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ON THE GEOMETRY OF SOME SOLVABLE EXTENSIONS OF THE HEISENBERG GROUP

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Abstract. In this paper we first classify left-invariant generalized Ricci solitons on some solvable extensions of the Heisenberg group in both Riemannian and Lorentzian cases. Then we obtain the exact form of all left-invariant unit time-like vector fields which are spatially harmonic. We also calculate the energy of an arbitrary left-invariant vector field X on these spaces and obtain all vector fields which are critical points for the energy functional restricted to vector fields of the same length. Furthermore, we determine all homogeneous Lorentzian structures and their types on these spaces and give a complete and explicit description of all parallel and totally geodesic hypersurfaces of these spaces. The non-existence of harmonic maps in the non-abelian case is proved and it is shown that the existence of Einstein, Einstein-like metrics and some equations in the Riemannian case can not be extended to their Lorentzian analogues.

Keywords: generalized Ricci soliton; harmonicity of vector field; homogeneous Lorentzian structure; parallel hypersurfaces

MSC 2010: 53C30, 53C50, 53C43

1. Introduction

Heisenberg groups play an important role in geometric analysis, physics and quantum mechanics. Among Heisenberg groups, the three-dimensional Heisenberg group H_3 has attracted a special attention of geometers. For example, in the Riemannian case we refer to [19], [4] and in the Lorentzian case we refer to [17], [3]. Also recently the existence of major differences on the three-dimensional Heisenberg group in Riemannian and Lorentzian cases has been shown in [18]. These differences motivate us to obtain a comparison between Riemannian results and their Lorentzian analogues on some solvable extensions of the Heisenberg group, developing our understanding of which properties are strictly related to the metric signature and which ones are more general. We prove that among five geometric properties, i.e.,

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parallel and totally geodesic hypersurfaces, harmonicity of invariant vector fields, left-invariant generalized Ricci solitons, Einstein-like metrics and homogeneous Lorentzian structures, which we shall investigate on these spaces, four of them have different behaviours in Riemannian and Lorentzian cases. Moreover, we show that the existence of Einstein, Einstein-like metrics, the special Einstein-Weyl equation (E-W) and the vacuum near-horizon geometry equation (VN-H) on these spaces in the Riemannian case cannot be extended to the Lorentzian case.

The structure of the paper is as follows. In Section 2 we report Riemannian curvature properties of some solvable extensions of the Heisenberg group which are given in [2]. We also obtain the Levi-Civita connection and the Ricci tensor in the Lorentzian case on these spaces. In Section 3 we classify the left-invariant generalized Ricci solitons on these spaces in both Riemannian and Lorentzian cases. These were recently introduced in [16] and left-invariant examples in dimension three were classified in [7]. We also show that these spaces cannot even admit a nontrivial left-invariant Ricci soliton, although in the Riemannian case by putting some additional conditions they are Einstein manifolds. Besides, we investigate Einstein-like metrics on these spaces and show that the existence of some results in the Riemannian case cannot extend to their Lorentzian analogues. In Section 4, following the method developed in [5] and [6], we calculate the energy of an arbitrary left-invariant vector field X on these spaces and obtain all the vector fields which are critical points for the energy functional restricted to vector fields of the same length. We also determine all left-invariant unit time-like vector fields which are spatially harmonic and show that critical points for the space-like energy are never harmonic maps. In Section 5 we obtain all of the descriptions of homogeneous Lorentzian structures on these spaces and determine their types, obtaining the Lorentzian result corresponding to the classification of Riemannian structures proved in [2]. Totally geodesic and parallel hypersurfaces of a given manifold enrich our knowledge and understanding of its geometry (see for example [9]). In Section 6 we obtain the complete classification of parallel and totally geodesic hypersurfaces of these spaces in both Riemannian and Lorentzian cases. Throughout this paper we use Maple software for checking the computations.

2. Curvature of some solvable extensions of the Heisenberg group

In this section, we provide the information needed for the study of the geometry of some solvable extensions of the Heisenberg group in both Riemannian and Lorentzian cases.

One-dimensional extensions of the Heisenberg group: These spaces are four-dimensional and denoted by $A_4(\lambda, \mu)$, where λ and μ are positive real numbers.

In [2], the left-invariant Riemannian metric $g_{\lambda,\mu}$ on $A_4(\lambda,\mu)$ is given by

$$g_{\lambda,\mu} = \mathrm{e}^{-4\mu x_4}\Big\{\Big(\mathrm{e}^{2\mu x_4} + \frac{\lambda^2}{4}x_2^2\Big)\,\mathrm{d}x_1^2 + \Big(\mathrm{e}^{2\mu x_4} + \frac{\lambda^2}{4}x_1^2\Big)\,\mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \frac{\lambda^2}{2}x_1x_2\,\mathrm{d}x_1\,\mathrm{d}x_2 + \lambda(x_2\,\mathrm{d}x_1\,\mathrm{d}x_3 - x_1\,\mathrm{d}x_2\,\mathrm{d}x_3)\Big\} + \mathrm{d}x_4^2.$$

Thus the Lie algebra $a_4(\lambda,\mu)$ of the Lie group $A_4(\lambda,\mu)$ with respect to the Riemannian metric $g_{\lambda,\mu}$ has an orthonormal basis

$$(2.1)\qquad e_1 = \mathrm{e}^{\mu x_4}\Big(\partial_{x_1} - \frac{\lambda}{2}x_2\partial_{x_3}\Big),\quad e_2 = \mathrm{e}^{\mu x_4}\Big(\partial_{x_2} + \frac{\lambda}{2}x_1\partial_{x_3}\Big),\quad e_3 = \mathrm{e}^{2\mu x_4}\partial_{x_3},\quad e_4 = \partial_{x_4}.$$

Considering the above left-invariant vector fields, we can equip these spaces with the following left-invariant Lorentzian metric:

$$\widehat{g_{\lambda,\mu}} = \mathrm{e}^{-4\mu x_4}\Big\{\Big(\mathrm{e}^{2\mu x_4} + \frac{\lambda^2}{4}x_2^2\Big)\,\mathrm{d}x_1^2 + \Big(\mathrm{e}^{2\mu x_4} + \frac{\lambda^2}{4}x_1^2\Big)\,\mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \frac{\lambda^2}{2}x_1x_2\,\mathrm{d}x_1\,\mathrm{d}x_2 + \lambda(x_2\,\mathrm{d}x_1\,\mathrm{d}x_3 - x_1\,\mathrm{d}x_2\,\mathrm{d}x_3)\Big\} - \mathrm{d}x_4^2,$$

where e_1, e_2, e_3 are space-like and e_4 is time-like. Also, by using (2.1), the nonzero Lie brackets are given by $[e_1,e_2]=\lambda e_3$, $[e_4,e_1]=\mu e_1$, $[e_4,e_2]=\mu e_2$ and $[e_4,e_3]=2\mu e_3$.

Riemannian case: By [2], the nonzero components of the Levi-Civita connection are given by

$$(2.2)\qquad \begin{gathered} \nabla_{e_1}e_1 = \nabla_{e_2}e_2 = \frac{1}{2}\nabla_{e_3}e_3 = \mu e_4,\quad \nabla_{e_1}e_4 = -\mu e_1,\\ \nabla_{e_1}e_2 = -\nabla_{e_2}e_1 = \frac{1}{2}\lambda e_3,\quad \nabla_{e_3}e_2 = \nabla_{e_2}e_3 = \frac{\lambda}{2}e_1,\\ \nabla_{e_3}e_1 = \nabla_{e_1}e_3 = -\frac{\lambda}{2}e_2,\quad \nabla_{e_3}e_4 = -2\mu e_3,\quad \nabla_{e_2}e_4 = -\mu e_2, \end{gathered}$$

and the nonzero components of the Ricci tensor are given by

$$\varrho_{11} = \varrho_{22} = -\Big(\frac{1}{2}\lambda^2 + 4\mu^2\Big),\quad \varrho_{33} = \frac{1}{2}\lambda^2 - 8\mu^2 \quad\text{and}\quad \varrho_{44} = -6\mu^2.$$

Lorentzian case: Using Koszul's formula

$$2\langle\nabla_{e_i}e_j, e_k\rangle = \langle[e_i,e_j],e_k\rangle - \langle[e_j,e_k],e_i\rangle + \langle[e_k,e_i],e_j\rangle$$

for all e_i, e_j, e_k in the Lie algebra $a_4(\lambda,\mu)$ of $A_4(\lambda,\mu)$, the nonzero components of the Levi-Civita connection are given by

$$\begin{aligned}\nabla_{e_1}e_1=\nabla_{e_2}e_2=\frac{1}{2}\nabla_{e_3}e_3=-\mu e_4,\quad \nabla_{e_1}e_4=-\mu e_1,\\ \nabla_{e_1}e_2=-\nabla_{e_2}e_1=\frac{1}{2}\lambda e_3,\quad \nabla_{e_3}e_2=\nabla_{e_2}e_3=\frac{1}{2}\lambda e_1,\\ \nabla_{e_3}e_1=\nabla_{e_1}e_3=-\frac{1}{2}\lambda e_2,\quad \nabla_{e_3}e_4=-2\mu e_3,\quad \nabla_{e_2}e_4=-\mu e_2.\end{aligned}\tag{2.3}$$

Also, by using the Ricci tensor formula $\varrho_{ij}=\sum\limits_{t=1}^{4}\varepsilon_t g(R(e_i,e_t)e_j,e_t)$, where $\varepsilon_t=\langle e_t,e_t\rangle=\pm1$ and $R(e_i,e_j)=\nabla_{[e_i,e_j]}-[\nabla_{e_i},\nabla_{e_j}]$, the nonzero components of the Ricci tensor are given by $\varrho_{11}=\varrho_{22}=-\lambda^2/2+4\mu^2$, $\varrho_{33}=8\mu^2+\lambda^2/2$ and $\varrho_{44}=-6\mu^2$.

Two-dimensional extensions of the Heisenberg group. These spaces are five-dimensional and denoted by $A_5(\lambda,\mu,\nu)$, where $\lambda,\mu,\nu\in\mathbb{R}$ and $\lambda,\mu>0$. In [2], the left-invariant Riemannian metric $g_{\lambda,\mu,\nu}$ on $A_5(\lambda,\mu,\nu)$ is given by

$$\begin{aligned}g_{\lambda,\mu,\nu}={}&\mathrm{e}^{-4\mu x_4}\Big\{\Big(\mathrm{e}^{2\mu x_4}+\frac{\lambda^2}{4}x_2^2\Big)\,\mathrm{d}x_1^2+\Big(\mathrm{e}^{2\mu x_4}+\frac{\lambda^2}{4}x_1^2\Big)\,\mathrm{d}x_2^2\\ &+\mathrm{d}x_3^2-\frac{\lambda^2}{2}x_1x_2\,\mathrm{d}x_1\,\mathrm{d}x_2+\lambda(x_2\,\mathrm{d}x_1\,\mathrm{d}x_3-x_1\,\mathrm{d}x_2\,\mathrm{d}x_3)\Big\}\\ &+\mathrm{d}x_4^2+\mathrm{e}^{-2\nu x_4}\,\mathrm{d}x_5^2.\end{aligned}$$

Thus the Lie algebra $a_5(\lambda,\mu,\nu)$ of $A_5(\lambda,\mu,\nu)$ with respect to the Riemannian metric $g_{\lambda,\mu,\nu}$ has an orthonormal basis

$$\begin{aligned}e_1=\mathrm{e}^{\mu x_4}\Big(\partial_{x_1}-\frac{\lambda}{2}x_2\partial_{x_3}\Big),\quad e_2=\mathrm{e}^{\mu x_4}\Big(\partial_{x_2}+\frac{\lambda}{2}x_1\partial_{x_3}\Big),\\ e_3=\mathrm{e}^{\mu x_4}\partial_{x_3},\quad e_4=\partial_{x_4},\quad e_5=\mathrm{e}^{\nu x_4}\partial_{x_5}.\end{aligned}\tag{2.4}$$

Now we can equip these spaces with the following left-invariant Lorentzian metric:

$$\begin{aligned}\widehat{g_{\lambda,\mu,\nu}}={}&\mathrm{e}^{-4\mu x_4}\Big\{\Big(\mathrm{e}^{2\mu x_4}+\frac{\lambda^2}{4}x_2^2\Big)\,\mathrm{d}x_1^2+\Big(\mathrm{e}^{2\mu x_4}+\frac{\lambda^2}{4}x_1^2\Big)\,\mathrm{d}x_2^2\\ &+\mathrm{d}x_3^2-\frac{\lambda^2}{2}x_1x_2\,\mathrm{d}x_1\,\mathrm{d}x_2+\lambda(x_2\,\mathrm{d}x_1\,\mathrm{d}x_3-x_1\,\mathrm{d}x_2\,\mathrm{d}x_3)\Big\}\\ &-\mathrm{d}x_4^2+\mathrm{e}^{-2\nu x_4}\,\mathrm{d}x_5^2,\end{aligned}$$

where e_1, e_2, e_3, e_5 are space-like and e_4 is time-like. Also, by using (2.4), the nonzero brackets are given by $[e_1,e_2]=\lambda e_3$, $[e_4,e_1]=\mu e_1$, $[e_4,e_2]=\mu e_2$, $[e_4,e_3]=2\mu e_3$ and $[e_4,e_5]=\nu e_5$.

Riemannian case: By [2], the nonzero components of the Levi-Civita connection are given by equations (2.2) and $\nabla_{e_5} e_4 = -\nu e_5$, $\nabla_{e_5} e_5 = \nu e_4$. Also, the nonzero components of the Ricci tensor can be described by $\varrho_{11} = \varrho_{22} = -(\lambda^2/2 + 4\mu^2 + \mu\nu)$, $\varrho_{33} = \lambda^2/2 - 8\mu^2 - 2\mu\nu$, $\varrho_{44} = -6\mu^2 - \nu^2$ and $\varrho_{55} = -4\mu\nu - \nu^2$.

Lorentzian case: The nonzero components of the Levi-Civita connection are given by equations (2.3) and $\nabla_{e_5} e_4 = -\nu e_5$, $\nabla_{e_5} e_5 = -\nu e_4$. Also, the nonzero components of the Ricci tensor are given by $\varrho_{11} = \varrho_{22} = -\lambda^2/2 + 4\mu^2 + \mu\nu$, $\varrho_{33} = 8\mu^2 + \lambda^2/2 + 2\mu\nu$, $\varrho_{44} = -6\mu^2 - \nu^2$ and $\varrho_{55} = 4\mu\nu + \nu^2$.

3. Left-invariant generalized Ricci solitons and Einstein-like metrics on some solvable extensions of the Heisenberg group

As it is introduced in [16], a generalized Ricci soliton is a pseudo-Riemannian manifold (M, g) admitting a smooth vector field X such that

$$L_X g + 2\alpha X^b \odot X^b - 2\beta\varrho = 2\delta g, \tag{3.1}$$

where α, β, δ are real constants, L_X is the Lie derivative in the direction of X, X^b is a 1-form which is defined by $X^b(Y) = g(X, Y)$ and ϱ is the Ricci tensor. In the special case that $M = G$ is a Lie group, g is a left-invariant metric on G and the equation (3.1) holds with respect to a left-invariant vector field X, we say that (G, g) is a left-invariant generalized Ricci soliton. Recently, left-invariant generalized Ricci solitons in two- and three-dimensional Lie groups have been determined, respectively, in [16] and [7]. These are helpful to obtain the following result.

Theorem 3.1.

(I) *Consider the Lie algebra $a_4(\lambda, \mu)$ of $A_4(\lambda, \mu)$ with respect to the orthonormal basis $\{e_1, \dots, e_4\}$. Then the nontrivial left-invariant generalized Ricci solitons on $a_4(\lambda, \mu)$ are the following:*

(1) $\beta \neq 0, \quad \delta = 6\beta\mu^2, \quad \lambda = 2\mu, \quad X = 0$ *for all* α,

(2) $\beta \neq 0, \quad \alpha = \dfrac{3\mu^2}{2\beta(\lambda^2 - 4\mu^2)} \neq 0, \quad \delta = \dfrac{3}{2}\beta\lambda^2, \quad \lambda \neq 2\mu, \quad X = -\dfrac{\beta(\lambda^2 - 4\mu^2)}{\mu} e_4.$

(II) *Consider the Lie algebra $a_4(\lambda, \mu)$ of $A_4(\lambda, \mu)$ with respect to the pseudo-orthonormal basis $\{e_1, \dots, e_4\}$, with e_4 time-like. Then the nontrivial left-invariant generalized Ricci solitons on $a_4(\lambda, \mu)$ are the following:*

(1) $\alpha \neq 0 = \beta, \quad \delta = -\dfrac{\mu^2}{\alpha}, \quad X = \dfrac{\mu}{\alpha}(e_4 \pm e_3),$

(2) $\beta \neq 0 \neq \alpha = -\dfrac{3\mu^2}{2\beta(4\mu^2 + \lambda^2)}, \quad \delta = \dfrac{3}{2}\beta\lambda^2, \quad X = -\dfrac{\beta(4\mu^2 + \lambda^2)}{\mu} e_4.$

(III) *Consider the Lie algebra $a_5(\lambda,\mu,\nu)$ of $A_5(\lambda,\mu,\nu)$ with respect to the orthonormal basis $\{e_1,\dots,e_5\}$. Then the nontrivial left-invariant generalized Ricci solitons on $a_5(\lambda,\mu,\nu)$ are the following:*

$$(1)\qquad \alpha=\beta=\delta=\nu=0,\quad X=k_5e_5,$$

$$(2)\qquad \beta\neq 0,\quad \lambda=\frac{\sqrt{22}}{2}\mu,\quad \nu=\frac{3}{2}\mu,\quad \delta=\frac{33}{4}\beta\mu^2,\quad X=0\quad \text{for all } \alpha,$$

$$(3)\qquad \beta\neq 0\neq\alpha=\frac{3\mu^2}{\beta(2\lambda^2-11\mu^2)},\quad \nu=\frac{3}{2}\mu,\quad \lambda\neq\frac{\sqrt{22}}{2}\mu,\quad \delta=\frac{3}{2}\beta\lambda^2,$$
$$X=-\frac{\beta(\lambda^2-11\mu^2/2)}{\mu}e_4,$$

$$(4)\qquad 0\neq\beta\neq-\frac{1}{2\alpha},\quad 0\neq\alpha=\frac{6\beta\mu^2}{k_5^2},\quad \delta=6\beta\mu^2,\quad \lambda=2\mu,\quad \nu=0,\quad X=k_5e_5,$$

$$(5)\qquad 0\neq\beta\neq-\frac{1}{2\alpha},-\frac{1}{4\alpha},\quad \delta=\frac{3(20\beta\alpha+44\beta^2\alpha^2+3)\mu^2}{4(4\beta\alpha+1)\alpha},\quad \nu=\frac{3\mu(1+2\beta\alpha)}{(4\beta\alpha+1)},$$
$$\lambda=t\delta,\quad t^2=\frac{8\alpha(4\alpha\beta+1)}{9(20\alpha\beta+44\alpha^2\beta^2+3)\mu^2\beta},\quad X=\sum_{i=4}^{5}k_ie_i,$$
$$k_4^2=\frac{9(1+2\beta\alpha)^2\mu^2}{4(4\beta\alpha+1)^2\alpha^2},\quad k_5^2=-\frac{3\mu^2(20\alpha\beta+44\alpha^2\beta^2+3)}{4\alpha^2(4\alpha\beta+1)^2}.$$

(IV) *Consider the Lie algebra $a_5(\lambda,\mu,\nu)$ of $A_5(\lambda,\mu,\nu)$ with respect to the pseudo-orthonormal basis $\{e_1,\dots,e_5\}$, with e_4 time-like. Then the nontrivial left-invariant generalized Ricci solitons on $a_5(\lambda,\mu,\nu)$ are the following:*

$$(1)\qquad \alpha=\beta=\delta=\nu=0,\ X=k_5e_5,$$

$$(2)\qquad \beta\neq 0\neq\alpha=-\frac{3\mu^2}{\beta(11\mu^2+2\lambda^2)},\ \nu=\frac{3}{2}\mu,\ \delta=\frac{3}{2}\beta\lambda^2,\ X=-\frac{\beta(\frac{11}{2}\mu^2+\lambda^2)}{\mu}e_4,$$

$$(3)\qquad \alpha\neq 0=\beta,\quad \delta=-\frac{\mu^2}{\alpha},\quad \nu=\mu,\quad X=\frac{\mu}{\alpha}(e_4\pm e_3).$$

P r o o f. Assume that $X=k_1e_1+\ldots+k_5e_5$ is an arbitrary left-invariant vector field on $A_5(\lambda,\mu,\nu)$. Then with respect to the pseudo-Riemannian basis $\{e_1,\dots,e_5\}$ with e_4 time-like we have

$$L_X\widehat{g_{\lambda,\mu,\nu}}=\begin{pmatrix} -2\mu k_4 & 0 & \lambda k_2 & \mu k_1 & 0\\ 0 & -2\mu k_4 & -\lambda k_1 & \mu k_2 & 0\\ \lambda k_2 & -\lambda k_1 & -4\mu k_4 & 2\mu k_3 & 0\\ \mu k_1 & \mu k_2 & 2\mu k_3 & 0 & \nu k_5\\ 0 & 0 & 0 & \nu k_5 & -2\nu k_4 \end{pmatrix}.$$

Therefore, by equation (3.1), where $X^b \odot X^b(e_i, e_j) = \varepsilon_i\varepsilon_j k_i k_j$ we obtain

$$\begin{cases} -2\mu k_4 + 2\alpha k_1^2 - 2\beta\Big(-\dfrac{1}{2}\lambda^2 + 4\mu^2 + \mu\nu\Big) = 2\delta, \quad 2\mu k_3 - 2\alpha k_3 k_4 = 0, \\ 2\alpha k_1 k_2 = 0, \quad \lambda k_2 + 2\alpha k_1 k_3 = 0, \mu k_1 - 2\alpha k_1 k_4 = 0, \quad 2\alpha k_1 k_5 = 0, \\ -2\mu k_4 + 2\alpha k_2^2 - 2\beta\Big(-\dfrac{1}{2}\lambda^2 + 4\mu^2 + \mu\nu\Big) = 2\delta, \quad 2\alpha k_3 k_5 = 0, \end{cases}$$

$$\begin{cases} -\lambda k_1 + 2\alpha k_2 k_3 = 0, \quad \mu k_2 - 2\alpha k_2 k_4 = 0, \quad 2\alpha k_2 k_5 = 0, \quad \nu k_5 - 2\alpha k_4 k_5 = 0, \\ 2\alpha k_4^2 - 2\beta(-6\mu^2 - \nu^2) = -2\delta, \quad -2\nu k_4 + 2\alpha k_5^2 - 2\beta(4\mu\nu + \nu^2) = 2\delta, \\ -4\mu k_4 + 2\alpha k_3^2 - 2\beta\Big(8\mu^2 + \dfrac{1}{2}\lambda^2 + 2\mu\nu\Big) = 2\delta. \end{cases}$$

Solving the above system of equations we obtain case (IV) of the theorem. The remaining cases have similar proofs. □

Recall that equation (3.1) has a general form of several important equations. In fact by considering special values of α, β, δ in equation (3.1) we obtain one of the following cases: (K) the Killing vector field equation, if $\alpha = \beta = \delta = 0$; (H) the homothetic vector field equation, if $\alpha = \beta = 0$; (RS) the Ricci soliton equation, if $\alpha = 0$ and $\beta = 1$; (E-W) a special case of the Einstein-Weyl equation in conformal geometry, if $\alpha = 1$ and $\beta = -1/(n-2)$, $n > 2$; (PS) the equation for a metric projective structure with a skew-symmetric Ricci tensor representative in the projective class, if $\alpha = 1$, $\beta = -1/(n-1)$ and $\delta = 0$; (VN-H) the vacuum near-horizon geometry equation of a space-time, if $\alpha = 1$, $\beta = 1/2$ and δ plays the role of the cosmological constant (see [7], [16]). Thus we obtain the following result.

Theorem 3.2.

(i) *The Riemannian Lie group* $(A_4(\lambda,\mu), g_{\lambda,\mu})$ *gives left-invariant solutions to the special Einstein-Weyl equation* (E-W) *and to the vacuum near-horizon geometry equation* (VN-H). *However, the Lorentzian Lie group* $(A_4(\lambda,\mu), \widehat{g_{\lambda,\mu}})$ *never gives any left-invariant solution to these equations.*

(ii) *The Riemannian Lie group* $(A_5(\lambda,\mu,\nu), g_{\lambda,\mu,\nu})$ *gives left-invariant solutions to the Killing vector field equation* (K), *to the homothetic vector field equation* (H), *to the special Einstein-Weyl equation* (E-W) *and to the vacuum near-horizon geometry equation* (VN-H). *However, the Lorentzian Lie group* $(A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$ *only gives left-invariant solutions to the Killing and homothetic vector field equations* (K) *and* (H).

P r o o f. To prove (i) we notice that by Theorem 3.1 part (I) for the Riemannian Lie group $(A_4(\lambda,\mu), g_{\lambda,\mu})$ we have two cases. The first case gives us the trivial solution $X = 0$. Thus we consider the second case which implies that for $\alpha = 1$,

$\beta = 1/2$ and $\alpha = 1$, $\beta = -1/2$ we have $\lambda^2 = 7\mu^2$ and $\lambda^2 = \mu^2$, respectively. Thus $\lambda \neq 2\mu$ and the left-invariant solutions are compatible with (E-W) and (VN-H). Now we consider the Lorentzian Lie group $(A_4(\lambda, \mu), \widehat{g_{\lambda,\mu}})$. Thus by Theorem 3.1 part (II) we obtain the following result. Since we have $\alpha \neq 0$, the solutions are not compatible with (K), (H) and (RS). Also, since $\alpha = 1$ and $\beta \neq 0$ give us $-3\mu^2 = 2\beta(4\mu^2+\lambda^2)$, the solutions are not compatible with (E-W) and (VN-H). Also, since $\alpha = 1$, $\beta = -1/3$ and $\delta = 0$ give us the contradiction $\lambda = 0$ or $\mu = 0$, the solutions are never compatible with (PS). These complete case (i) of theorem. Case (ii) can be proved by a similar argument. □

Thus by Theorem 3.2 we have the following result.

Corollary 3.3. *The Riemannian and the Lorentzian Lie groups* $(A_4(\lambda, \mu), g_{\lambda,\mu})$, $(A_4(\lambda, \mu), \widehat{g_{\lambda,\mu}})$, $(A_5(\lambda, \mu, \nu), g_{\lambda,\mu,\nu})$ *and* $(A_5(\lambda, \mu, \nu), \widehat{g_{\lambda,\mu,\nu}})$ *do not admit any non-trivial left-invariant Ricci soliton.*

Ricci solitons are natural generalizations of Einstein manifolds. In the Riemannian case it is proved in [2] that the metric Lie groups $(A_4(\lambda, \mu), g_{\lambda,\mu})$ and $(A_5(\lambda, \mu, \nu), g_{\lambda,\mu,\nu})$ by putting some additional conditions are Einstein. In the Lorentzian case by using the Einstein equation $\varrho = \lambda g$ and the nonzero Ricci tensor components, which are given in Section 2, on these spaces we obtain a contradiction and hence the following result.

Theorem 3.4. *The Riemannian Lie groups* $(A_5(\sqrt{22}\mu/2, \mu, 3\mu/2), g_{\mu\sqrt{22}/2,\mu,3\mu/2})$ *and* $(A_4(2\mu, \mu), g_{2\mu,\mu})$ *are Einstein while the Lorentzian Lie groups* $(A_4(\lambda, \mu), \widehat{g_{\lambda,\mu}})$ *and* $(A_5(\lambda, \mu, \nu), \widehat{g_{\lambda,\mu,\nu}})$ *are never Einstein manifolds.*

Einstein-like metrics, which were introduced by Gray in [13], are generalizations of Einstein metrics. Thus the above result makes it interesting to investigate Einstein-like metrics on these spaces. Recall that Einstein-like metrics on a pseudo-Riemannian manifold (M, g) are defined through conditions on the Ricci tensor. In fact a pseudo-Riemannian manifold (M, g) belongs to the classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{P} = \mathcal{A} \cap \mathcal{B}$, respectively, if and only if its Ricci tensor is cyclic-parallel, i.e., $\nabla_i \varrho_{jk} + \nabla_j \varrho_{ki} + \nabla_k \varrho_{ij} = 0$, is Codazzi tensor, that is, $\nabla_i \varrho_{jk} = \nabla_j \varrho_{ik}$, and is parallel, which means $\nabla_i \varrho_{jk} = 0$, where $\nabla_i \varrho_{jk} = -\sum_t (\varepsilon_j B_{ijt} \varrho_{tk} + \varepsilon_k B_{ikt} \varrho_{tj})$, such that the components B_{ijk} can be obtained by the relation $\nabla_{e_i} e_j = \sum_k \varepsilon_j B_{ijk} e_k$ (for more details see [15]). Thus we can obtain the following result.

Theorem 3.5. *Among all four- or five-dimensional solvable Riemannian and Lorentzian Lie groups* $(A_4(\lambda, \mu), g_{\lambda,\mu})$, $(A_5(\lambda, \mu, \nu), g_{\lambda,\mu,\nu})$, $(A_5(\lambda, \mu, \nu), \widehat{g_{\lambda,\mu,\nu}})$ *and*

$(A_4(\lambda,\mu),\ \widehat{g_{\lambda,\mu}})$, *Lie groups which are equipped with Einstein-like metrics are* $(A_5(2\mu,\mu,0),\ g_{2\mu,\mu,0})$, $(A_4(2\mu,\mu), g_{2\mu,\mu})$ *and* $(A_5(\sqrt{22}\mu/2,\mu,3\mu/2), g_{\sqrt{22}\mu/2,\mu,3\mu/2})$, *whose Ricci tensors are parallel.*

Theorems 3.2, 3.4 and 3.5 give us the following result.

Corollary 3.6. *Einstein, Einstein-like metrics, the special Einstein-Weyl equation* (E-W) *and the vacuum near-horizon geometry equation* (VN-H) *exist on the Riemannian Lie groups* $(A_4(\lambda,\mu), g_{\lambda,\mu,\nu})$ *and* $(A_5(\lambda,\mu,\nu), g_{\lambda,\mu,\nu})$. *However, none of these properties exist on Lorentzian Lie groups* $(A_4(\lambda,\mu),\widehat{g_{\lambda,\mu,\nu}})$ *and* $(A_5(\lambda,\mu,\nu),\widehat{g_{\lambda,\mu,\nu}})$.

4. Harmonicity of invariant vector fields on some solvable extensions of the Heisenberg group

Let (M,g) be a compact connected and oriented n-dimensional pseudo-Riemannian manifold and (TM,g^s) be its tangent bundle with the Sasakian metric g^s. Then the energy of the smooth vector field $X\colon (M,g)\to(TM,g^s)$ is defined by

$$E(X)=\frac{n}{2}\mathrm{vol}(M,g)+\frac{1}{2}\int_M \|\nabla X\|^2\,\mathrm{d}v. \tag{4.1}$$

In the non-compact case, one works over relatively compact domains (for more details see [1], [6]). Thus we obtain the following result.

Proposition 4.1. *Let* (G,g) *be one of two families of four- or five-dimensional solvable Lie groups* $A_4(\lambda,\mu)$ *and* $A_5(\lambda,\mu,\nu)$ *and* $X=\sum\limits_i k_ie_i$ *be a left-invariant vector field on* G. *Also let* D *be a relatively compact domain of* G *and* $E_D(X)$ *be the energy of* $X|_D$.

(i) *If* (G,g) *is* $(A_4(\lambda,\mu), g_{\lambda,\mu})$, *then* $E_D(X)=\Big\{2+\frac12\mu^2\|X\|^2+\frac14\lambda^2\sum\limits_{i=1}^3 k_i^2+\frac12(3\mu^2k_3^2+5\mu^2k_4^2)\Big\}\mathrm{vol}(D)$.

(ii) *If* (G,g) *is* $(A_4(\lambda,\mu),\widehat{g_{\lambda,\mu}})$, *then* $E_D(X)=\Big\{2-\frac12\mu^2\|X\|^2+\frac14\lambda^2\sum\limits_{i=1}^3 k_i^2+\frac12(5\mu^2k_4^2-3\mu^2k_3^2)\Big\}\mathrm{vol}(D)$.

(iii) *If* (G,g) *is* $(A_5(\lambda,\mu,\nu), g_{\lambda,\mu,\nu})$, *then* $E_D(X)=\Big\{\frac52+\frac12\mu^2\|X\|^2+\frac14\lambda^2\sum\limits_{i=1}^3 k_i^2+\frac32\mu^2k_3^2+\frac12(5\mu^2+\nu^2)k_4^2+\frac12(\nu^2-\mu^2)k_5^2\Big\}\mathrm{vol}(D)$.

(iv) *If* (G,g) *is* $(A_5(\lambda,\mu,\nu),\widehat{g_{\lambda,\mu,\nu}})$, *then* $E_D(X)=\Big\{\frac{5}{2}-\frac{1}{2}\mu^2\|X\|^2+\frac{1}{4}\lambda^2\sum\limits_{i=1}^{3}k_i^2-\frac{3}{2}\mu^2k_3^2+\frac{1}{2}(5\mu^2+\nu^2)k_4^2-\frac{1}{2}(\nu^2-\mu^2)k_5^2\Big\}\operatorname{vol}(D)$.

Proof. For $(A_5(\lambda,\mu,\nu),\widehat{g_{\lambda,\mu,\nu}})$ we have

$$\begin{aligned}\|\nabla X\|^2&=\sum_{i=1}^{5}\varepsilon_i\langle\nabla_{e_i}X,\nabla_{e_i}X\rangle\\&=-3\mu^2k_3^2-\mu^2\|X\|^2+\frac{1}{2}\lambda^2\sum_{i=1}^{3}k_i^2+(5\mu^2+\nu^2)k_4^2+(\mu^2-\nu^2)k_5^2,\end{aligned}$$

where $\|X\|^2=k_1^2+k_2^2+k_3^2-k_4^2+k_5^2$. Considering (4.1) gives us case (iv) of the proposition. The remaining cases can be obtained by similar calculations. □

The critical points for the energy functional are harmonic maps. These vector fields are characterized by Euler-Lagrange equations. In fact, a vector field X defines a nonzero harmonic map from a pseudo-Riemannian manifold (M,g) to its tangent bundle (TM,g^s) if and only if X satisfies conditions $\nabla^*\nabla X=0$ and $\operatorname{tr}[R(\nabla_. X,X).]=0$, where $\nabla^*\nabla X=\sum\limits_i\varepsilon_i(\nabla_{e_i}\nabla_{e_i}X-\nabla_{\nabla_{e_i}e_i}X)$, $\operatorname{tr}[R(\nabla_. X,X).]=\sum\limits_i\varepsilon_iR(\nabla_{e_i}X,X)e_i$ and $\varepsilon_i=\langle e_i,e_i\rangle=\pm1$. Moreover by denoting $\chi^{\varrho}(M)=\{W\in\chi(M)\colon\ \|W\|^2=\varrho^2\}$, where $\varrho\neq0$ is a real constant, one can consider vector fields $X\in\chi^{\varrho}(M)$ which are critical points for the energy functional restricted to vector fields of the same length $E|_{\chi^{\varrho}(M)}$. The Euler-Lagrange equations of this variational condition given by $\nabla^*\nabla X$ is collinear to X (see [6]). Thus we can prove the following result.

Theorem 4.2.

(a) *A left-invariant vector field* $X=\sum\limits_{i=1}^{4}k_ie_i$ *on the Riemannian and Lorentzian Lie groups* $(A_4(\lambda,\mu),g_{\lambda,\mu})$ *and* $(A_4(\lambda,\mu),\widehat{g_{\lambda,\mu}})$ *is a critical point for the energy functional restricted to vector fields of the same length if and only if* $X=\sum\limits_{i=1}^{2}k_ie_i$, $X=k_3e_3$ *or* $X=k_4e_4$. *However, no vector field is a harmonic map.*

(b) *A left-invariant vector field* $X=\sum\limits_{i=1}^{5}k_ie_i$ *on the Riemannian and Lorentzian Lie groups* $(A_5(\lambda,\mu,\nu),g_{\lambda,\mu,\nu})$ *and* $(A_5(\lambda,\mu,\nu),\widehat{g_{\lambda,\mu,\nu}})$ *is a critical point for the energy functional restricted to vector fields of the same length if and only if* $X=k_4e_4$, $X=k_1e_1+k_2e_2+k_5e_5$ *or* $X=k_3e_3+k_5e_5$. *However, the only nontrivial harmonic map is* $X=k_5e_5$, *where* $\nu=0$ *(i.e., when* $A_5(\lambda,\mu,\nu)$ *is abelian).*

P r o o f. For the Lorentzian Lie group $(A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$ we obtain $\nabla^*\nabla X = (\mu^2-\lambda^2/2)\sum_{i=1}^{2} k_i e_i + (4\mu^2-\lambda^2/2)k_3e_3 + (6\mu^2+\nu^2)k_4e_4 + \nu^2 k_5 e_5$. Thus $\nabla^*\nabla X = \delta X$ if and only if we have $(\mu^2-\lambda^2/2)k_i = \delta k_i$, $(4\mu^2-\lambda^2/2)k_3 = \delta k_3$, $(6\mu^2+\nu^2)k_4 = k_4\delta$ and $\nu^2 k_5 = k_5\delta$, where $i = 1,2$. Solutions of these equations give us the first part of (b). To prove the second part we notice that $\nabla^*\nabla X = 0$ if and only if we have one of the cases

(i) $X = k_3e_3 + k_5e_5$, where $\lambda = 2\sqrt{2}\mu$, $\nu = 0$ and $k_3 \neq 0$;

(ii) $X = k_1e_1 + k_2e_2 + k_5e_5$, where $\lambda = \sqrt{2}\mu$, $\nu = 0$ and $k_1, k_2 \neq 0$;

(iii) $X = k_5e_5$, where $k_5 \neq 0$ and $\nu = 0$.

Checking whether vector fields listed in the cases (i)–(iii) satisfy the condition $\operatorname{tr}[R(\nabla_{.}X, X).] = 0$, we obtain the result. The case (a) can be obtained in a similar manner. □

Suppose that (M,g) is a Lorentzian manifold and X is a unit time-like vector field on M (i.e., $\|X\| = -1$). Then by the Euler-Lagrange equations X is spatially harmonic if and only if $\widehat{X}_X = \delta X$, where $\delta \in \mathbb{R}$ and for $\operatorname{div} X = \sum_i \varepsilon_i g(\nabla_{e_i}X, e_i)$ and $(\nabla X)^t\nabla_X X = \sum_i \varepsilon_i g(\nabla_X X, \nabla_{e_i}X)e_i$, $\widehat{X}_X$ is defined by $\widehat{X}_X = -\nabla^*\nabla X - \nabla_X\nabla_X X - \operatorname{div} X \cdot \nabla_X X + (\nabla X)^t\nabla_X X$. Thus we can obtain the exact form of all left-invariant unit-time like vector fields as follows.

Proposition 4.3.

(a) *A left-invariant unit time-like vector field $X = \sum_{i=1}^{4} k_i e_i$ on the Lorentzian Lie group $(A_4(\lambda,\mu), \widehat{g_{\lambda,\mu}})$ is spatially harmonic if and only if X has one of the following forms:*

(a$_1$) $$X = \pm e_4.$$

(a$_2$) $$X = k_3e_3 + k_4e_4, \quad \textit{with} \quad k_3^2 = \frac{4\mu^2-\lambda^2}{16\mu^2}, \quad k_4^2 = k_3^2 + 1.$$

(a$_3$) $$X = k_ie_i + k_3e_3 + k_4e_4, \quad \textit{with} \quad k_3^2 = -\frac{\lambda^4 + 3\lambda^2\mu^2 + 12\mu^4}{2(10\lambda^2\mu^2 - 7\mu^4 + \lambda^4)},$$
$$k_4^2 = \frac{\mu^2(10\lambda^2 + 21\mu^2)}{10\lambda^2\mu^2 - 7\mu^4 + \lambda^4}, \quad k_i^2 = k_4^2 - k_3^2 - 1,\ i = 1,2.$$

(b) *A left-invariant unit time-like vector field $X = \sum_{i=1}^{5} k_i e_i$ on the Lorentzian Lie group $(A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$ is spatially harmonic if and only if X has one* *of the following forms:*

(b$_1$) $X = k_3e_3 + k_4e_4 + k_5e_5,$

$$\text{with} \quad k_3^2 = \frac{-24\mu^4 + 28\mu^3\nu + 8\mu^2\nu^2 - 4\mu\nu^3 - 2\nu^4 - \lambda^2\nu^2}{4(\nu + 3\mu)(-\nu + 2\mu)^2\mu},$$

$$k_4^2 = \frac{-8\mu^2 + \lambda^2 + 2\nu^2}{4\mu(2\mu - \nu)}, \quad k_5^2 = k_4^2 - k_3^2 - 1.$$

(b$_2$) $X = \pm e_4.$

(b$_3$) $X = k_3e_3 + k_4e_4, \quad \text{with} \quad k_4^2 = k_3^2 + 1, \quad k_3^2 = \dfrac{-\lambda^2 + 4\mu^2 + 4\mu\nu - 2\nu^2}{16\mu^2}.$

(b$_4$) $X = k_4e_4 + k_5e_5, \quad \text{with} \quad k_5^2 = \dfrac{\mu(2\nu - 3\mu)}{\nu^2}, \quad k_4^2 = k_5^2 + 1.$

(b$_5$) $X = \sum_{i=1}^{2} k_ie_i + \sum_{i=4}^{5} k_ie_i,$

$$\text{with} \quad \sum_{i=1}^{2} k_i^2 = \frac{\lambda\nu^2 + 6\mu\nu^3 - 30\mu^3\nu + 4\mu^2\nu^2 + 2\nu^4 + 18\mu^4}{-6\mu\nu^3 - 3\mu^2\nu^2 + 24\mu^3\nu - 15\mu^4},$$

$$k_4^2 = \frac{2\nu^2 + \lambda^2 - 2\mu^2}{6\mu(\mu - \nu)}, \quad k_5^2 = k_4^2 - (1 + k_1^2 + k_2^2).$$

(b$_6$) $X = k_1e_1 + k_2e_2 + k_4e_4, \quad \text{with} \quad k_1^2 + k_2^2 = \dfrac{-4\mu^2 + 2\mu\nu - \lambda^2 - 2\nu^2}{4\mu^2},$

$$k_4^2 = 1 + k_1^2 + k_2^2.$$

(b$_7$) $X = \sum_{i=1}^{5} k_ie_i, \quad \text{with} \quad k_1^2 + k_2^2 = \dfrac{123\mu^2 + 8\lambda^2}{4(45\mu^2 - 4\lambda^2)}, \quad k_4^2 = \dfrac{7\lambda^2 + 720\mu^2}{16(45\mu^2 - 4\lambda^2)},$

$$k_5^2 = \frac{7(-171\mu^2 + \lambda^2)}{16(45\mu^2 - 4\lambda^2)}, \quad k_3^2 = k_4^2 - (1 + k_1^2 + k_2^2 + k_5^2).$$

(b$_8$) $X = \sum_{i=1}^{4} k_ie_i, \quad \text{with} \quad k_4^2 = \dfrac{\mu^2(9\mu^2 - 34\lambda^2)}{9\mu^4 + 10\lambda^2\mu^2 + \lambda^4},$

$$k_1^2 + k_2^2 = -\frac{\lambda^4 - 132\mu^4 + 41\mu^2\lambda^2}{2(9\mu^4 + 10\mu^2\lambda^2 + \lambda^4)}, \quad k_3^2 = -1 - k_1^2 - k_2^2 + k_4^2.$$

(b$_9$) $X = \sum_{i=1}^{2} k_ie_i + \sum_{i=4}^{5} k_ie_i, \quad \text{with} \quad k_2^2 + k_1^2 = \dfrac{16\lambda^2 + 330\mu^2}{225\mu^2},$

$$k_4^2 = \frac{30\lambda^2 + 900\mu^2}{900\mu^2}, \quad k_5^2 = k_4^3 - 1 - k_1^2 - k_2^2.$$

(b$_{10}$) $X = k_1e_1 + k_2e_2 + k_4e_4, \quad \text{with} \quad k_1^2 + k_2^2 = -\dfrac{4\lambda^2 + 44\mu^2}{\mu^2}$

$$\text{and} \quad k_4^2 = 1 + k_1^2 + k_2^2.$$

In cases $(\mathrm{b}_5),\ldots,(\mathrm{b}_{10})$ *we have either* $k_1\neq 0$ *and* $k_2\neq 0$ *or* $k_1=0\neq k_2$ *or* $k_2=0\neq k_1$.

Proof. To prove case (b) we notice that since X is a unit time-like vector field, we have $k_1^2+k_2^2+k_3^2-k_4^2+k_5^2=-1$, and hence $k_4\neq 0$. Thus, by using the condition $\widehat{X}_X=\delta X$ and doing some calculations we obtain that X is spatially harmonic if and only if there is a real constant δ such that

$$\begin{cases}\text{(i)} & \lambda^2k_1k_3^2+(4\lambda\mu+\lambda\nu)k_2k_3k_4-(3\mu^2+\mu\nu)k_1k_4^2+(\frac12\lambda^2-\mu^2)k_1=\delta k_1,\\ \text{(ii)} & \lambda^2k_2k_3^2-(4\lambda\mu+\lambda\nu)k_1k_3k_4-(3\mu^2+\mu\nu)k_2k_4^2+(\frac12\lambda^2-\mu^2)k_2=\delta k_2,\\ \text{(iii)} & \frac12(2\lambda^2k_1^2+2\lambda^2k_2^2-8\mu^2k_4^2-4\mu\nu k_4^2+\lambda^2-8\mu^2)k_3=\delta k_3,\\ \text{(iv)} & \sum\limits_{i=1}^{2}k_i^2(5\mu^2+\mu\nu)+2(6\mu^2+\mu\nu)k_3^2+2(2\mu\nu+\mu^2)k_5^2+6\mu^2+\nu^2=-\delta,\\ \text{(v)} & (4\mu k_4^2+\nu)\nu k_5=-\delta k_5.\end{cases}$$

To solve the above system of equations, we consider four cases:

(a) $k_1=k_2=0$,

(b) k_1 and k_2 are nonzero,

(c) $0=k_1$, $k_2\neq 0$ and

(d) $0\neq k_1$, $k_2=0$.

Thus we obtain the result. Conversely, assume that X has one of the forms given in case (b) of the theorem. Then we get $\widehat{X}_X=\delta_{b_i}X$, where

$$\delta_{b_1}=\frac{\nu(-8\mu^2+2\mu\nu+\lambda^2+\nu^2)}{\nu-2\mu},\quad \delta_{b_2}=-6\mu^2-\nu^2,$$
$$\delta_{b_3}=\frac{-72\mu^3-28\nu\mu^2+6\mu\lambda^2+2\nu^3+\nu\lambda^2}{8\mu},\quad \delta_{b_4}=\frac{12\mu^3-8\mu^2\nu-4\mu\nu^2-\nu^3}{\nu},$$
$$\delta_{b_5}=\frac{-\nu(3\mu\nu+\nu^2+2\lambda^2-4\mu^2)}{3(\mu-\nu)},\quad \delta_{b_6}=\frac{-6\nu\mu^2+5\mu\lambda^2+4\mu\nu^2+\nu\lambda^2+2\nu^3-4\mu^3}{4\mu},$$
$$\delta_{b_7}=\frac{71\lambda^2\mu^2}{45\mu^2-4\lambda^2},\quad \delta_{b_8}=-\frac{(39\lambda^2+211\mu^2)\lambda^2\mu^2}{2(10\lambda^2\mu^2+9\mu^4+\lambda^4)},\quad \delta_{b_9}=\frac{8\lambda^2}{15},\quad \delta_{b_{10}}=\frac{\lambda^2}{4}-11\mu^2.$$

Thus each X which is given in case (b) of the theorem is a spatially harmonic vector field. Case (a) can be proved in a similar manner. □

The space-like energy of a unit time-like vector field X on a Lorentzian manifold (M,g) is the integral of the square norm of the restriction of ∇X to the space-like distribution $X^{\perp}$. By [12], a unit time-like vector field X is said to be spatially harmonic if it is a critical point of the space-like energy. Thus as an immediate consequence of Theorems 4.2 and 4.3 we obtain the following result.

Corollary 4.4. *A critical point of the space-like energy for the Lorentzian Lie groups* $(A_4(\lambda,\mu),\widehat{g_{\lambda,\mu}})$ *and* $(A_5(\lambda,\mu,\nu),\widehat{g_{\lambda,\mu,\nu}})$ *is never a harmonic map.*

5. Homogeneous Lorentzian structures on some solvable extensions of the Heisenberg group

A homogeneous pseudo-Riemannian structure on a pseudo-Riemannian manifold (M, g) is a tensor field S of type $(1,2)$ such that the connection $\widetilde{\nabla} = \nabla - S$ satisfies $\widetilde{\nabla} g = 0$, $\widetilde{\nabla} R = 0$ and $\widetilde{\nabla} S = 0$, where the condition $\widetilde{\nabla} R = 0$ is equivalent to

$$(\nabla_T R)(X,Y,Z,W) = -R(S_T X, Y, Z, W) - R(X, S_T Y, Z, W) - R(X, Y, S_T Z, W) - R(X, Y, Z, S_T W). \tag{5.1}$$

If g is a Lorentzian metric, then we say that S is a homogeneous Lorentzian structure. Here we obtain the following result which extends the study of all homogeneous Riemannian structures on these spaces in [2].

Theorem 5.1.

(i) *All homogeneous Lorentzian structures on the Lorentzian Lie group $(A_4(\lambda,\mu), \widehat{g_{\lambda,\mu}})$ are given by*

$$S = \theta \otimes (e^1 \wedge e^2) + \frac{\lambda}{2}(e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3)) + \mu((e^1 \otimes (e^1 \wedge e^4) + e^2 \otimes (e^2 \wedge e^4)) + 2\mu e^3 \otimes (e^3 \wedge e^4), \tag{5.2}$$

where $\theta = ae^3 + be^4$, with $a, b \in \mathbb{R}$ and $\{e^1, \ldots, e^4\}$ is dual to the basis $\{e_1, \ldots, e_4\}$.

(ii) *If $\nu \neq 0$, all homogeneous Lorentzian structures on the Lorentzian Lie group $(A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$ are given by*

$$S = \theta \otimes (e^1 \wedge e^2) - \nu e^5 \otimes (e^4 \wedge e^5) + \frac{\lambda}{2}(e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) + \mu(e^1 \otimes (e^1 \wedge e^4) + e^2 \otimes (e^2 \wedge e^4)) + 2\mu e^3 \otimes (e^3 \wedge e^4), \tag{5.3}$$

where $\theta = ae^3 + be^4 + ce^5$, such that $a, b, c \in \mathbb{R}$ and $\{e^1, \ldots, e^5\}$ is dual to the basis $\{e_1, \ldots, e_5\}$. If $\nu = 0$, all the homogeneous Lorentzian structures on the Lorentzian Lie group $(A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$ are given by (5.2), where $\theta = ae^3 + be^4 + ce^5$, with $a, b, c \in \mathbb{R}$.

P r o o f. To prove (i) we replace (X,Y,Z,W) in (5.1) by (e_1,e_2,e_1,e_3), (e_1,e_2,e_2,e_4), (e_1,e_2,e_1,e_4), (e_1,e_2,e_2,e_3) and (e_1,e_3,e_1,e_4). Then by some computations we get $S_{Ze_2e_4} = \mu e^2(Z)$, $S_{Ze_1e_3} = -\lambda e^2(Z)/2$, $S_{Ze_1e_4} = \mu e^1(Z)$, $S_{Ze_2e_3} = \lambda e^1(Z)/2$ and $S_{Ze_3e_4} = 2\mu e^3(Z)$. Thus by using $\widetilde{\nabla} S = 0$ we obtain case (i) of the theorem. Case (ii) can be proved by a similar argument. □

In [11], a classification for the homogeneous pseudo-Riemannian structures is given and it is proved that if V is a real vector space of dimension n endowed with an inner product $\langle\,,\rangle$ of signature $(k, n-k)$, $S(V) = \{S \in \bigotimes^3 V^*\colon S_{XYZ} = -S_{XZY},\ X, Y, Z \in V\}$ and $\dim V \geqslant 3$, then

$$S(V) = S_1(V) \oplus S_2(V) \oplus S_3(V),$$

where for $S_{XYZ} = \langle S_X Y, Z\rangle$, $c_{12}(S)(Z) = \sum\limits_{i=1}^{n} \varepsilon_i S_{e_i e_i Z}$, $\langle e_i, e_i\rangle = \varepsilon_i$ and the cyclic sum $\underset{XYZ}{\mathfrak{S}}$ we have

$$\begin{aligned}
&S_1(V) = \{S \in S(V)\colon\ S_{XYZ} = \langle X, Y\rangle\omega(Z) - \langle X, Z\rangle\omega(Y),\ \omega \in V^*\},\\
&S_2(V) = \{S \in S(V)\colon\ \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0,\ c_{12}(S) = 0\},\\
&S_3(V) = \{S \in S(V)\colon\ S_{XYZ} + S_{YXZ} = 0\},\\
&S_1(V) \oplus S_2(V) = \{S \in S(V)\colon\ \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0\},\\
&S_2(V) \oplus S_3(V) = \{S \in S(V)\colon\ c_{12}(S) = 0\},\\
&S_1(V) \oplus S_3(V) = \{S \in S(V)\colon\ S_{XYZ} + S_{YXZ} = 2\langle X, Y\rangle\omega(Z) - \langle X, Z\rangle\omega(Y)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \langle Y, Z\rangle\omega(X),\ \omega \in V^*\}.
\end{aligned}$$

Homogeneous structures belonging to $S_1 \oplus S_2$ are as different as possible from the naturally reductive ones. The study of this kind of structures was recently undertaken in [10] and [8].

Corollary 5.2.

(i) *The homogeneous Lorentzian structures on $(A_4(\lambda, \mu), \widehat{g_{\lambda,\mu}})$ given in* (5.2) *are of the type $S_1 \oplus S_2$ if for them we have $a = -\lambda$ and $b = 0$. Also they are not of the types $S_2 \oplus S_3$ or $S_1 \oplus S_3$.*

(ii) *The homogeneous Lorentzian structures on $(A_5(\lambda, \mu, \nu), \widehat{g_{\lambda,\mu,\nu}})$ given in* (5.3) *are of the type $S_2 \oplus S_3$, but not of the type $S_1 \oplus S_3$. Also they are of the type $S_1 \oplus S_2$ if for them we have $a = -\lambda$ and $b = c = 0$.*

P r o o f. To prove part (ii), by equation (5.3) in Theorem 5.1 we have $c_{12}(S)(e_4) = 4\mu + \nu = 0$, which implies that for $\nu = -4\mu$ the homogeneous Lorentzian structures are of type $S_2 \oplus S_3$. Also if $a = -\lambda$ and $b = c = 0$, then we have $\underset{e_i e_j e_k}{\mathfrak{S}} S_{e_i e_j e_k} = 0$ which implies that they are of types $S_1 \oplus S_2$ and S_2. Moreover, they are not of types S_1 and $S_1 \oplus S_3$ (if they were, then $S_{e_1 e_1 e_4} = \langle e_1, e_1\rangle\omega(e_4)$ and $S_{e_3 e_3 e_4} = \langle e_3, e_3\rangle\omega(e_4)$ give us the contradiction $\mu = 0$). Also the relation $S_{e_2 e_2 e_4} + S_{e_2 e_2 e_4} = 2\mu \neq 0$ shows that they are not of type S_3. A similar proof will be used for case (i). $\square$

6. Parallel and totally geodesic hypersurfaces of some solvable extensions of the Heisenberg group

In this section we show that Riemannian and Lorentzian left-invariant metrics of our examples have a similar behaviour with regard to totally geodesic and parallel hypersurfaces. The study of these hypersurfaces is a natural problem and enriches our understanding of the geometry of a given manifold (see for example [9]). We use a similar argument as the one given in Lemma 3.1 in [14]. Thus we obtain the following result

Lemma 6.1.

(i) *Let $F\colon M \to (A_4(\lambda,\mu), g_{\lambda,\mu})$ ($F\colon M \to (A_4(\lambda,\mu), \widehat{g_{\lambda,\mu}})$) be a non-degenerate hypersurface of the Riemannian (or Lorentzian) Lie group $(A_4(\lambda,\mu), g_{\lambda,\mu})$ (or $(A_4(\lambda,\mu), \widehat{g_{\lambda,\mu}})$, respectively). Then the second fundamental form of this immersion is not a Codazzi tensor.*

(ii) *Let $F\colon M \to (A_5(\lambda,\mu,\nu), g_{\lambda,\mu,\nu})$ ($F\colon M \to (A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$), be a parallel hypersurface of the Riemannian (Lorentzian) Lie group $(A_5(\lambda,\mu,\nu), g_{\lambda,\mu,\nu})$ ($(A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$). If ξ is a ε-unit normal vector field on M, then $\xi = \pm e_5$, where $\{e_1,\ldots,e_5\}$ is a (pseudo-)orthonormal basis of the Lie algebra $a_5(\lambda,\mu,\nu)$.*

Using Lemma 6.1, we can prove the following classification result.

Theorem 6.2.

(i) *There are no non-degenerate parallel hypersurfaces in the Riemannian (or Lorentzian) Lie group $(A_4(\lambda,\mu), g_{\lambda,\mu})$ (or $(A_4(\lambda,\mu), \widehat{g_{\lambda,\mu}})$, respectively).*

(ii) *Let $F\colon M \to (A_5(\lambda,\mu,\nu), g_{\lambda,\mu,\nu})$ ($F\colon M \to (A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$), be a parallel hypersurface of the Riemannian (Lorentzian) Lie group $(A_5(\lambda,\mu,\nu), g_{\lambda,\mu,\nu})$ ($(A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$). Then there exist local coordinates (w_1,w_2,w_3,w_4) on M such that this immersion with respect to these coordinates, up to isometries, is given by*

$$F(w_1,\ldots,w_4) = (\mathrm{e}^{\mu w_4} w_1, \mathrm{e}^{\mu w_4} w_2, \mathrm{e}^{2\mu w_4} w_3, w_4, 0). \tag{6.1}$$

Conversely, this hypersurface is parallel.

Proof. Case (i) is an immediate consequence of Lemma 6.1. For case (ii) we assume that M is a parallel hypersurface of $(A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})$. Then by Lemma 6.1 we have $\xi = \pm e_5$. Thus the vectors $Y_1 = e_1$, $Y_2 = e_2$, $Y_3 = e_3$, $Y_4 = e_4$ span the tangent space to M at each point. Using the formula of Gauss $\nabla_X Y = \nabla^M_X Y + h(X,Y)\xi$, we find out that the Levi-Civita connection of M and the second fundamental form of the immersion are determined by $\nabla^M_{Y_i} Y_j = 0$ and $h(Y_i, Y_j) = 0$, respectively (where $i, j \in \{1, \ldots, 4\}$). Now we put $\partial_{w_1} = Y_1, \ldots, \partial_{w_4} = Y_4$ and denote $F\colon M \to (A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}})\colon (w_1, \ldots, w_4) \mapsto (F_1(w_1, \ldots, w_4), \ldots, F_5(w_1, \ldots, w_4))$ as the immersion of the hypersurface. Then by using (2.4), we obtain

$$\begin{aligned}
(\partial_{w_1} F_1, \ldots, \partial_{w_1} F_5) &= \bigl(\mathrm{e}^{\mu F_4}, 0, -\lambda \mathrm{e}^{\mu F_4} F_2/2, 0, 0\bigr),\\
(\partial_{w_2} F_1, \ldots, \partial_{w_2} F_5) &= \bigl(0, \mathrm{e}^{\mu F_4}, \lambda \mathrm{e}^{\mu F_4} F_1/2, 0, 0\bigr),\\
(\partial_{w_3} F_1, \ldots, \partial_{w_3} F_5) &= (0, 0, \mathrm{e}^{2\mu F_4}, 0, 0),\\
(\partial_{w_4} F_1, \ldots, \partial_{w_4} F_5) &= (0, 0, 0, 1, 0).
\end{aligned}$$

These equations give us the immersion which is isometric to the immersion given in the theorem. We can verify the converse by a straightforward computation. $\square$

Considering the proof of Theorem 6.2 we obtain the following result.

Corollary 6.3.

(i) *There are no totally geodesic hypersurfaces in the Riemannian (Lorentzian) Lie group* $(A_4(\lambda,\mu),\ g_{\lambda,\mu})$ $((A_4(\lambda,\mu),\ \widehat{g_{\lambda,\mu}}))$.

(ii) *The only totally geodesic hypersurface of the Riemannian (Lorentzian) Lie group* $(A_5(\lambda,\mu,\nu), g_{\lambda,\mu,\nu})$ $((A_5(\lambda,\mu,\nu), \widehat{g_{\lambda,\mu,\nu}}))$ *is given by* (6.1) *and conversely this hypersurface is totally geodesic.*

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References

[1] *M. Aghasi, M. Nasehi*: On the geometrical properties of solvable Lie groups. Adv. Geom. *15* (2015), 507–517. zbl MR doi

[2] *W. Batat, P. M. Gadea, J. A. Oubiña*: Homogeneous Riemannian structures on some solvable extensions of the Heisenberg group. Acta Math. Hung. *138* (2013), 341–364. zbl MR doi

[3] *W. Batat, S. Rahmani*: Isometries, geodesics and Jacobi fields of Lorentzian Heisenberg group. Mediterr. J. Math. *8* (2011), 411–430. zbl MR doi

[4] *R. R. Bouckaert*: A probabilistic line breaking algorithm. Advances in Artificial Intelligence. Proc. 16th Australian Conf. on AI, Perth, 2003 (T. D. Gedeon et al., eds.). Lecture Notes in Comput. Sci. 2903. Lecture Notes in Artificial Intelligence. Springer, Berlin, 2003, pp. 390–401. zbl MR doi

[5] *G. Calvaruso*: Harmonicity properties of invariant vector fields on three-dimensional Lorentzian Lie groups. J. Geom. Phys. *61* (2011), 498–515. zbl MR doi

[6] *G. Calvaruso*: Harmonicity of vector fields on four-dimensional generalized symmetric spaces. Cent. Eur. J. Math. *10* (2012), 411–425. zbl MR doi

[7] *G. Calvaruso*: Three-dimensional homogeneous generalized Ricci solitons. Avaible at arXiv:1503.07767v2 [math.DG].

[8] *G. Calvaruso, M. Castrillón López*: Cyclic Lorentzian Lie groups. Geom. Dedicata *181* (2016), 119–136. zbl MR doi

[9] *B. De Leo, J. Van Der Veken*: Totally geodesic hypersurfaces of four-dimensional generalized symmetric spaces. Geom. Dedicata *159* (2012), 373–387. zbl MR doi

[10] *P. M. Gadea, J. C. González-Dávila, J. A. Oubiña*: Cyclic metric Lie groups. Monatsh. Math. *176* (2015), 219–239. zbl MR doi

[11] *P. M. Gadea, J. A. Oubiña*: Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures. Houston J. Math. *18* (1992), 449–465. zbl MR

[12] *O. Gil-Medrano, A. Hurtado*: Spacelike energy of timelike unit vector fields on a Lorentzian manifold. J. Geom. Phys. *51* (2004), 82–100. zbl MR doi

[13] *A. Gray*: Einstein-like manifolds which are not Einstein. Geom. Dedicata *7* (1978), 259–280. zbl MR doi

[14] *M. Nasehi*: Parallel and totally geodesic hypersurfaces of 5-dimensional 2-step homogeneous nilmanifolds. Czech. Math. J. *66* (2016), 547–559. zbl MR doi

[15] *M. Nasehi, M. Aghasi*: On the geometrical properties of hypercomplex four-dimensional Lie groups. Georgian Math. J. *25,* (2018), 1–10. doi

[16] *P. Nurowski, M. Randall*: Generalized Ricci solitons. J. Geom. Anal. *26* (2016), 1280–1345. zbl MR doi

[17] *S. Rahmani*: Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension trois. J. Geom. Phys. *9* (1992), 295–302. (In French.) zbl MR doi

[18] *N. Rahmani, S. Rahmani*: Lorentzian geometry of the Heisenberg group. Geom. Dedicata *118* (2006), 133–140. zbl MR doi

[19] *H. Shin, Y. W. Kim, S.-E. Koh, H. Y. Lee, S.-D. Yang*: Ruled minimal surfaces in the three-dimensional Heisenberg group. Pac. J. Math. *261* (2013), 477–496. zbl MR doi

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