Ondrej Hutník; Jozef Pócs

On $\star$- associated comonotone functions

Kybernetika, Vol. 54 (2018), No. 2, 268–278

Persistent URL: `http://dml.cz/dmlcz/147193`

KYBERNETIKA — VOLUME *54* (2018), NUMBER 2, PAGES 268–278

ON $\star$-ASSOCIATED COMONOTONE FUNCTIONS

Ondrej Hutník and Jozef Pócs

We give a positive answer to two open problems stated by Boczek and Kaluszka in their paper [1]. The first one deals with an algebraic characterization of comonotonicity. We show that the class of binary operations solving this problem contains any strictly monotone right-continuous operation. More precisely, the comonotonicity of functions is equivalent not only to $+$-associatedness of functions (as proved by Boczek and Kaluszka), but also to their $\star$-associatedness with $\star$ being an arbitrary strictly monotone and right-continuous binary operation. The second open problem deals with an existence of a pair of binary operations for which the generalized upper and lower Sugeno integrals coincide. Using a fairly elementary observation we show that there are many such operations, for instance binary operations generated by infima and suprema preserving functions.

Keywords: comonotone functions, binary operation, $\star$-associatedness, Sugeno integral

Classification: 26A48, 28E10

1. INTRODUCTION

In this note we deal with measurable spaces $(X, \mathcal{A})$, where $\mathcal{A}$ is a σ-algebra of subsets of a non-empty set X. Two functions $f, g : X \to Y$ are called *comonotone*[1] on $D \subseteq X$ if $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in D$. In what follows we will consider only non-negative (real) functions, therefore $Y \subseteq [0, +\infty]$. Nowadays, comonotonicity is an important property in analysis, uncertainty theory, economics, financial mathematics, actuarian science, etc. There are some generalizations of comonotonicity in the available literature related to different contexts, see for instance [10] for several multivariate extensions of comonotonicity, or [5, 6] for a connection with a characterization of Sugeno integral on bounded distributive lattices. Recently, in [1, Definition 2.1] the following relation between two functions, having a significant role in the theory of generalized Sugeno integrals, was introduced.

Definition 1.1. Given a binary operation $\star : Y^2 \to Y$, we say that $f, g : X \to Y$ are

DOI: 10.14736/kyb-2018-2-0268

[1] an abbreviation of "common monotone"; however, comonotonicity was already used under different names by many authors, see e. g. [9, Chapter IX], [3, Chapter 4] and [4, Chapter 4] for more details as well as for different characterizations of comonotonicity

⋆-associated on $D \subseteq X$ if for any non-empty measurable subset $A \subset D$,

$$\inf_{x \in A} \{f(x) \star g(x)\} = \inf_{x \in A} f(x) \star \inf_{x \in A} g(x). \tag{1}$$

In the case $D = X$ we simply say "f, g are comonotone" and "f, g are ⋆-associated". Observe that the condition (1) is purely algebraic in its nature and it is based on the preservation of the (infinitary) operation of infimum. Considering different subsets for which the infimum have to be preserved, different relations of measurable functions can be obtained, e. g., if an operation ⋆ preserves all possible infima, then all pairs of functions are ⋆-associated. It can be easily seen that the operation min represents such a function, hence any two measurable functions are min-associated. Boczek and Kaluszka proved in [1] that the comonotonicity of real functions is equivalent to +-associatedness with + being the standard addition on the real line $\mathbb{R}$. Then they formulated the following

OPEN PROBLEM 1: *Is there any non-trivial binary operation $\star \neq +$ on Y such that the ⋆-associatedness is equivalent to the comonotonicity condition?*

In this note we answer the question in the positive by showing that there are many such non-trivial binary operations (addition being just one special case). It turns out that for relating comonotonicity with ⋆-associatedness the property of infimum preservation of downward directed subsets plays a crucial role. In Section 2 we first show that this condition for the operation ⋆ is equivalent to the property that any comonotone functions are also ⋆-associated. Then we show that any ⋆-associated pair of measurable functions is comonotone provided the binary operation ⋆ is strictly monotone on Y. We also provide a sufficient condition for fulfilling the equivalence between the comonotonicity and ⋆-associatedness of functions. For this reason, injective infimum preserving functions are characterized. Using a similar method, involving supremum and infimum preserving mappings, we also positively answer the second open problem stated by Boczek and Kaluszka in [1] about the existence of a pair of binary operations such that the generalized upper Sugeno integral equals the generalized lower Sugeno integral for each measurable function and each monotone measure. All the necessary notions and the corresponding result are included in Section 3.

2. ⋆-ASSOCIATEDNESS VERSUS COMONOTONICITY

This section deals with a detailed study of relationship between the two notions of comonotonicity and ⋆-associatedness mainly from an algebraic point of view. During the way to provide a description of operations ⋆ for which ⋆-associatedness implies comonotonicity we restate some basic observations from [1] from a more general (algebraic) point of view.

2.1. Comonotonicity implies ⋆-associatedness

We say that a binary operation $\star \colon Y^2 \to Y$ is *≤-monotone* (or, order-preserving) if for all $a, b, c \in Y$ the condition $a \leq b$ implies $a \star c \leq b \star c$ and $c \star a \leq c \star b$. It is said to be *strictly monotone* if $a < b$ implies $a \star c < b \star c$ and $c \star a < c \star b$ for any $c \in Y$.

For $i = 1, 2$ let $\pi_i \colon Y^2 \to Y$ be the ith projection, i. e., $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $x, y \in Y$. A subset $T \subseteq Y^2$ is said to be *downward directed*, if for any $(x_1, y_1) \in T$ and $(x_2, y_2) \in T$ there is $(x, y) \in T$ such that $x \le x_1, x_2$ and $y \le y_1, y_2$. Let us note, that the notion of downward directness[2] can be defined in any partially ordered set. From this point of view the introduced notion coincides with that on the direct product $Y \times Y$ of posets.

A binary operation $\star \colon Y^2 \to Y$ *preserves* downward directed infima if

$$\inf_{(x,y)\in T} \{x \star y\} = \Big(\inf_{x\in\pi_1(T)} x\Big) \star \Big(\inf_{y\in\pi_2(T)} y\Big)$$

for any non-empty downward directed subset $T \subseteq Y^2$. First we show the important property concerning downward directed subsets of Y^2. Hereafter, $c_n \searrow c$ means that $\lim_{n\to\infty} c_n = c$ and $c_n \ge c_{n+1}$ for all $n \in \mathbb{N}$. Recall that in this case $\lim_{n\to\infty} c_n = \inf_{n\in\mathbb{N}} c_n$.

Lemma 2.1. *Let $T \subseteq Y^2$ be a downward directed subset and $a = \inf \pi_1(T)$, $b = \inf \pi_2(T)$. Then there is a sequence $\{(a_n, b_n) : n \in \mathbb{N}\} \subseteq T$ such that $a_n \searrow a$ and $b_n \searrow b$.*

Proof. As $a = \inf \pi_1(T)$, there is $a'_n \searrow a$ such that $a'_n \in \pi_1(T)$ for all $n \in \mathbb{N}$. Note that such a sequence can be found inductively, e. g., we choose $a'_1 \in \pi_1(T)$ arbitrarily, and for $n \ge 1$ we put $a'_{n+1} = \min\{a'_n, c_n\}$, where $c_n \in \pi_1(T)$ satisfies $a \le c_n < a + 2^{-n}$. Similarly one can find $b''_n \searrow b$ with $b''_n \in \pi_2(T)$ for all $n \in \mathbb{N}$, thus we have two sequences $\{(a'_n, b'_n) : n \in \mathbb{N}\} \subseteq T$ and $\{(a''_n, b''_n) : n \in \mathbb{N}\} \subseteq T$.

Using these two sequences we define a sequence (a_n, b_n) inductively as follows: choose $(a_1, b_1) \in T$ arbitrarily, and for $n \ge 1$ we put $(a_{n+1}, b_{n+1}) \in T$ an element satisfying $a_{n+1} \le \min\{a_n, a'_n, a''_n\}$ and $b_{n+1} \le \min\{b_n, b'_n, b''_n\}$. Note that the existence of such a pair follows from the fact that T is a downward directed subset of Y^2. □

Given a binary operation on Y^2, the infimum preservation of downward directed sets represents an algebraic property commonly used in the theory of partially ordered sets. We provide a characterization of such property also in terms of mathematical analysis. Recall that a binary operation $\star : Y^2 \to Y$ is *right-continuous* if $\lim_{n\to\infty} (a_n \star b_n) = a \star b$ for all $a_n, b_n, a, b \in Y$ such that $a_n \searrow a$ and $b_n \searrow b$.

Lemma 2.2. *A binary operation $\star \colon Y^2 \to Y$ preserves downward directed infima if and only if it is right-continuous and monotone.*

Proof. Assume that $\star$ preserves downward directed infima. If $a_1 \le a_2$ and $b_1 \le b_2$, then the set $\{(a_1, b_1), (a_2, b_2)\}$ is downward directed, hence

$$\inf\{a_1 \star b_1, a_2 \star b_2\} = \inf\{a_1, a_2\} \star \inf\{b_1, b_2\} = a_1 \star b_1,$$

i. e., $a_1 \star b_1 \le a_2 \star b_2$.

[2] non-empty subset $D \subseteq P$ of a partially ordered set P is said to be *downward directed* if for any two elements $x, y \in D$ there is $z \in D$ satisfying $z \le x$ and $z \le y$.

If $a_n \searrow a$ and $b_n \searrow b$, then the set $T = \{(a_n, b_n) : n \in \mathbb{N}\}$ is a downward directed subset of Y^2. As $\star$ is monotone, the sequence $(a_n \star b_n)_{n=1}^{\infty}$ is non-increasing and we obtain

$$\lim_{n\to\infty}(a_n \star b_n) = \inf_{n\in\mathbb{N}}\{a_n \star b_n\} = \left(\inf_{n\in\mathbb{N}} a_n\right) \star \left(\inf_{n\in\mathbb{N}} b_n\right) = \lim_{n\to\infty} a_n \star \lim_{n\to\infty} b_n,$$

thus the operation $\star$ is right-continuous.

Conversely, we assume that $T \subseteq Y^2$ is downward directed, denote $a = \inf \pi_1(T)$ and $b = \inf \pi_2(T)$. According to Lemma 2.1, there are $a_n \searrow a$ and $b_n \searrow b$ such that $\{(a_n, b_n) : n \in \mathbb{N}\} \subseteq T$. As $\star$ is monotone, we have $a \star b \le x \star y$ for all $(x, y) \in T$. Hence

$$a \star b \le \inf_{(x,y)\in T}\{x \star y\} \le \inf_{n\in\mathbb{N}}\{a_n \star b_n\},$$

where the second inequality follows from $\{(a_n, b_n) : n \in \mathbb{N}\} \subseteq T$. Consequently we obtain

$$a\star b \le \inf_{(x,y)\in T}\{x\star y\} \le \inf_{n\in\mathbb{N}}\{a_n\star b_n\} = \lim_{n\to\infty}(a_n\star b_n) = a\star b = \left(\inf_{x\in\pi_1(T)} x\right) \star \left(\inf_{y\in\pi_2(T)} y\right).$$

Hence the right-continuity and monotonicity imply that $\star$ preserves downward directed infima. □

Using this characterization we now prove that $\star$-associatedness is a necessary condition for comonotonicity for any binary operation $\star : Y^2 \to Y$ preserving downward directed infima. For a measurable space $(X, \mathcal{A})$ we denote by $\mathcal{F}^Y_{(X,\mathcal{A})}$ the set of all $\mathcal{A}$-measurable functions $f : X \to Y$. We also use the notation

$$\mathcal{C}_{(X,\mathcal{A})} := \left\{(f, g) : f, g \in \mathcal{F}^Y_{(X,\mathcal{A})} \text{ are comonotone}\right\},$$
$$\mathcal{C}^{\star}_{(X,\mathcal{A})} := \left\{(f, g) : f, g \in \mathcal{F}^Y_{(X,\mathcal{A})} \text{ are} \star\text{-associated}\right\}.$$

Theorem 2.3. Let $\star \colon Y^2 \to Y$ be a binary operation. Then the following assertions are equivalent:

(i) $\mathcal{C}_{(X,\mathcal{A})} \subseteq \mathcal{C}^{\star}_{(X,\mathcal{A})}$ for any measurable space $(X, \mathcal{A})$;

(ii) the operation $\star$ preserves downward directed infima.

Proof. (i)⇒(ii) First we show that $\star$ is monotone. For this consider the two-element space $X = \{1, 2\}$ with $\mathcal{A} = \mathbf{P}(X)$, i. e., all subsets being measurable. For any elements $a, b, c \in Y$ such that $a \le b$ the functions $f, g \colon X \to Y$ (given as vectors) $f = (a, b)$ and $g = (c, c)$ are comonotone. Since $\mathcal{C}_{(X,\mathcal{A})} \subseteq \mathcal{C}^{\star}_{(X,\mathcal{A})}$ for any measurable space $(X, \mathcal{A})$, the functions f, g are also $\star$-associated, and particularly for the set X from (1) we obtain

$$\inf\{a \star c, b \star c\} = \inf\{a, b\} \star \inf\{c, c\} = a \star c,$$

which yields $a \star c \le b \star c$. The inequality $c \star a \le c \star b$ can be proved analogously. Thus, $\star$ is a monotone binary operation.

Further consider the space $\mathbb{N}$ with $\mathcal{A} = \mathbf{P}(\mathbb{N})$. For any $a_n \searrow a$ and $b_n \searrow b$, the functions $f, g\colon \mathbb{N} \to Y$ given by $f(n) = a_n$ and $g(n) = b_n$ for all $n \in \mathbb{N}$ are comonotone. Consequently, since $\star$ is monotone, we obtain

$$\lim_{n\to\infty} a_n \star b_n = \inf_{n\in\mathbb{N}} \{a_n \star b_n\} = \left(\inf_{n\in\mathbb{N}} a_n\right) \star \left(\inf_{n\in\mathbb{N}} b_n\right) = a \star b,$$

which shows that $\star$ is right-continuous. Thus, by Lemma 2.2 the binary operation $\star$ preserves downward directed infima.

(ii)$\Rightarrow$(i) Suppose that $f, g\colon X \to Y$ are comonotone and $A \in \mathcal{A}$ is non-empty. Then the set of pairs $\{(f(x), g(x)) : x \in A\}$ forms a chain in Y^2, i. e., for any two elements $(f(x), g(x))$, $(f(y), g(y))$ with $x, y \in A$, we have at once $f(x) \le f(y)$ and $g(x) \le g(y)$, or $f(y) \le f(x)$ and $g(y) \le g(x)$, respectively. Since $\star$ preserves downward directed infima, we get

$$\inf_{x\in A} \{f(x) \star g(x)\} = \inf_{x\in A} f(x) \star \inf_{x\in A} g(x),$$

which yields that f, g are $\star$-associated. □

Note that Theorem 2.3 gives a full explanation for [1, Example 2.3] providing a characterization of inclusion $\mathcal{C}_{(X,\mathcal{A})} \subseteq \mathcal{C}^{\star}_{(X,\mathcal{A})}$ for any measurable space $(X, \mathcal{A})$ in terms of properties of $\star$.

2.2. $\star$-associatedness implies comonotonicity

Now we show that each strictly monotone binary operation is a good candidate for solving the open problem.

Theorem 2.4. Let $\star\colon Y^2 \to Y$ be a strictly monotone binary operation. Then $\mathcal{C}^{\star}_{(X,\mathcal{A})} \subseteq \mathcal{C}_{(X,\mathcal{A})}$ for any measurable space $(X, \mathcal{A})$.

Proof. If $(X, \mathcal{A})$ is a trivial measurable space, i. e., it admits only constant measurable functions, then any pair of measurable functions is comonotone as well as $\star$-associated, thus the assertion is valid.

Further assume that $(X, \mathcal{A})$ is non-trivial. We show that if $f, g \in \mathcal{F}^{Y}_{(X,\mathcal{A})}$ are not comonotone, then f, g are not $\star$-associated. Indeed, let $x, y \in X$ be elements satisfying $(f(x) - f(y)) \cdot (g(x) - g(y)) < 0$. Put $a_1 = f(x)$, $a_2 = f(y)$, $b_1 = g(x)$, $b_2 = g(y)$ and $A_1 = f^{-1}(a_1)$, $A_2 = f^{-1}(a_2)$, $B_1 = g^{-1}(b_1)$, $B_2 = g^{-1}(b_2)$. Then $A_1 \cap B_1 \in \mathcal{A}$ and $A_2 \cap B_2 \in \mathcal{A}$ are non-empty disjoint subsets and for the values of the function f, and g respectively, on the set $A = (A_1 \cap B_1) \cup (A_2 \cap B_2)$ we obtain:

$$f(z) = \begin{cases} a_1, & \text{if } z \in A_1 \cap B_1, \\ a_2, & \text{if } z \in A_2 \cap B_2, \end{cases}$$

$$g(z) = \begin{cases} b_1, & \text{if } z \in A_1 \cap B_1, \\ b_2, & \text{if } z \in A_2 \cap B_2. \end{cases}$$

Since $(a_1 - a_2) \cdot (b_1 - b_2) < 0$, we may assume without loss of generality that $a_1 > a_2$ and $b_1 < b_2$. From this assumption we obtain

$$\inf_{z \in A} \{f(z) \star g(z)\} = \inf\{a_1 \star b_1, a_2 \star b_2\}$$

and

$$\inf_{z \in A} f(z) \star \inf_{z \in A} g(z) = a_2 \star b_1.$$

However, the strict monotonicity of $\star$ implies that $a_2 \star b_1 < a_1 \star b_1$ as well as $a_2 \star b_1 < a_2 \star b_2$, which yields

$$\inf_{z \in A} f(z) \star \inf_{z \in A} g(z) = a_2 \star b_1 < \inf\{a_1 \star b_1, a_2 \star b_2\} = \inf_{z \in A} \{f(z) \star g(z)\},$$

i. e., the functions f and g are not $\star$-associated. □

Example 2.5. Typical examples of operations $\star$ fitting Theorem 2.4 are strict pseudo-additions $\oplus$ on $Y = [0, +\infty]$, see e. g. [12]. In that case there exists an increasing bijection $g : [0, +\infty] \to [0, +\infty]$ such that

$$x \oplus y = g^{-1} (g(x) + g(y)).$$

For example, with an arbitrary $p \in (0, +\infty)$ the operation $\oplus_p$ given by $x \oplus_p y = (x^p + y^p)^{1/p}$ is appropriate. Also the operation $(x + \epsilon) \cdot (x + \delta)$ for $\epsilon, \delta > 0$ is an example of strictly monotone binary operation on $Y = [0, +\infty)$. If $Y = [a, +\infty)$ with $a > 0$, then the standard product $\cdot$ is strictly monotone on Y as well.

Let us note that it is quite possible that also other than strictly monotone binary operations $\star$ satisfy the condition $\mathcal{C}^\star_{(X,\mathcal{A})} \subseteq \mathcal{C}_{(X,\mathcal{A})}$ for any measurable space $(X, \mathcal{A})$. However, currently we are not able to give an example of such operation.

Remark 2.6. If $(X, \mathcal{A})$ is a nontrivial measurable space and $\star \colon Y^2 \to Y$ is a binary operation with an annihilator element, i. e., there is an element $c \in Y$ such that $c \star a = a \star c = c$ for all $a \in Y$, then there is a pair of functions which are $\star$-associated, but not comonotone. Indeed, let c be an annihilator element and $\emptyset \neq A \subsetneq X$ be a measurable subset. Put

$$f(x) = \begin{cases} c, & \text{if } x \in A, \\ c + 1, & \text{if } x \in X \setminus A, \end{cases}$$

$$g(x) = \begin{cases} c + 1, & \text{if } x \in A, \\ c, & \text{if } x \in X \setminus A. \end{cases}$$

Obviously, f and g are not comonotone, whereas it can be easily seen that f and g are $\star$-associated. Note that [1, Example 2.5] is a special case of this observation. From this viewpoint, the triangular norms[3] cannot be used as operations $\star$ on $[0, 1]$ in order to get the equivalence between comonotonicity and $\star$-associatedness.

[3] a function $T : [0, 1]^2 \to [0, 1]$ is a *triangular norm* if and only if the triple $([0, 1], T, \leq)$ is a fully ordered commutative semigroup with neutral element 1 and annihilator 0, see [8]

Summarizing the results of this section we get the solution to Open problem 1: there are many non-trivial binary operations $\star$ on Y such that comonotonicity is equivalent to $\star$-associatedness on each measurable space $(X,\mathcal{A})$. More precisely, combining Theorem 2.3 and Theorem 2.4 we conclude

Theorem 2.7. Let $\star\colon Y^2 \to Y$ be a strictly monotone right-continuous binary operation. Then $\mathcal{C}^{\star}_{(X,\mathcal{A})} = \mathcal{C}_{(X,\mathcal{A})}$ for any measurable space $(X,\mathcal{A})$.

2.3. $\star$-associatedness-comonotonicity preserving operations

Now we provide a sufficient condition for fulfilling the property of equivalence between the comonotonicity and $\star$-associatedness. We say that a binary operation $\star\colon Y^2 \to Y$ *fulfills the property* (E) if

(E) $\mathcal{C}^{\star}_{(X,\mathcal{A})} = \mathcal{C}_{(X,\mathcal{A})}$ for any measurable space $(X,\mathcal{A})$.

In the previous subsection we have shown that any strictly monotone and right continuous operation on Y fulfills the property (E). In what follows we prove that this property is "hereditary" in some sense. First we recall that a function φ is *infimum preserving* if

$$\varphi\left(\inf_{t\in T} t\right) = \inf_{t\in T}\{\varphi(t)\}$$

for any $\emptyset \neq T \subseteq Y$.

Theorem 2.8. Let $\star\colon Y^2 \to Y$ be a binary operation satisfying the property (E) and $\varphi\colon Y \to Y$ be an injective infimum preserving function. Then the binary operation $\diamond$ given by

$$x \diamond y := \varphi(x \star y), \quad x, y \in Y, \tag{2}$$

fulfills the property (E).

Proof. Let $(X,\mathcal{A})$ be a measurable space and $f, g \in \mathcal{F}^{Y}_{(X,\mathcal{A})}$. Assume that f and g are comonotone. Then for any non-empty measurable subset $A \subseteq X$, the $\star$-associatedness of f and g yields

$$\inf_{x\in A}\{f(x) \star g(x)\} = \inf_{x\in A} f(x) \star \inf_{x\in A} g(x).$$

Consequently,

$$\begin{aligned}\inf_{x\in A}\{x \diamond y\} &= \inf_{x\in A}\{\varphi(f(x) \star g(x))\} = \varphi\left(\inf_{x\in A}\{f(x) \star g(x)\}\right)\\ &= \varphi\left(\inf_{x\in A} f(x) \star \inf_{x\in A} g(x)\right) = \inf_{x\in A} f(x) \diamond \inf_{x\in A} g(x),\end{aligned}$$

hence, f and g are $\diamond$-associated.

Conversely, suppose that f and g are not comonotone. Then

$$\inf_{x\in A}\{f(x) \star g(x)\} \neq \inf_{x\in A} f(x) \star \inf_{x\in A} g(x),$$

for some non-empty $A \in \mathcal{A}$. Since φ is injective, we obtain

$$\inf_{x\in A}\{x \diamond y\} = \varphi\left(\inf_{x\in A}\{f(x) \star g(x)\}\right) \neq \varphi\left(\inf_{x\in A} f(x) \star \inf_{x\in A} g(x)\right) = \inf_{x\in A} f(x) \diamond \inf_{x\in A} g(x),$$

i. e., f and g are not $\diamond$-associated. □

Observe that if an operation $\star$ is right-continuous and strictly monotone, then the operation $\diamond$ in (2), determined by an injective infimum preserving function φ, has the same properties as the former one. However, if there is an operation, which violates the assumptions of Theorem 2.7 but fulfills the condition (E), using the described construction one may obtain a lot of new operations (parametrized by infimum preserving functions) violating the mentioned assumptions and fulfilling (E). From this point of view, the method described in Theorem 2.8 can be seen as a fundamental process of generating new operations satisfying (E) and having certain properties invariant with respect to infimum preserving functions.

Since infimum preserving functions on a set Y play an important role in Theorem 2.8, we can give their easy algebraic-analytic characterization. The proof is very similar to the proof of Lemma 2.2, and therefore we omit it.

Theorem 2.9. A function $\varphi\colon Y \to Y$ is infimum preserving if and only if it is order-preserving and right-continuous.

Remark 2.10. Note that an analogous version of Theorem 2.9 holds for supremum preserving functions: a function preserves existing suprema if and only if it is order-preserving and left-continuous. Hence, functions which preserve existing suprema and infima are precisely order-preserving continuous functions. We use this characterization in the next section.

3. ON GENERALIZED UPPER AND LOWER SUGENO INTEGRAL

In this section we use the results in the context of non-additive integrals. We continue considering a measurable space $(X, \mathcal{A})$. A monotone measure on $\mathcal{A}$ is a non-decreasing set function $\mu : \mathcal{A} \to [0, +\infty]$ with $\mu(\emptyset) = 0$. We also put $\mu(\mathcal{A}) = \{\mu(A) : A \in \mathcal{A}\}$. The *generalized upper Sugeno integral* of a measurable function $f : X \to Y$ on a set $D \in \mathcal{A}$ with respect to a monotone measure μ and a non-decreasing operation $\circ : Y \times \mu(\mathcal{A}) \to [0, +\infty]$ is defined as

$$\int_{\circ,D} f \,\mathrm{d}\mu := \sup_{t\in Y}\{t \circ \mu(D \cap \{f \geq t\})\}.$$

Here $\{f \geq t\} = \{x \in X : f(x) \geq t\}$ is the weak upper level set of f for the level $t \in Y$. Note that for comonotone functions $f, g \in \mathcal{F}^Y_{(X,\mathcal{A})}$ on D and for any level $t \in Y$ we have either $(D \cap \{f \geq t\}) \subset (D \cap \{g \geq t\})$ or $(D \cap \{g \geq t\}) \subset (D \cap \{f \geq t\})$.

On the other hand, the *generalized lower Sugeno integral* of a measurable function $f : X \to Y$ on a set $D \in \mathcal{A}$ with respect to a monotone measure μ and a non-decreasing operation $\diamond : Y \times \mu(\mathcal{A}) \to [0, +\infty]$ is defined as

$$\int_{\diamond,D} f \,\mathrm{d}\mu := \inf_{t\in Y}\{t \diamond \mu(D \cap \{f > t\})\},$$

where $\{f > t\} = \{x \in X : f(x) > t\}$ is the strict upper level set of f for the level $t \in Y$. It is well-known that for any $Y = [0, m] \subseteq [0, +\infty]$ and a pair $(\min, \max)$ the generalized upper and lower integrals coincide with the Sugeno integral[4], see e. g. [7]. Therefore, Boczek and Kaluszka in [1] stated the following

OPEN PROBLEM 2: *Is there any pair of operations* $(\triangle, \triangledown) \neq (\min, \max)$ *such that*

$$\int_{\triangle,D} f\,\mathrm{d}\mu = \fint_{\triangledown,D} f\,\mathrm{d}\mu \tag{3}$$

for all measurable functions $f : X \to Y$ *and all monotone measures* μ*?*

Recently, in [2, Corollary 1] authors partially answer the problem in the negative for the class of seminormed and semiconormed fuzzy integrals, see [11]. More precisely, the equality (3) with $\triangle = S$ (a semicopula) and $\triangledown = S^*$ (a dual semicopula) is satisfied for all measurable functions $f : X \to [0, 1]$ and all capacities μ[5] if and only if $S = \min$ and $S^* = \max$. Knowing that the equality (3) holds for the Sugeno integral, i. e.,

$$(\mathrm{Su})\int_D f\,\mathrm{d}\mu := \sup_{t\in Y} \min\{t, \mu(D \cap \{f \geq t\})\} = \inf_{t\in Y} \max\{t, \mu(D \cap \{f > t\})\}, \tag{4}$$

we apply the idea of infima and suprema preserving functions to get the following positive answer to the problem.

Theorem 3.1. Let $\varphi\colon [0, +\infty] \to [0, +\infty]$ be an order-preserving continuous function. Then the pair $(\triangledown, \triangle)$ of operators $\triangledown, \triangle\colon Y \times [0, +\infty] \to [0, +\infty]$, where

$$x \triangledown y = \varphi(\max\{x, y\}),$$
$$x \triangle y = \varphi(\min\{x, y\}), \text{ for all } x \in Y,\ y \in [0, +\infty],$$

fulfills the equality (3) for all measurable functions and all monotone measures.

Proof. According to Remark 2.10 the function φ preserves existing suprema and infima of non-empty subsets, i. e., $\varphi(\sup T) = \sup \varphi(T)$ and $\varphi(\inf T) = \inf \varphi(T)$ for any $\emptyset \neq T \subseteq [0, +\infty]$. Hence, using these equalities and (4) we obtain

$$\begin{aligned}\fint_{\triangledown,D} f\,\mathrm{d}\mu &= \inf_{t\in Y}\{t \triangledown \mu(D \cap \{f > t\})\} = \inf_{t\in Y} \varphi\big(\max\{t, \mu(D \cap \{f > t\})\}\big)\\ &= \varphi\left(\inf_{t\in Y} \max\{t, \mu(D \cap \{f > t\}\}\right) = \varphi\left(\sup_{t\in Y} \min\{t, \mu(D \cap \{f \geq t\}\}\right)\\ &= \sup_{t\in Y} \varphi\big(\min\{t, \mu(D \cap \{f \geq t\})\}\big) = \sup_{t\in Y}\{t \triangle \mu(D \cap \{f \geq t\})\}\\ &= \int_{\triangle,D} f\,\mathrm{d}\mu,\end{aligned}$$

which completes the proof. □

[4] it is enough to realize that in both formulas the value of the Sugeno integral $(\mathrm{Su})\int_D f\,\mathrm{d}\mu$ is obtained by the intersection point of the decumulative function $\mu(D \cap \{f \geq t\})$ and the diagonal, no matter whether $\geq$ or $>$ is taken

[5] *capacity* is a normalized monotone measure μ, i. e., $\mu(X) = 1$

Remark 3.2. Due to the fact that the mapping φ is suprema and infima preserving, the following can easily be seen

$$⨍_{\triangledown,D} f\,\mathrm{d}\mu = \inf_{t\in Y} \varphi\big(\max\{t, \mu(D\cap\{f>t\})\}\big) = \inf_{t\in Y} \big(\max\{\varphi(t), \varphi(\mu(D\cap\{f>t\}))\}\big).$$

Indeed, it means that $⨍_{\triangledown,D} f\,\mathrm{d}\mu$ is the Sugeno integral of $\varphi\circ f$ with respect to $\varphi\circ\mu$. Similarly, using the dual expression for the Sugeno integrals, we obtain the equality

$$\int_{\vartriangle,D} f\,\mathrm{d}\mu = (\mathrm{Su})\int_D (\varphi\circ f)\,\mathrm{d}(\varphi\circ\mu).$$

Theorem 3.1 provides an easy way how to generate pairs of operators for which the equality (3) holds for all measurable functions and all monotone measures. However, we do not have any example of operations, which are not generated by a continuous suprema and infima preserving function from the operations max and min respectively, but the generalized upper and lower Sugeno integrals coincide. As a consequence, a characterization of all the pairs of operations $(\vartriangle, \triangledown)$ on Y for which the equality

$$\int_{\vartriangle,D} f\,\mathrm{d}\mu = ⨍_{\triangledown,D} f\,\mathrm{d}\mu$$

is fulfilled for all measurable functions $f: X\to Y$ and all monotone measures μ is still unknown to us.

ACKNOWLEDGEMENT

Authors thank the anonymous reviewers for their suggestions and improvements of the previous version of the paper. This work was supported by the Slovak Research and Development Agency under the contract No. APVV-16-0337 and No. APVV-16-0073. The second author also acknowledges the support by the IGA project of the Faculty of Science Palacký University Olomouc under the contract No. PrF 2017012.

(Received June 7, 2017)

REFERENCES

[1] M. Boczek and M. Kaluszka: On the Minkowski–Hölder type inequalities for generalized Sugeno integrals with an application. Kybernetika *52* (2016), 3, 329–347. DOI:10.14736/kyb-2016-3-0329

[2] M. Boczek and M. Kaluszka: On conditions under which some generalized Sugeno integrals coincide: A solution to Dubois problem, Fuzzy Sets and Systems *326* (2017), 81–88. DOI:10.1016/j.fss.2017.06.004

[3] D. Denneberg: Non-Additive Measure and Integral. Kluwer Academic Publishers, Dordrecht/Boston/London, 1994. DOI:10.1007/978-94-017-2434-0

[4] M. Grabisch: Set Functions, Games and Capacities in Decision Making. Theory and Decision Library C 46, Springer International Publishing 2016.

[5] R. Halaš, R. Mesiar, and J. Pócs: Congruences and the discrete Sugeno integrals on bounded distributive lattices. Inform. Sci. *367-368* (2016), 443–448. DOI:10.1016/j.ins.2016.06.017

[6] R. Halaš, R. Mesiar, and J. Pócs: Generalized comonotonicity and new axiomatizations of Sugeno integrals on bounded distributive lattices. Int. J. Approx. Reason. *81* (2017), 183–192. DOI:10.1016/j.ijar.2016.11.012

[7] A. Kandel and W. J. Byatt: Fuzzy sets, fuzzy algebra and fuzzy statistics. Proc. IEEE *66* (1978), 1619–1639. DOI:10.1109/proc.1978.11171

[8] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Trends in Logic. Studia Logica Library *8*, Kluwer Academic Publishers, 2000. DOI:10.1007/978-94-015-9540-7

[9] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink: Classical and New Inequalities in Analysis. Kluwer Academic Publishers, Dordrecht/Boston/London, 1993. DOI:10.1007/978-94-017-1043-5

[10] G. Puccetti and M. Scarsini: Multivariate comonotonicity. J. Multivariate Anal. *101* (2010), 1, 291–304. DOI:10.1016/j.jmva.2009.08.003

[11] F. Suárez-García and P. Álvarez-Gil: Two families of fuzzy integrals. Fuzzy Sets and Systems *18* (1986), 67-81. DOI:10.1016/0165-0114(86)90028-x

[12] M. Sugeno and T. Murofushi: Pseudo-additive measures and integrals. J. Math. Anal. Appl. *122* (1987), 197–222. DOI:10.1016/0022-247x(87)90354-4

[13] Z. Wang and G. Klir: Generalized Measure Theory. IFSR International Series on Systems Science and Engineering, Vol. 25, Springer, 2009. DOI:10.1007/978-0-387-76852-6

Ondrej Hutník, Institute of Mathematics, Faculty of Science, Pavol Jozef Šafárik University in Košice, Jesenná 5, 040 01 Košice. Slovak Republic.
e-mail: ondrej.hutnik@upjs.sk

Jozef Pócs, Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, Slovakia AND Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc. Czech Republic.
e-mail: pocs@saske.sk