous observer

$$\dot{\eta}_i = -L\ell_{i-1}\hat{z}_i, \quad \hat{z}_i = (\eta_i + \ell_{i-1}\hat{z}_{i-1})^{\frac{r_i}{r_{i-1}}}, \quad i = 2, \dots, n, \quad (36)$$

and the output feedback control law with the same structure as (24),

$$v(\hat{z}) = -\beta_n \left( \hat{z}_n^{\frac{1}{r_n}} + \beta_{n-1}^{\frac{1}{r_n}} \left( \hat{z}_{n-1}^{\frac{1}{r_{n-1}}} + \dots + \beta_2^{\frac{1}{r_3}} \left( \hat{z}_2^{\frac{1}{r_2}} + \beta_1^{\frac{1}{r_2}} \hat{z}_1^{\frac{1}{r_1}} \right) \dots \right) \right)^{r_{n+1}}, \quad (37)$$

where the gains  $\ell_i > 0$ ,  $i = 1, \dots, n-1$  are the same as those selected in Theorem 3.1.

**Theorem 3.9.** For the driving subsystem (2), if Assumption 2.1 holds, then the closed-loop system composed by (2), (36) and  $u = L^n v(\hat{z})$  is GFTSiP.

Proof. The closed-loop system consisting of (35), (36) and (37) can be written as the following compact form

$$dZ = \Xi(Z) dt + F(Z) dt + G^T(Z) d\omega, \tag{38}$$

where  $\Xi(Z)$  is defined in (32),  $F(Z) = (f_1, \frac{f_2}{L}, \dots, \frac{f_n}{L^{n-1}}, 0, \dots, 0)^T$  and  $G(Z) = (g_1, \frac{g_2}{L}, \dots, \frac{g_n}{L^{n-1}}, 0, \dots, 0)$ .

It follows from  $W(Z)$ , Definition 2.1 and (33) that

$$\mathcal{L}W \leq -L\theta_2 \|Z\|_{\Delta}^{4+\tau} + \frac{\partial W}{\partial Z} F(Z) + \frac{1}{2} \text{trace} \left\{ G(Z) \frac{\partial^2 W}{\partial Z^2} G^T(Z) \right\}. \tag{39}$$

Under the coordinate transformation (34), we deduce from Assumption 2.1 and the fact  $L > 1$  that

$$\left| \frac{f_i(\cdot)}{L^{i-1}} \right| \leq c_1 L^{1 - \frac{1}{2(i-1)\tau+1}} \left( |z_1|^{\frac{r_i+\tau}{r_1}} + \dots + |z_i|^{\frac{r_i+\tau}{r_i}} \right) \leq c_1 L^{1-\lambda_{i1}} \|Z\|_{\Delta}^{r_i+\tau}, \tag{40}$$

$$\left| \frac{g_i(\cdot)}{L^{i-1}} \right| \leq c_2 L^{\frac{1}{2} - \frac{1}{4(i-1)\tau+2}} \left( |z_1|^{\frac{2r_i+\tau}{2r_1}} + \dots + |z_i|^{\frac{2r_i+\tau}{2r_i}} \right) \leq c_2 L^{\frac{1}{2}-\lambda_{i2}} \|Z\|_{\Delta}^{r_i+\frac{\tau}{2}}, \tag{41}$$

where  $\lambda_{i1} = \frac{1}{2(i-1)\tau+1}, \lambda_{i2} = \frac{1}{4(i-1)\tau+2} \in (0, 1)$ .

As  $\frac{\partial W}{\partial Z_i}$  is homogeneous of degree  $4 - r_i$ . By Lemma 2.6, it is easy to show that there is a  $\delta_{i1}$  such that

$$\frac{\partial W}{\partial Z} F(Z) = \sum_{i=1}^n \frac{\partial W}{\partial Z_i} \left| \frac{f_i(\cdot)}{L^{i-1}} \right| \leq \sum_{i=1}^n \delta_{i1} L^{1-\lambda_{i1}} \|Z\|_{\Delta}^{4+\tau} \leq \delta_1 L^{1-\lambda_1} \|Z\|_{\Delta}^{4+\tau}, \tag{42}$$

where  $\lambda_1 = \min_{1 \leq i \leq n} \{\lambda_{i1}\} > 0$  and  $\delta_1 > 0$ .

Similarly, it can be shown that

$$\begin{aligned} \frac{1}{2} \text{trace} \left\{ G(Z) \frac{\partial^2 W}{\partial Z^2} G^T(Z) \right\} &\leq \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^n \left| \frac{\partial^2 W}{\partial Z_i \partial Z_j} \right| |G_i(Z)| |G_j(Z)| \\ &\leq \frac{1}{2} m \sqrt{m} c_2^2 L^{1-\lambda_{i2}-\lambda_{j2}} \sum_{i,j=1}^n \|Z\|_{\Delta}^{4-r_i-r_j} \|Z\|_{\Delta}^{r_i+\frac{\tau}{2}} \|Z\|_{\Delta}^{r_j+\frac{\tau}{2}} \\ &\leq \delta_2 L^{1-\lambda_2} \|Z\|_{\Delta}^{4+\tau}, \end{aligned} \tag{43}$$

where  $\lambda_2 = \min_{1 \leq i,j \leq n} \{\lambda_{i2} + \lambda_{j2}\}$  is a positive constant.

Substituting (42) and (43) into (39) yields

$$\begin{aligned} \mathcal{L}W &\leq -L\theta_2 \|Z\|_{\Delta}^{4+\tau} + \delta_1 L^{1-\lambda_1} \|Z\|_{\Delta}^{4+\tau} + \delta_2 L^{1-\lambda_2} \|Z\|_{\Delta}^{4+\tau} \\ &\leq -L(\theta_2 - (\delta_1 + \delta_2)L^{-\lambda_0}) \|Z\|_{\Delta}^{4+\tau} \end{aligned} \tag{44}$$

where  $\lambda_0 = \min\{\lambda_1, \lambda_2\}$ . Obviously, if we choose  $L > L^* = \max \left\{ ((\delta_1 + \delta_2)/\theta_2)^{\frac{1}{\lambda_0}}, 1 \right\}$ , then we have

$$\mathcal{L}W \leq -\tilde{\mu} \|Z\|_{\Delta}^{4+\tau}, \tag{45}$$

for a positive constant  $\tilde{\mu}$ . By (33) and Lemma 2.7, it is not difficult to show that there is a constant  $\mu > 0$  such that  $\mathcal{L}W \leq -\mu W^\gamma$  with  $\gamma = \frac{4+\tau}{4} \in (0, 1)$ . By Lemmas 2.3 and 2.4, the closed-loop system (35), (36) and (37) is GFTSiP.

Note that coordinate transformation does not change the properties of system, therefore, the closed-loop system consisting of (2), (36) and  $u = L^n v(\hat{z})$  is GFTSiP. This completes the proof of Theorem 3.9.  $\square$

Now, we are in the position to propose our main results of this paper.

**Theorem 3.10.** For SNCS (1)–(2), if Assumptions 2.1 and 2.2 hold, then under the control law  $u = L^n v(\hat{z})$ , the closed-loop system is GFTSiP.

*Proof.* According to Theorem 3.9, we know that the subsystem (2) can be globally stabilized by  $u = L^n v(\hat{z})$  in finite-time in probability. To show the global finite-time stability of the closed-loop system composed by (1), (2), (36) and  $u = L^n v(\hat{z})$ . We only need to show that the states of driven subsystem (1) are bounded on the finite time interval  $[0, T_0]$  (here we assume that  $T_0$  is the stochastic settling time of driving subsystem (2)). In fact, based on Assumptions 2.1 and 2.2, we can prove the global boundedness of driven subsystem (1).

By Assumption 2.2 and Lemma 2.8, for system (8) we have that for all  $\sigma > r \times \max\{s_1, \dots, s_m\}$ , there exist constant  $a > 0$  and a  $\mathbb{C}^r$  Lyapunov function  $V_0(\zeta)$  such that

$$\mathcal{L}V_0(\zeta) \leq -aV_0^{\frac{\sigma+\tau_1}{\sigma}} \tag{46}$$

and

$$V_0(\Delta_\varepsilon^s \zeta) = \varepsilon^\sigma V_0(\zeta) \tag{47}$$

with  $s = (s_1, \dots, s_m)$  and  $\varepsilon > 0$ .

Consider Lyapunov function

$$V(\zeta, Z) = V_0(\zeta) + \frac{b}{\tilde{\mu}} W(Z) \tag{48}$$

where  $b > 0$  is a positive constant to be defined later.

By (45) and (46), we have

$$\begin{aligned} \mathcal{L}V(\zeta, Z) &\leq -\varphi_1(\zeta) - b\varphi_2(Z) + \sum_{i=1}^m F_{0i}(\zeta, z_1) \left| \frac{\partial V_0(\zeta)}{\partial \zeta_i} \right| \\ &\quad + \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\zeta)}{\partial \zeta_i \partial \zeta_j} \right| G_{0i}(\zeta, z_1) G_{0j}(\zeta, z_1) \end{aligned} \tag{49}$$

where

$$\varphi_1(\zeta) = aV_0^{\frac{\sigma+\tau_1}{\sigma}}(\zeta), \quad \varphi_2(Z) = \|Z\|_\Delta^{4+\tau}. \tag{50}$$

Let  $\varphi(\zeta, Z) = \varphi_1(\zeta) + b\varphi_2(Z)$ . By (46), it is easy to show that

$$\varphi_1(\Delta_\varepsilon^s \zeta) = \varepsilon^{\sigma+\tau_1} \varphi_1(\zeta). \tag{51}$$

Similarly, we also can show that

$$\varphi_2(\Delta_\varepsilon^l Z) = \varepsilon^{\sigma+\tau_1} \varphi_2(Z) \tag{52}$$

with  $l = (l_1, \dots, l_{2n-1})$ ,  $l_1 = \dots, l_{2n-1} = \frac{\sigma+\tau_1}{4+\tau}$ .

It follows from (51) and (52) that

$$\varphi(\Delta_\varepsilon^s \zeta, \Delta_\varepsilon^l Z) = \varepsilon^{\sigma+\tau_1} \varphi(\zeta, Z). \tag{53}$$

Let

$$\begin{aligned} \Psi(\zeta, Z) = & \varphi(\zeta, Z) - \sum_{i=1}^m F_{0i}(\zeta, z_1) \left| \frac{\partial V_0(\zeta)}{\partial \zeta_i} \right| \\ & - \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\zeta)}{\partial \zeta_i \partial \zeta_j} \right| G_{0i}(\zeta, z_1) G_{0j}(\zeta, z_1), \end{aligned} \tag{54}$$

and  $\mathbb{D} = \{(\zeta^T, Z^T) | \varphi(\zeta, Z) = 1, (\zeta^T, Z^T) \in \mathbb{R}^{2n+m-1}\}$ . Obviously,  $\mathbb{D}$  is a bounded closed set. Let  $\varepsilon = \varphi^{\frac{1}{\sigma+\tau_1}}(\zeta, Z)$ , by (53), we have

$$\begin{aligned} & \varphi((\varepsilon^{-1})^{s_1} \zeta_1, \dots, (\varepsilon^{-1})^{s_m} \zeta_m, (\varepsilon^{-1})^{l_1} z_1, \dots, (\varepsilon^{-1})^{l_{2n-1}} z_{2n-1}) \\ & = \varphi(\varphi(\zeta, Z)^{\frac{-s_1}{\sigma+\tau_1}} \zeta_1, \dots, \varphi(\zeta, Z)^{\frac{-s_m}{\sigma+\tau_1}} \zeta_m, \varphi(\zeta, Z)^{\frac{-l_1}{\sigma+\tau_1}} z_1, \dots, \varphi(\zeta, Z)^{\frac{-l_{2n-1}}{\sigma+\tau_1}} z_{2n-1}) \\ & = (\varphi(\zeta, Z)^{\frac{-1}{\sigma+\tau_1}})^{\sigma+\tau_1} \varphi(\zeta, Z) = 1. \end{aligned} \tag{55}$$

Define

$$\begin{aligned} \bar{\zeta} &= (\bar{\zeta}_1, \dots, \bar{\zeta}_m)^T, & \bar{Z} &= (\bar{z}_1, \dots, \bar{z}_{2n-1})^T \\ \bar{\zeta}_1 &= \varphi(\zeta, Z)^{\frac{-s_1}{\sigma+\tau_1}} \zeta_1, & \dots, & \bar{\zeta}_m = \varphi(\zeta, Z)^{\frac{-s_m}{\sigma+\tau_1}} \zeta_m, \\ \bar{z}_1 &= \varphi(\zeta, Z)^{\frac{-l_1}{\sigma+\tau_1}} z_1, & \dots, & \bar{z}_{2n-1} = \varphi(\zeta, Z)^{\frac{-l_{2n-1}}{\sigma+\tau_1}} z_{2n-1}. \end{aligned} \tag{56}$$

By (55), it is clear that  $\varphi(\bar{\zeta}, \bar{Z}) = 1$ , and  $(\bar{\zeta}^T, \bar{Z}^T) \in \mathbb{D}$ . By (55), we obtain that for all  $(\zeta^T, Z^T) \in \mathbb{R}^{2n+m-1} \setminus \{0\}$ , there exist  $\varepsilon = \varphi^{\frac{1}{\sigma+\tau_1}}(\zeta, Z)$  and  $(\bar{\zeta}^T, \bar{Z}^T) = (\bar{\zeta}_1, \dots, \bar{\zeta}_m, \bar{z}_1, \dots, \bar{z}_{2n-1}) \in \mathbb{D}$  such that

$$\begin{aligned} (\zeta^T, Z^T) &= (\zeta_1, \dots, \zeta_m, z_1, \dots, z_{2n-1}) = (\varepsilon^{s_1} \bar{\zeta}_1, \dots, \varepsilon^{s_m} \bar{\zeta}_m, \varepsilon^{l_1} \bar{z}_1, \dots, \varepsilon^{l_{2n-1}} \bar{z}_{2n-1}) \\ &= (\Delta_\varepsilon^s \bar{\zeta}, \Delta_\varepsilon^l \bar{Z}). \end{aligned} \tag{57}$$

Thus, we have

$$F_{0i}(\zeta, z_1) = F_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \varepsilon^{l_1} \bar{z}_1), \quad G_{0i}(\zeta, z_1) = G_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \varepsilon^{l_1} \bar{z}_1) \tag{58}$$



with  $(\bar{\zeta}^T, \bar{Z}^T) \in \mathbb{D}$ , which also implies

$$\begin{aligned} \lim_{\varepsilon \rightarrow +\infty} \frac{F_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1)}{\varepsilon^{s_i + \tau_0}} &= \lim_{\varepsilon \rightarrow +\infty} \frac{F_{0i}(\varepsilon^{s_1} \bar{\zeta}_1, \dots, \varepsilon^{s_m} \bar{\zeta}_m, \bar{z}_1)}{\varepsilon^{s_i + \tau_0}}, \\ \lim_{\varepsilon \rightarrow +\infty} \frac{G_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1)}{\varepsilon^{s_i + \frac{\tau_0}{2}}} &= \lim_{\varepsilon \rightarrow +\infty} \frac{G_{0i}(\varepsilon^{s_1} \bar{\zeta}_1, \dots, \varepsilon^{s_m} \bar{\zeta}_m, \bar{z}_1)}{\varepsilon^{s_i + \frac{\tau_0}{2}}} \end{aligned} \tag{59}$$

with  $\tau_0 = 0$  or  $-\min\{s_1, \dots, s_m\} \leq \tau_0 < \tau_1$ . According to Assumption 2.2, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow +\infty} \frac{F_{0i}(\varepsilon^{s_1} \bar{\zeta}_1, \dots, \varepsilon^{s_m} \bar{\zeta}_m, \bar{z}_1)}{\varepsilon^{s_i + \tau_0}} &= M_{1i}(\bar{\zeta}, \bar{z}_1), \\ \lim_{\varepsilon \rightarrow +\infty} \frac{G_{0i}(\varepsilon^{s_1} \bar{\zeta}_1, \dots, \varepsilon^{s_m} \bar{\zeta}_m, \bar{z}_1)}{\varepsilon^{s_i + \frac{\tau_0}{2}}} &= M_{2i}(\bar{\zeta}, \bar{z}_1). \end{aligned} \tag{60}$$

Let

$$k_1 = \max_{(\zeta, Z) \in \mathbb{D}} \left( \sum_{i=1}^m \left| \frac{\partial V_0(\zeta)}{\partial \zeta_i} \right| \right), \quad k_2 = \max_{(\zeta, Z) \in \mathbb{D}} \left( \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\zeta)}{\partial \zeta_i \partial \zeta_j} \right| \right). \tag{61}$$

By (60), it is can be proved that

$$0 < \lambda_0 < \min \left\{ \frac{1}{2k_1}, \frac{1}{\sqrt{2k_2}} \right\}, \tag{62}$$

there exist a constant  $\varepsilon_{\lambda_0}$  such that for all  $\varepsilon > \varepsilon_{\lambda_0}$

$$\left| \frac{F_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1)}{\varepsilon^{s_i + \tau_0}} - M_{1i}(\bar{\zeta}, \bar{z}_1) \right| \leq \lambda_0, \quad \left| \frac{G_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1)}{\varepsilon^{s_i + \frac{\tau_0}{2}}} - M_{2i}(\bar{\zeta}, \bar{z}_1) \right| \leq \lambda_0. \tag{63}$$

If we choose  $\varepsilon_0 = \max\{\varepsilon_{\lambda_0}, 1\}$ , then it follows from (63) that for all  $\varepsilon > \varepsilon_0$ ,

$$F_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1) \leq (M_{1i}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \varepsilon^{s_i + \tau_0}, \quad G_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1) \leq (M_{2i}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \varepsilon^{s_i + \frac{\tau_0}{2}}. \tag{64}$$

Let

$$\phi_{1i}(\bar{\zeta}) = \frac{\partial V_0(\bar{\zeta})}{\partial \bar{\zeta}_i}, \quad \phi_{2i}(\bar{\zeta}) = \frac{\partial^2 V_0(\bar{\zeta})}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j}, \tag{65}$$

then we can verify that

$$\begin{aligned} \phi_{1i}(\Delta_\varepsilon^s \bar{\zeta}) &= \varepsilon^{\sigma - s_i} \phi_{1i}(\bar{\zeta}) = \varepsilon^{\sigma - s_i} \frac{\partial V_0(\bar{\zeta})}{\partial \bar{\zeta}_i}, \\ \phi_{2i}(\Delta_\varepsilon^s \bar{\zeta}) &= \varepsilon^{\sigma - s_i - s_j} \phi_{2i}(\bar{\zeta}) = \varepsilon^{\sigma - s_i - s_j} \frac{\partial^2 V_0(\bar{\zeta})}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j}. \end{aligned} \tag{66}$$

Combining (53), (57), (64) and (66) together, we obtain from (54) that for  $\varepsilon > \varepsilon_0 \geq 1$

$$\begin{aligned}
 \Psi(\zeta, Z) &= \Psi(\Delta_\varepsilon^s \bar{\zeta}, \Delta_\varepsilon^l \bar{Z}) \\
 &= \varphi(\Delta_\varepsilon^s \bar{\zeta}, \Delta_\varepsilon^l \bar{Z}) - \sum_{i=1}^m F_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1) |\phi_{1i}(\Delta_\varepsilon^s \bar{\zeta})| \\
 &\quad - \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m |\phi_{2i}(\Delta_\varepsilon^s \bar{\zeta})| G_{0i}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1) G_{0j}(\Delta_\varepsilon^s \bar{\zeta}, \bar{z}_1) \\
 &\geq \varepsilon^{\sigma+\tau_0} \varphi(\bar{\zeta}, \bar{Z}) - \varepsilon^{\sigma+\tau_0} \left( \sum_{i=1}^m (M_{1i}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \left| \frac{\partial V_0(\bar{\zeta})}{\partial \bar{\zeta}_i} \right| \right. \\
 &\quad \left. + \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\bar{\zeta})}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j} \right| (M_{2i}(\bar{\zeta}, \bar{z}_1) + \lambda_0) (M_{2j}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \right). \tag{67}
 \end{aligned}$$

Next, we show that for all  $\varepsilon > \varepsilon_0$ , the following inequality holds

$$\begin{aligned}
 \Lambda(\bar{\zeta}, \bar{Z}) &= \varphi(\bar{\zeta}, \bar{Z}) - \sum_{i=1}^m (M_{1i}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \left| \frac{\partial V_0(\bar{\zeta})}{\partial \bar{\zeta}_i} \right| \\
 &\quad - \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\bar{\zeta})}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j} \right| (M_{2i}(\bar{\zeta}, \bar{z}_1) + \lambda_0) (M_{2j}(\bar{\zeta}, \bar{z}_1) + \lambda_0) > 0. \tag{68}
 \end{aligned}$$

Define  $\Omega = \{(\bar{\zeta}, \bar{Z}) | \Lambda(\bar{\zeta}, \bar{Z}) > 0, \varepsilon > \varepsilon_0, (\bar{\zeta}, \bar{Z}) \in \mathbb{D}\}$ ,  $\mathbb{D}_1 = \mathbb{D} \cap \Omega$ ,  $\mathbb{D}_2 = \mathbb{D} \cap \Omega^c$ , where  $\Omega^c$  is a complementary set of  $\Omega$ . It follows that  $\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2$ ,  $\mathbb{D}_1 \cap \mathbb{D}_2 = \emptyset$ .

By the definition of  $\mathbb{D}_1$ , we can show that for all  $\varepsilon > \varepsilon_0$ ,

$$\Lambda(\bar{\zeta}, \bar{Z}) > 0, \forall (\bar{\zeta}, \bar{Z}) \in \mathbb{D}_1. \tag{69}$$

Next, we show that for all  $\varepsilon > \varepsilon_0$ ,  $\Lambda(\bar{\zeta}, \bar{Z}) > 0, \forall (\bar{\zeta}, \bar{Z}) \in \mathbb{D}_2$ . Let  $\Gamma_1 = \min_{(\bar{\zeta}^T, \bar{Z}^T) \in \mathbb{D}} \{\varphi_2(\bar{Z})\}$ , then we have  $\Gamma_1 \geq 0$ . If  $\Gamma_1 = 0$ , then we have  $\varphi_2(\bar{Z}) = 0$ , which means that  $\bar{Z} = 0$ . Thus, there exists  $\bar{\mathbb{D}} = \{(\bar{\zeta}, \bar{Z}) | \bar{\zeta} \neq 0, \bar{Z} = 0\} \subset \mathbb{D}_2 = \mathbb{D} \cap \Omega^c$ . Let  $(\bar{\zeta}_0^T, 0) |_{\bar{\zeta}_0 \neq 0} \in \bar{\mathbb{D}}$ , then we have  $(\bar{\zeta}_0^T, 0) \in \mathbb{D}$  and  $(\bar{\zeta}_0^T, 0) \in \Omega^c$ . By (58), if  $\bar{Z} = 0$ , then one has  $Z = 0$ . According (60), if  $\bar{Z} = 0$ , then we have  $\bar{z}_1 = 0$  and  $M_{1i}(\bar{\zeta}, 0) = M_{2i}(\bar{\zeta}, 0) = 0$ . It means that  $\varphi(\bar{\zeta}_0, 0) = 1$  and  $\varphi(\bar{\zeta}_0, 0) \leq \left( \sum_{i=1}^m \lambda_0 \left| \frac{\partial V_0(\bar{\zeta})}{\partial \bar{\zeta}_i} \right| + \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\bar{\zeta})}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j} \right| \lambda_0^2 \right) |_{\bar{\zeta}=\bar{\zeta}_0}$ . By the definition of  $\varphi(\zeta, Z)$ , it follows that  $\varphi_1(\bar{\zeta}_0) = 1$  and

$$\varphi_1(\bar{\zeta}_0, 0) \leq \left( \sum_{i=1}^m \lambda_0 \left| \frac{\partial V_0(\bar{\zeta})}{\partial \bar{\zeta}_i} \right| + \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\bar{\zeta})}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j} \right| \lambda_0^2 \right) |_{\bar{\zeta}=\bar{\zeta}_0}.$$

By (62), we have  $\varphi_1(\bar{\zeta}_0, 0) \leq \lambda_0 k_1 + \lambda_0^2 k_2 < 1$ , which is a contradiction. Therefore, we have  $\Gamma_1 > 0$ . Note that  $\zeta$  and  $Z$  are bounded, thus there exists a constant  $\Gamma_2$  such

that

$$\begin{aligned}
 & - \sum_{i=1}^m (M_{1i}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \left| \frac{\partial V_0(\bar{\zeta})}{\partial \bar{\zeta}_i} \right| \\
 & - \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\bar{\zeta})}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j} \right| (M_{2i}(\bar{\zeta}, \bar{z}_1) + \lambda_0)(M_{2j}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \geq \Gamma_2,
 \end{aligned}$$

which means that  $\Gamma_2 < 0$ . Then we have

$$\begin{aligned}
 \Lambda(\bar{\zeta}, \bar{Z}) &= \varphi_1(\bar{\zeta}) + b\varphi_2(\bar{Z}) - \sum_{i=1}^m (M_{1i}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \left| \frac{\partial V_0(\bar{\zeta})}{\partial \bar{\zeta}_i} \right| \\
 & - \frac{1}{2} m \sqrt{m} \sum_{i,j=1}^m \left| \frac{\partial^2 V_0(\bar{\zeta})}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j} \right| (M_{2i}(\bar{\zeta}, \bar{z}_1) + \lambda_0)(M_{2j}(\bar{\zeta}, \bar{z}_1) + \lambda_0) \\
 & > b\Gamma_1 + \Gamma_2, \quad \forall (\bar{\zeta}^T, \bar{Z}^T) \in \mathbb{D}_2.
 \end{aligned} \tag{70}$$

Let  $b = \frac{-\Gamma_2 + \Gamma_3}{\Gamma_1}$ ,  $\Gamma_3 > 0$ . By (70), we have that for all  $\varepsilon > \varepsilon_0$ ,

$$\Lambda(\bar{\zeta}, \bar{Z}) > \Gamma_3 > 0, \quad \forall (\bar{\zeta}^T, \bar{Z}^T) \in \mathbb{D}_2. \tag{71}$$

By (69) and (71), we have that  $\forall \varepsilon > \varepsilon_0$ ,  $\Lambda(\bar{\zeta}, \bar{Z}) > 0$ . It follows from (67) that for all  $\varepsilon > \varepsilon_0$ ,  $\Psi(\zeta, Z) \geq \varepsilon^{\sigma + \tau_0} \Lambda(\bar{\zeta}, \bar{Z})$ .

Define  $\bar{\Omega} = \{(\zeta^T, Z^T) | (\zeta^T, Z^T) = (\Delta_\varepsilon^s \bar{\zeta}^T, \Delta_\varepsilon^l \bar{Z}^T), \varepsilon > \varepsilon_0, (\bar{\zeta}^T, \bar{Z}^T) \in \mathbb{D}\}$ , then we have  $\mathcal{L}V(\zeta, Z) \leq -\Psi(\zeta, Z) < 0$ ,  $(\zeta^T, Z^T) \in \bar{\Omega}$ , which implies that the states of system (1) are bounded. This completes the proof of the Theorem.  $\square$

**Remark 3.11.** Obviously, when  $\tau = 0$ , Assumption 2.1 reduces to the lower-triangular linear growth condition. Then the closed-loop system composed by (2), (36) and  $u = L^n v(\hat{z})$  is globally asymptotically stable in probability, instead of GFTSiP. Thus, in this case, the whole closed-loop system is globally asymptotically stable in probability.

### 3.3. Discussions and Extensions

It should be pointed out that Theorems 3.9 and 3.10 are obtained based on the assumption that system (2) is in the lower-triangular form. In fact, many practical systems may not have lower-triangular structures. Next, we will show that the result obtained in the preceding subsection can be further extended to a more general class of SNCSS which do not need driving subsystem (2) in the lower-triangular form. Due to the domination nature of the homogeneous domination approach, the results obtained can be extended if the driving system (2)'s equivalent system (35) meets the following more general assumption.

**Assumption 3.1.** There are constants  $\tau \in (-\frac{1}{2n}, 0)$ ,  $c_1 > 0, c_2 > 0$  and  $\alpha \in (0, 1)$  such that for  $i = 1, \dots, n$

$$\frac{|f_i(\cdot)|}{L^{i-1}} \leq c_1 L^\alpha \left( |z_1|^{\frac{r_i+\tau}{r_1}} + |z_2|^{\frac{r_i+\tau}{r_2}} + \dots + |z_n|^{\frac{r_i+\tau}{r_n}} \right), \tag{72}$$

$$\frac{|g_i(\cdot)|}{L^{i-1}} \leq c_2 L^{\alpha/2} \left( |z_1|^{\frac{2r_i+\tau}{2r_1}} + |z_2|^{\frac{2r_i+\tau}{2r_2}} + \dots + |z_n|^{\frac{2r_i+\tau}{2r_n}} \right), \tag{73}$$

where  $r_1 \in (0, \frac{1}{2})$  is a ratio of two odd numbers.

**Theorem 3.12.** For SNCS (1)–(2), if Assumptions 2.2 and 3.1 hold, then the closed-loop system composed by (1), (2), (36) and  $u = L^n v(\hat{z})$  is GFTSiP.

*Proof.* Under Assumption 3.1, we use the same coordination (34), observer (36) and control law (37) for stochastic nonlinear system (35). Under the new growth condition, the relations (40), (41), (42), (43) and consequently (44) need to be revised. Similar to that of Theorem 3.9, we can prove that by choosing  $L > L^* = \max \left\{ ((\delta_1 + \delta_2)/\theta_2)^{\frac{1}{1-\alpha}}, 1 \right\}$ , the closed-loop system consisting of (1), (36) and  $u = L^n v(\hat{z})$  is GFTSiP. Following the same line of the proof of Theorem 3.10, it is not difficult to show that the closed-loop system consisting of (1), (2), (36) and  $u = L^n v(\hat{z})$  is GFTSiP. This completes the proof of Theorem 3.12.  $\square$

**Remark 3.13.** It should be pointed out that the control parameters  $\beta_i$  and  $l_{i-1}, i = 1, \dots, n$  will increase significantly along with the dimension augment of each subsystem due to the nature of the adding a power integrator technique and the domination approach. The proposed design method is quite conservative for the neatness of the proof. The high-gain parameter  $L$  is utilised to dominate the drift and diffusion terms. Following general rules provided by the design procedure, one can choose some smaller parameters by trial and error methods in practice.

#### 4. SIMULATION EXAMPLE

Consider the following stochastic nonlinear cascaded system

$$d\zeta = -\frac{3}{2}\zeta^{\frac{1}{3}} dt + \zeta y dt + \frac{3}{10}\zeta \sin(y) d\omega \tag{74}$$

$$\begin{cases} dx_1 = x_2 dt + \frac{1}{5}x_1^{10/11} d\omega \\ dx_2 = x_3 dt + \frac{3}{10}x_2^{7/9} dt - \frac{3}{10}\sin(y) d\omega \\ dx_3 = u dt + \frac{1}{5}\cos(x_1)x_3^{6/7} d\omega, y = x_1 \end{cases} \tag{75}$$

In this simulation, we choose  $r_1 = \frac{11}{23}, \tau = -\frac{2}{23}$ , which imply that  $r_2 = \frac{9}{23}, r_3 = \frac{7}{23}$  and  $r_4 = \frac{5}{23}$ . It is easy to show the nonlinear terms in subsystem (75) satisfy Assumption 2.1. Noticed that, we have verified that the nonlinear terms in subsystem (74) satisfy Assumption 2.2 in Remark 2.14. Therefore, by Theorem 3.10, the global

output feedback finite-time control law for stochastic nonlinear cascaded system (74)–(75) can be designed as

$$\begin{aligned}
 u &= -L^3 b_3 \left( \hat{z}_3^{23/7} + b_2^{23/7} \left( \hat{z}_2^{23/9} + b_1^{23/9} y^{23/11} \right) \right)^{5/23} \\
 \dot{\eta}_2 &= -L\ell_1 \hat{z}_2, \quad \hat{z}_2 = (\eta_2 + \ell_1 y)^{9/11} \\
 \dot{\eta}_3 &= -L\ell_2 \hat{z}_3, \quad \hat{z}_3 = (\eta_3 + \ell_2 \hat{z}_2)^{7/9}
 \end{aligned} \tag{76}$$

where  $b_1, b_2, b_3, \ell_1, \ell_2$  and  $L$  ( $L > 1$ ) are appropriate positive constants. The simulation is carried out with following choice:  $b_1 = 0.8, b_2 = 1.5, b_3 = 2, \ell_1 = 8, \ell_2 = 5, L = 2.5$ , and  $(\zeta(0), x_1(0), x_2(0), x_3(0), \eta_2(0), \eta_3(0)) = (5, 1.5, 2, -10, -3, -2)$ . The closed-loop responses of (74)–(75)–(76) are shown in Fig.1.

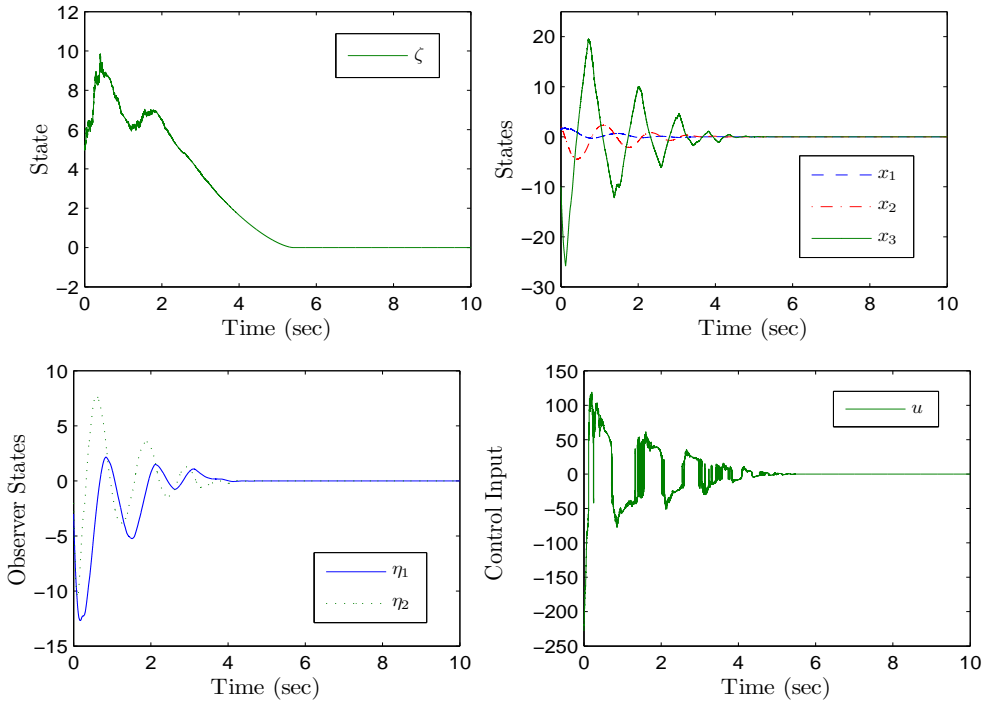


Fig. 1. Response curves of the closed-loop system (74)–(75)–(76).

### 5. CONCLUSIONS

In this paper, we have investigated the global output-feedback finite-time stabilization problems for a class of stochastic nonlinear cascaded systems based on the recently developed finite-time stability theory for stochastic nonlinear systems [14, 35, 36]. By utilizing the adding a power integrator technique and homogeneous system theory, it has

been proved that global output-feedback finite-time stabilization problems for a class of stochastic nonlinear cascaded systems can be solved by partial state feedback. Although the obtained result is preliminary at this stage, it is hoped that the results obtained pave a way to study the output-feedback finite-time stabilization for more general stochastic nonlinear cascaded systems.

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#### REFERENCES

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- [1] Z. Chen and J. Huang: A Lyapunov's direct method for the global robust stabilization of nonlinear cascaded systems. *Automatica* *44* (2008), 745–752. DOI:10.1016/j.automata.2007.06.027
  - [2] H. Deng and M. Krstic: Output-feedback stochastic nonlinear stabilization. *IEEE Trans. Automat. Control* *44* (1999), 328–333. DOI:10.1109/9.746260
  - [3] S. Ding, A. Levant, and S. Li: Simple homogeneous sliding-mode controller. *Automatica* *67* (2016), 22–32. DOI:10.1016/j.automata.2016.01.017
  - [4] S. Ding, S. Li, and W. Zheng: Nonsmooth stabilization of a class of nonlinear cascaded systems. *Automatica* *48* (2012), 2597–2606. DOI:10.1016/j.automata.2012.06.060
  - [5] S. Ding, J. Wang, and W. Zheng: Second-order sliding mode control for nonlinear uncertain systems bounded by positive functions. *IEEE Trans. Industrial Electronics* *62* (2015), 5899–5909. DOI:10.1109/tie.2015.2448064
  - [6] H. Du, Y. He, and Y. Cheng: Finite-time cooperative tracking control for a class of second-order nonlinear multi-agent systems. *Kybernetika* *49* (2013), 507–523.
  - [7] H. Du, Y. He, and Y. Cheng: Finite-time synchronization of a class of second-order nonlinear multi-agent systems using output feedback control. *IEEE Trans. Circ. Syst.* *61* (2014), 1778–1788. DOI:10.1109/tcsi.2013.2295012
  - [8] H. Du, G. Wen, Y. Cheng, Y. He, and Ruting Jia: Distributed Finite-Time Cooperative Control of Multiple High-Order Nonholonomic Mobile Robots. *IEEE Trans. Neu. Net. Lear. Sys. PP* (2016), 99, 1–9. DOI:10.1109/tnnls.2016.2610140
  - [9] H. Du, G. Wen, X. Yu, S. Li, and M. Chen: Finite-time consensus of multiple non-holonomic chained-form systems based on recursive distributed observer. *Automatica* *62* (2015), 236–242. DOI:10.1016/j.automata.2015.09.026
  - [10] N. Duan and X. Xie: Further results on output-feedback stabilization for a class of stochastic nonlinear systems. *IEEE Trans. Automat. Control* *56* (2011), 1208–1213. DOI:10.1109/tac.2011.2107112
  - [11] Y. Hong, J. Huang, and Y. Xu: On an output feedback finite-time stabilization problem. *IEEE Trans. Automat. Control* *46* (2001), 305–309. DOI:10.1109/9.905699
  - [12] Y. Hong, Z. Jiang, and G. Feng: Finite-time input-to-state stability and applications to finite-time control design. *SIAM J. Control Optim.* *48* (2010), 4395–4418. DOI:10.1137/070712043

- [13] Z. Jiang and I. Mareels: A small-gain control method for nonlinear cascaded systems with dynamic uncertainties. *IEEE Trans. Autom. Control* *42* (1997), 292–308. DOI:10.1109/9.557574
- [14] S. Khoo, J. Yin, Z. Man, and X. Yu: Finite-time stabilization of stochastic nonlinear systems in strict-feedback form. *Automatica* *49* (2013), 1403–1410. DOI:10.1016/j.automatica.2013.01.054
- [15] Q. Lan and S. Li: Global output-feedback stabilization for a class of stochastic nonlinear systems via sampled-data control. *Int. J. Robust Nonlinear Control* *27*, 17, (2017), 3643–3658. DOI:10.1002/rnc.3758
- [16] Q. Lan, S. Li, S. Khoo, and P. Shi: Global finite-time stabilisation for a class of stochastic nonlinear systems by output feedback. *Int. J. Control* *83* (2015), 494–506. DOI:10.1080/00207179.2014.962766
- [17] Z. Lendek, R. Babuska and B. Schutter: Stability of cascaded fuzzy systems and observers. *IEEE Trans. Fuzzy Syst.* *17* (2009), 641–653. DOI:10.1109/tfuzz.2008.924353
- [18] J. Li, C. Qian, and S. Ding: Global finite-time stabilisation by output feedback for a class of uncertain nonlinear systems. *Int. J. Control* *83* (2010), 2241–2252. DOI:10.1080/00207179.2010.511658
- [19] S. Li and Y. Tian: Finite-time stability of cascaded time-varying systems. *Int. J. Control* *80* (2007), 646–657. DOI:10.1080/00207170601148291
- [20] W. Li, X. Xie, and S. Zhang: Output-feedback stabilization of stochastic high-order nonlinear systems under weaker conditions. *SIAM J. Control Optim.* *49* (2011), 1262–1282. DOI:10.1137/100798259
- [21] S. Liu and J. Zhang: Output-feedback control of a class of stochastic nonlinear systems with linearly bounded unmeasurable states. *Int. J. Robust Nonlinear Control* *18* (2008), 665–687. DOI:10.1002/rnc.1255
- [22] X. Mao: *Stochastic Differential Equations and Their Applications*. Second edition. Horwood Publishing 1997.
- [23] F. Mazenc, L. Praly, and W. Dayawansa: Global stabilization by output feedback: examples and counterexamples. *Syst. Control Lett.* *23* (1994), 119–125. DOI:10.1016/0167-6911(94)90041-8
- [24] M. Ou, S. Gu, X. Wang, and K. Dong: Finite-Time tracking control of multiple non-holonomic mobile robots with external disturbances. *Kybernetika* *49* (2013), 1049–1067. DOI:10.14736/kyb-2015-6-1049
- [25] L. Rosier: Homogeneous Lyapunov function for homogeneous continuous vector field. *Syst. Control Lett.* *19* (1992), 467–473. DOI:10.1016/0167-6911(92)90078-7
- [26] H. Sun, L. Hou, and G. Zong: Continuous finite time control for static var compensator with mismatched disturbances. *Nonlinear Dynamics* *85* (2016), 2159–2169. DOI:10.1007/s11071-016-2821-2
- [27] Z. Sun, L. Xue, and K. Zhang: A new approach to finite-time adaptive stabilization of high-order uncertain nonlinear system. *Automatica* *58* (2015), 60–66. DOI:10.1016/j.automatica.2015.05.005
- [28] Z. Sun, M. Yun, and T. Li: A new approach to fast global finite-time stabilization of high-order nonlinear terms. *Automatica* *81* (2017), 455–463. DOI:10.1016/j.automatica.2017.04.024

- [29] C. Qian and W. Lin: A continuous feedback approach to global strong stabilization of nonlinear systems. *IEEE Trans. Automat. Control* *46* (2001), 1061–1079. DOI:10.1109/9.935058
- [30] C. Qian and W. Lin: Output feedback control for a class of nonlinear systems: a non-separation principle paradigm. *IEEE Trans. Automat. Control* *47* (2002), 1710–1715. DOI:10.1109/9.935058
- [31] Y. Wu, F. Gao, and Z. Liu: Finite-time state-feedback stabilisation of non-holonomic systems with low-order non-linearities. *IET Control Theory Appl.* *9* (2015), 1553–1560. DOI:10.1049/iet-cta.2014.1001
- [32] Y. Wei and W. Zheng: Finite-time stochastic stabilisation of Markovian jump non-linear quadratic systems with partially known transition probabilities. *IET Control Theory Appl.* *8* (2014), 311–318. DOI:10.1049/iet-cta.2013.0570
- [33] Z. Wu, X. Xie, P. Shi, and Y. Xia: Backstepping controller design for a class of stochastic nonlinear systems with Markovian switching. *Automatica* *45* (2009), 997–1004. DOI:10.1016/j.automatica.2008.12.002
- [34] Y. Wu, J. Yu, and Y. Zhao: Output feedback regulation control for a class of cascaded nonlinear systems and its applications to fan speed control. *Nonlinear Anal.: Real World Appl.* *13* (2012), 1278–1291. DOI:10.1016/j.nonrwa.2011.10.005
- [35] J. Yin and S. Khoo: Conitnuous finite-time state feedback stabilizers for some nonlinear stochastic systems. *Int. J. Robust Nonlinear Control* *25* (2015), 1581–1600. DOI:10.1002/rnc.3161
- [36] J. Yin, S. Khoo, Z. Man, and X. Yu: Finite-time stability and instability of stochastic nonlinear systems. *Automatica* *47* (2011), 2671–2677. DOI:10.1016/j.automatica.2011.08.050
- [37] W. Zha, J. Zhai, S. Fei, and Y. Wang: Finite-time stabilization for a class of stochastic nonlinear systems via output feedback. *ISA Trans.* *53* (2014), 709–716. DOI:10.1016/j.isatra.2014.01.005
- [38] J. Zhou, C. Wen, and T. Li: Adaptive output feedback control of uncertain nonlinear systems with hysteresis nonlinearity. *IEEE Trans. Automat. Control* *57* (2012), 2627–2633. DOI:10.1109/tac.2012.2190208

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