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SOLVING A CLASS OF HAMILTON–JACOBI–BELLMAN EQUATIONS USING PSEUDOSPECTRAL METHODS

Mohsen Mehrali-Varjani, M. Shamsi and Alaeddin Malek

This paper presents a numerical approach to solve the Hamilton–Jacobi–Bellman (HJB) problem which appears in feedback solution of the optimal control problems. In this method, first, by using Chebyshev pseudospectral spatial discretization, the HJB problem is converted to a system of ordinary differential equations with terminal conditions. Second, the time-marching Runge–Kutta method is used to solve the corresponding system of differential equations. Then, an approximate solution for the HJB problem is computed. In addition, to get more efficient and accurate method, the domain decomposition strategy is proposed with the pseudospectral spatial discretization. Five numerical examples are presented to demonstrate the efficiency and accuracy of the proposed hybrid method.

Keywords: nonlinear optimal control, pseudospectral method, Hamilton–Jacobi–Bellman equation

Classification: 49J20, 65M70, 35F21

1. INTRODUCTION

Here, a Hamilton–Jacobi–Bellman (HJB) problem is a first order partial differential equation together with a terminal condition which arises in the solving of optimal control problems [20, 21, 22, 42]. In this way, the HJB problem will be derived from a Bolza type minimization problem when the solution to HJB problem is called value function [19]. The feedback form of the optimal control function is evaluated by means of the value function. This kind of the optimal control function is a decision rule which expresses the optimal control solution as a function of the current time and the current state. Of course, there exist other methods such as Pontryagin minimum principle and direct methods to solve the optimal control problems. However, the HJB method is very helpful because, firstly, it provides the feedback form of the solution which is much preferred in many engineering applications [24]. Secondly, in HJB method, the global solution of the optimal control problem can be evaluated, even for nonconvex optimal problems. Despite above utilities, in general, deriving an analytical solution or computing a numerical solution for HJB problem is difficult, especially when one deals with large scale problems. Accordingly, despite the advances in development of numerical methods to solve partial

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differential equations, a numerical solution for HJB problem has remained a challenge [19].

These computational complexities have led many research efforts towards achieving a suitably designed technique to solve HJB problem. In the year 2000, an upwind explicit finite difference method for the approximation of viscosity solutions to HJB is presented by Wang et al. [39]. Stability of the method under some mild conditions is proved. Six years later, Huang et al. [16] proposed a collocation method as a semi-meshless discretization scheme based on radial basis functions for approximating viscosity solutions of the HJB. In the year 2010, HJB method is used to provide an initial guess for the indirect methods, which is based on the Pontryagin minimum principle [8]. Advantage of this technique to other initialization schemes is that to find an initial guess close to global minimum. In the year 2013, Swaidan and Hussin introduced an efficient algorithm to approximate HJB solution for nonlinear optimal control problems with quadratic cost functions [36]. Recently, the first asymptotically optimal feedback planning algorithm for nonholonomic systems and additive cost function presented by Yershov and Frazzoli [41]. In the year 2016, Reisinger and Forsyth [29] introduced a piecewise constant policy approximation to Hamilton–Jacobi–Bellman problem. Rakhshan et al. [28] solved a class of fractional optimal control problems using the Hamilton–Jacobi–Bellman equation. Some other various methods introduced to solve HJB problem (for example see [2, 9, 15, 40]).

On the other hand, Pseudospectral methods are a class of numerical methods which were introduced in 1970s [6, 7, 26]. Their application for solving engineering problems has become popular due to their computational feasibility and efficiency [12, 27, 31, 32, 34, 35]. In the pseudospectral method, the unknown solution is expanded as a global polynomial interpolant based on some suitable collocation points. Here, derivatives are approximated by discrete derivative operator (the differentiation matrix). Thus, designing of differentiation matrices and how to interpolate the unknown solution are the key tools in the pseudospectral method. It is a well-established fact that a proper choice of collocation points is crucial in terms of accuracy and computational stability of the interpolation and pseudospectral methods [5, 10]. As a typically good choice of such collocation points, we can refer to the well-known Chebyshev points, which lies on and are accumulated near the endpoints. The pseudospectral method based on Chebyshev points is referred to Chebyshev pseudospectral method and widely used successfully in numerical solution of practical problems [4, 11, 14, 17, 18, 30, 38]. This motivates us to apply the Chebyshev pseudospectral methods for solving the HJB problem.

In this paper, we present a new hybrid method based on the Chebyshev pseudospectral technique to solve the HJB problem. Note that, after applying discretization technique, we approximate the value function as a weighted sum of smooth basis functions, which are Lagrange polynomials. By applying spatial discretization, the HJB problem reduces to an ordinary differential equation with terminal condition. Now we collocate at Chebyshev–Gauss–Lobatto points. We solve the resulting system by the Runge–Kutta time marching method. This part of the approach is similar to the method of lines. Furthermore, to improve accuracy and efficiency, a multidomain strategy combines with the proposed hybrid method. To the best knowledge of the authors, this type of hybrid method has not been applied to the HJB problems.

The rest of the paper is organized as follows. In Section 2, problem formulation is presented. In Section 3, pseudospectral method and in Section 4, the hybrid method for solving the HJB problem is presented. In Section 5, by some typical examples, the effectiveness and performance of the proposed method for one and two dimensional linear and high nonlinear problems are reported.

2. PROBLEM FORMULATION

We consider the following optimal control problem of the Bolza type

$$\min_{u\in\mathcal{U}} \; J(s,\mathbf{x},u) = \int_s^{t_f} L(t,\mathbf{y}(t),u(t))\,\mathrm{d}t + h(\mathbf{y}(t_f)) \tag{1a}$$

$$\text{s.t.} \qquad \dot{\mathbf{y}}(t) = \mathbf{f}\left(t,\mathbf{y}(t),u(t)\right), \qquad t\in(s,t_f], \tag{1b}$$

$$\mathbf{y}(s) = \mathbf{x}, \tag{1c}$$

where $u:\mathbb{R}\to\mathbb{R}$ is the control function, p is a positive integer number, $\mathbf{y}:\mathbb{R}\to\mathbb{R}^p$ is the state function, $L(\cdot)$ is the running cost, $h(\mathbf{y}(t_f))$ is the terminal cost, $\mathbf{f}:\mathbb{R}^p\to\mathbb{R}^p$ is the vector-valued transition function, s is the initial time, x in the initial state, t_f is the final time, $(s,\mathbf{x})\in[0,t_f)\times\mathbb{R}^p$ and $\mathcal{U}$ is the set of admissible controls. We consider the value function v defined by

$$v(s,\mathbf{x}) = \inf_{u\in\mathcal{U}} J(s,\mathbf{x},u). \tag{2}$$

The value function v satisfies the following HJB problem [19]

$$-v_s + \sup_{u\in\mathcal{U}} \; \left(-v_{\mathbf{x}}\cdot\mathbf{f}\left(s,\mathbf{x},u\right) - L(s,\mathbf{x},u)\right) = 0, \tag{3}$$

$$v(t_f,\mathbf{x}) = h(\mathbf{x}), \tag{4}$$

where the notation "$\cdot$" stands for the inner product. This problem consists of the HJB partial differential equation (3) and the terminal condition (4).
To solve (3) and (4) simultaneously, in general, there are two types of approaches:

Type (i) Approach (Analytical): Here, one aims to derive a control function u as a Closed-Analytical-form-Solution with respect to the value function v. If the value function v is evaluated, then we say that the optimal control u^* is in the following form

$$u^* = \operatorname*{argsup}_{u\in\mathcal{U}} \; -v_{\mathbf{x}}\cdot\mathbf{f}\left(s,\mathbf{x},u\right) - L(s,\mathbf{x},u). \tag{5}$$

Type (ii) Approach (Numerical): There are several different numerical approaches that are beyond the scope of this paper. Here, we describe of course precisely the numerical technique that we use in this paper. First, we calculate u* by using a maximization solver for algebraic equation (3). In this sense the algebraic equation (3) together with condition (4) interprets as the ODE problem. After using the Chebyshev–Gauss–Lobatto points one expresses the problem as the canonical and matrix form. In this matrix form, the unknown coefficients are Lagrange multipliers of the approximated value function. Now, by the help of Runge–Kutta technique one will calculate the Lagrange multiplier coefficients. This leads to the spectral accuracy property for the proposed hybrid method, since here indeed we make a benefit of that related residual vanishes over the collocation points (The HJB problem that consists of Lagrange functions, at Chebyshev–Gauss–Lobatto points) over the corresponding finite dimensional space [6].

For both Types (i) and (ii), the first question is, how does one claim that u* is a feedback solution, and the second is, what is the superiority of current hybrid method when one uses numerical approach?

To answer the first question, let solve the HJB problem (3) – (4) analytically (Type (i)) or numerically (Type (ii)), since by (5) the optimal control is a function of the state and time, u^* may be written in the following form

$$u^*(t) = g\left(t, \mathbf{x}(t)\right). \tag{6}$$

The above solution is called the feedback form of optimal control and is desirable in practical applications [3]. One advantage of such a formulation lies in the fact that even when a trajectory is diverted from its optimal path, we still have a new optimal strategy with different initial conditions without having to resolve the problem from the beginning. In short, the feedback form of the optimal control has advantages in many engineering applications and to obtain it, we need to solve HJB equation. In the next section, a novel approach, based on the Chebyshev pseudospectral method, is introduced and we show how the pseudospectral approach can be extended to solve HJB problems.

A reasonable way of answering the second question mathematically is to interpret the novel method in details. In the next section, a novel approach, based on the Chebyshev pseudospectral method, is introduced and we show how the pseudospectral approach can be extended efficiently to solve HJB problems.

3. THE PSEUDOSPECTRAL METHOD

In pseudospectral methods, the unknown solution is expanded by global polynomial interpolants based on some suitable points. In addition, in some cases, the derivatives are approximated by differentiation matrix. Thus, the concept of approximation by interpolation and matrix differentiation seems to be necessary to describe.

3.1. Approximation by interpolation

Let $-1 = \xi_0, \xi_1, \ldots, \xi_n = 1$ be $n+1$ distinct nodes in $[-1, 1]$, and $\varphi_k(x), \quad k = 0, \ldots, n$ be the Lagrange interpolation polynomials based on these nodes, which are defined as

$$\varphi_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - \xi_j}{\xi_k - \xi_j}, \tag{7}$$

with the Kronecker property $\varphi_k(\xi_j) = \delta_{kj}$. The polynomials $\varphi_k(x), \quad k = 0, \ldots, n$ form a basis for P_n the space of the polynomials of degree less than or equal to n in $[-1, 1]$. A function $z(x)$ defined on $[-1, 1]$ may be approximated by Lagrange interpolation polynomials as

$$z(x) = \sum_{k=0}^{n} z(\xi_k)\varphi_k(x). \tag{8}$$

The above approximation can be written in the following matrix form

$$z(x) \simeq \boldsymbol{\phi}_n^T(x)\mathbf{z}, \tag{9}$$

where $\mathbf{z} = [z(\xi_0), \dots, z(\xi_n)]^T$ and $\boldsymbol{\phi}_n(x) = [\varphi_0(x), \dots, \varphi_n(x)]^T$. From the Kronecker property, we conclude that

$$\boldsymbol{\phi}_n(\xi_j) = \mathbf{e}_j, \;\; j = 0, \dots, n, \tag{10}$$

where $\mathbf{e}_j$ is the jth column of the identity matrix of dimension $n+1$.

It is well known that the proper distribution of nodes is necessary for both the accuracy of the approximating solution and the computational efforts [1]. A good choice for these nodes is the well-known Gauss-Lobatto points [37], where lie inside $[-1, 1]$ and are accumulated near the endpoints. In this paper, Chebyshev–Gauss–Lobatto nodes are used, which are defined as

$$\xi_j = \cos\left(\frac{\pi j}{n}\right), \quad j = 0, \dots, n. \tag{11}$$

According to [6], utilizing these nodes leads to stability and accuracy of interpolation and pseudospectral methods.

3.2. Differentiation matrix

In pseudospectral methods, it is necessary to express the $\frac{d}{dx}z(x)$ in term of $z(x)$ at the collocation points $\xi_j, \; j = 0, \dots, n$. This can be done by using the so-called differentiation matrix.

Lemma 3.1. Let z be a function with sufficient degree of smoothness. From (9), the first derivative of z can be approximated by

$$\tfrac{d}{dx}z(x) \simeq \boldsymbol{\phi}_n^T(x)\mathbf{D}\mathbf{z}, \tag{12}$$

where $\mathbf{D}$ is differentiation matrix and the entries of the differentiation matrix $\mathbf{D}$ are obtained by

$$d_{ij} = \tfrac{d}{dx}\varphi_j(x)\big|_{x=\xi_i}, \; i, j = 0, \dots, n. \tag{13}$$

P r o o f. See Ref. [13]. □

Moreover, according to [1], the following recursive formula can be used for computing the entries of differentiation matrix

$$d_{j+1,k+1} = \begin{cases} \frac{1}{\xi_k - \xi_j}\left(\frac{\lambda_k}{\lambda_j} d_{k+1,k+1} - d_{k+1,j+1}\right), & j \neq k, \\ -\sum\limits_{i=0, i\neq j}^{n} d_{j+1,i+1}, & j = k, \end{cases} \qquad j = 0, 1, \dots, n, \tag{14}$$

where constants λ_j are defined by

$$\lambda_j = \prod_{i=0, i\neq j}^{n} (\xi_j - \xi_i) . \tag{15}$$

It is shown that with these formulas, the effect of roundoff error is reduced and calculation of the differentiation matrix is performed accurately [1].

4. THE PROPOSED HYBRID METHOD

In this section, a numerical method based on the Chebyshev pseudospectral method and Runge–Kutta time-marching is presented.

Theorem 4.1. Consider problem (1) in one dimensional case, then the Chebyshev pseudospectral discretization of this problem leads to a system of ordinary differential equations with terminal conditions.

Proof. In view of (8), we approximate the solution of problem (3) – (4) as

$$v(s, x_1) \simeq \sum_{i=0}^{n} \alpha_i(s)\varphi_i(x_1) = \boldsymbol{\alpha}(s)^T \boldsymbol{\phi}_n(x_1), \tag{16}$$

in which $\alpha_i(s)$, $i=0,\ldots,n$ are unknown coefficient functions and $\boldsymbol{\alpha}(s)=[\alpha_0(s),\ldots,\alpha_n(s)]^T$ is the unknown coefficient vector that must be determined.

By differentiating with respect to s, x_1 and using Lemma 3.1, we have

$$v_s(s, x_1) = \dot{\boldsymbol{\alpha}}(s)^T \boldsymbol{\phi}_n(x_1), \tag{17}$$
$$v_{x_1}(s, x_1) = \boldsymbol{\alpha}(s)^T \dot{\boldsymbol{\phi}}_n(x_1) \tag{18}$$
$$= [\mathbf{D}\boldsymbol{\alpha}(s)]^T \boldsymbol{\phi}_n(x_1), \tag{19}$$

where $\mathbf{D}$ is differentiation matrix. Then, by replacing (17) – (19) in problem (3), we get

$$-\dot{\boldsymbol{\alpha}}(s)^T \boldsymbol{\phi}_n(x_1) + \sup_u \left\{ -[\mathbf{D}\boldsymbol{\alpha}(s)]^T \boldsymbol{\phi}_n(x_1)\mathbf{f}(s, x_1, u) - L(s, x, u) \right\} = 0. \tag{20}$$

Collocating this equation at Chebyshev–Gauss–Lobatto points ξ_i for $i = 0, \ldots, n$ leads to the following differential equations

$$-\dot{\alpha}_i(s) + \sup_u \{ -[\mathbf{D}\boldsymbol{\alpha}(s)]_i \mathbf{f}(s, \xi_i, u) - L(s, \xi_i, u) \} = 0, \quad i = 0, \ldots, n. \tag{21}$$

Moreover, by collocating the terminal condition (4) in the collocation points ξ_i, $i = 0, \ldots, n$, we get $v(t_f, \xi_i) = h(\xi_i)$, $i = 0, \ldots, n$. Using (16) and the Kronecker property, we conclude the following terminal conditions

$$\alpha_i(t_f) = h(\xi_i), \quad i = 0, \ldots, n. \tag{22}$$

□

In order to calculate $\alpha_0, \ldots, \alpha_n$ one may solve the ordinary differential equations (21) with terminal condition (22). By using (16), the approximation of $v(s, x_1)$ will be computed.

However, we note that because of the existence of 'sup' function, the differential equations (21) are not classical and we need to resolve the 'sup' operator. In some problems, by differentiating with respect to u, we can explicitly express the control function u based on $\boldsymbol{\alpha}(t)$ and s. Consequently, in these problems, the 'sup' problem is simply solved and the differential equation (21) is converted to a classical differential equation. On the other hand, in some other problems, the 'sup' problem in (21) may be hard to solve by differentiation. In such cases, we can use a numerical optimization solver to solve the 'sup' problem. More precisely, we note that, generally, problem (21) – (22) is solved by a numerical time-marching method such as Runge–Kutta and in each iteration of Runge–Kutta, the 'sup' problem is converted to an optimization problem which can be solved by an optimization solver.

Theorem 4.1 can be extended to the multi-dimensional problems. In the next theorem, we extend the above problem for the two dimensional problems. For this purpose, we define the Kronecker product.

Definition 4.2. If $\mathbf{A}$ is a $p \times q$ matrix and $\mathbf{B}$ is an $r \times s$ matrix, then the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is a $pr \times qs$ matrix defined by [33]

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \ldots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix}.$$

Theorem 4.3. For the two dimensional version of problem (1), the Chebyshev pseudospectral method leads to a system of ordinary differential equations with terminal conditions.

P r o o f. For two dimensional HJB problem (1) we have

$$-v_s + \sup_{u \in \mathcal{U}} \left(-v_{x_1} f_1\left(s, x_1, x_2, u\right) - v_{x_2} f_2\left(s, x_1, x_2, u\right) - L(s, x_1, x_2, u)\right) = 0, \quad \text{(23a)}$$

$$v(t_f, x_1, x_2) = h(x_1, x_2). \quad \text{(23b)}$$

Let to approximate the value function as

$$v(s, x_1, x_2) \simeq \boldsymbol{\alpha}(s)^T \left(\boldsymbol{\phi}_n(x_1) \otimes \boldsymbol{\phi}_n(x_2)\right). \quad (24)$$

Taking derivative from v with respect to s, x_1, x_2 and using Lemma 3.1 yield

$$v_s(s, x_1, x_2) = \dot{\boldsymbol{\alpha}}(s)^T \left(\boldsymbol{\phi}_n(x_1) \otimes \boldsymbol{\phi}_n(x_2)\right), \quad (25)$$

$$v_{x_1}(s, x_1, x_2) = \boldsymbol{\alpha}(s)^T \left(\mathbf{D}^T \boldsymbol{\phi}_n(x_1) \otimes \boldsymbol{\phi}_n(x_2)\right), \quad (26)$$

$$v_{x_2}(s, x_1, x_2) = \boldsymbol{\alpha}(s)^T \left(\boldsymbol{\phi}_n(x_1) \otimes \mathbf{D}^T \boldsymbol{\phi}_n(x_2)\right), \quad (27)$$

where the differentiation matrix $\mathbf{D}$ is defined in (14). Then, by substituting (25), (26) and (27) in problem (23a), we have

$$\begin{aligned}
&-\dot{\boldsymbol{\alpha}}(s)^T\left(\boldsymbol{\phi}_n(x_1)\otimes\boldsymbol{\phi}_n(x_2)\right)\\
&\quad+\sup_u\Big\{-\boldsymbol{\alpha}(s)^T\left(\mathbf{D}^T\boldsymbol{\phi}_n(x_1)\otimes\boldsymbol{\phi}_n(x_2)\right)f_1(s,x_1,x_2,u)\\
&\qquad-\boldsymbol{\alpha}(s)^T\left(\boldsymbol{\phi}_n(x_1)\otimes\mathbf{D}^T\boldsymbol{\phi}_n(x_2)\right)f_2(s,x_1,x_2,u)-L(s,x_1,x_2,u)\Big\}=0.
\end{aligned}$$

Collocating the above equation at $(n+1)\times(n+1)$ Chebyshev–Gauss–Lobatto points $(x_1,x_2)=(\xi_i,\xi_j)$, $i,j=0,\dots,n$ yields the following $(n+1)^2$ system of equations

$$\begin{aligned}
-\dot{\alpha}_{in+j}(s)&+\sup_u\Big\{-\boldsymbol{\alpha}(s)^T\left(\mathbf{D}^T\mathbf{e}_i\otimes\mathbf{e}_j\right)f_1(s,\xi_i,\xi_j,u)\\
&-\boldsymbol{\alpha}(s)^T\left(\mathbf{e}_i\otimes\mathbf{D}^T\mathbf{e}_j\right)f_2(s,\xi_i,\xi_j,u)-L(s,\xi_i,\xi_j,u)\Big\}=0, \qquad (28)\\
&i,j=0,\dots,n.
\end{aligned}$$

By collocating the terminal condition (23b) at the nodes $(x_1,x_2)=(\xi_i,\xi_j)$, $i,j=0,\dots,n$, we have

$$v(t_f,\xi_i,\xi_j)=h(\xi_i,\xi_j),\ \ i,j=0,\dots,n. \qquad (29)$$

Substituting v from (24) into (29) and by the Kronecker property, yields

$$\alpha_{in+j}(t_f)=h(\xi_i,\xi_j),\ \ i,j=0,\dots,n. \qquad (30)$$

□

Thanks to the Kronecker product, in a similar manner, we can extend Theorem 4.3 for the problems of three or more dimensions.

4.1. Domain decomposition

It is well-known that pseudospectral methods are efficient and accurate to solve the problems with smooth solutions [37]. However, in some HJB problems, the value function is not smooth and the efficiency of the hybrid method decreases for such problems. To overcome this difficulty, we can use the domain decomposition technique as follows.

We consider the value function for (23a)−(23b), where the domain is partitioned into m subdomains: $\Omega=[a,b]=[a_0,a_1]\cup\cdots\cup[a_{m-1},b]$

$$v(s,x)=\begin{cases} v^1(s,x), & x\in[a,a_0],\\ v^2(s,x), & x\in[a_0,a_1],\\ \quad\vdots & \\ v^{m+1}(s,x), & x\in[a_{m-1},b].\end{cases}$$

By applying the terminal condition of HJB problem for $x\in[a_{k-1},a_k]$ and $v^k(s,x)$ we have

$$\begin{cases} -v^k_s+\sup\limits_{u\in\mathcal{U}}\left(-v^k_x\cdot\mathbf{f}\,(s,x,u)-L(s,x,u)\right)=0,\\ v^k(t_f,x)=h(x), \\ x\in[a_{k-1},a_k],\end{cases}\qquad k=1,\dots,m. \qquad (31)$$

Solving HJB sub-problems in each sub-domain $[a_{k-1}, a_k]$, by the proposed hybrid method and assembling them we get the solution.

In the two dimensional HJB problem, with domain $\Omega = [a,b] \times [c,d]$, we partition $\Omega = ([a_0, a_1] \cup \cdots \cup [a_{m-1}, b]) \times ([c_0, c_1] \cup \cdots \cup [c_{m-1}, d])$ and the value function as

$$v(s, x_1, x_2) = \begin{cases} v^{1,1}(s, x_1, x_2), & (x_1, x_2) \in [a_0, a_1] \times [c_0, c_1], \\ v^{1,2}(s, x_1, x_2), & (x_1, x_2) \in [a_0, a_1] \times [c_1, c_2], \\ \quad \vdots & \\ v^{m+1,m+1}(s, x_1, x_2), & (x_1, x_2) \in [a_{m-1}, b] \times [c_{m-1}, d]. \end{cases}$$

Then, similar to the one dimensional case, we will solve the following HJB problems for $i, j = 1, \ldots, m$

$$\begin{cases} -v_s^{i,j} + \sup\limits_{u \in \mathcal{U}} \left(-v_{x_1}^{i,j} f_1\left(s, x_1, x_2, u\right) - v_{x_2}^{i,j} f_2\left(s, x_1, x_2, u\right) - L(s, x_1, x_2, u) \right) = 0, \\ v^{i,j}(t_f, x_1, x_2) = h(x_1, x_2) \\ x_1 \in [a_{i-1}, a_i], \quad x_2 \in [b_{j-1}, b_j]. \end{cases} \tag{32}$$

5. ILLUSTRATIVE EXAMPLES

In this section, we begin by demonstrating the performance of the proposed method developed in Section. MATLAB function `ode45` is used to solve the system of differential equations with terminal conditions. This solver controls the error by two parameters `RelTol` and `AbsTol`. We set `RelTol=1e-11` and `AbsTol=1e-9`.

To assess the accuracy of the method, the following averaged absolute error is reported:

$$E_n = \frac{1}{n+1} \| v_{Exact}(t, \mathbf{x}) - v(t, \mathbf{x}) \|_\infty, \tag{33}$$

where v_{Exact} and v are the exact and computed solutions, respectively.

5.1. Example 1 (Nonlinear)

In this example [25], we consider problem (1), in the one dimensional case with $L(t, y(t), u(t)) = u^2(t)$, $h(x) = x^2$, $h\left(y(t_f)\right) = y^2(t_f)$ and $\mathbf{f}\left(t, y(t), u(t)\right) = y(t) + u(t)$ for $x \in [-1, 1]$ and $t \in [s, t_f]$, in which $s = 0, t_f = 1$. In other words, the following optimal control problem is considered

$$\min \quad \int_0^1 u^2(t)\, \mathrm{d}t + y^2(t_f) \tag{34a}$$

$$\text{s.t.} \quad \dot{y}(t) = y(t) + u(t), \tag{34b}$$

$$y(s) = x. \tag{34c}$$

The corresponding HJB problem is as follows

$$-v_t + \sup_u \left(-v_x(x + u(t)) - u^2(t) \right) = 0, \tag{35a}$$

$$v(1, x) = x^2. \tag{35b}$$

The exact value function for HJB problem (35) is as follows [25]

$$v_{Exact}(t,x) = \frac{2x^2}{1+e^{2(t-1)}}. \tag{36}$$

Using the Chebyshev pseudospectral method, yields

$$-\dot{\alpha}_i(t) + \sup_u \left\{ -[\mathbf{D}\boldsymbol{\alpha}(t)]_i\,(\xi_i + u(t)) - u^2(t) \right\} = 0, \tag{37}$$

$$\alpha_i(1) = \xi_i^2. \tag{38}$$

Supremum $u(t)$ for the quadratic term $-[\mathbf{D}\boldsymbol{\alpha}(t)]_i\,(\xi_i + u(t)) - u^2(t)$ happens if

$$u(t) = -\frac{1}{2}[\mathbf{D}\boldsymbol{\alpha}(t)]_i. \tag{39}$$

Now, by substituting (39) in (37) for $i = 0, \ldots, n$, we get $n+1$ classical differential equations with $n+1$ terminal conditions

$$-\dot{\alpha}_i(t) - \xi_i[\mathbf{D}\boldsymbol{\alpha}(t)]_i + \frac{1}{4}[\mathbf{D}\boldsymbol{\alpha}(t)]_i^2 = 0, \tag{40a}$$

$$\alpha_i(1) = \xi_i^2. \tag{40b}$$

The above problem can be expressed as the following canonical and matrix form

$$\dot{\boldsymbol{\alpha}}(t) = \left(\frac{1}{4}\,[\mathbf{D}\boldsymbol{\alpha}(t)] - \boldsymbol{\xi}\right) \circ [\mathbf{D}\boldsymbol{\alpha}(t)], \tag{41}$$

$$\boldsymbol{\alpha}(1) = \boldsymbol{\xi} \circ \boldsymbol{\xi}, \tag{42}$$

where $\boldsymbol{\xi} = [\xi_0, \ldots, \xi_N]^T$ and $\circ$ denotes the Hadamard product, which for two matrices A, B of the same dimension $m \times n$ is defined by

$$(A \circ B)_{i,j} = (A)_{i,j}(B)_{i,j}.$$

Now, by by `ode45` of Matlab package, we solve the system (40a) – (40b) to calculate the coefficients $\alpha_0(t), \ldots, \alpha_n(t)$.

Table. 1 demonstrate the excellent results for even small values of n. This shows that using this hybrid method for the Example 1, one does not need to use high memory and thus the computational effort can reduce drastically.

In Figure 1, the value function and its error is depicted for $n = 20$. The computed averaged errors of approximate value function from the proposed method are given in Table 1. Table 1 demonstrates the high accuracy of the proposed method. This confirms that the method yields excellent results.

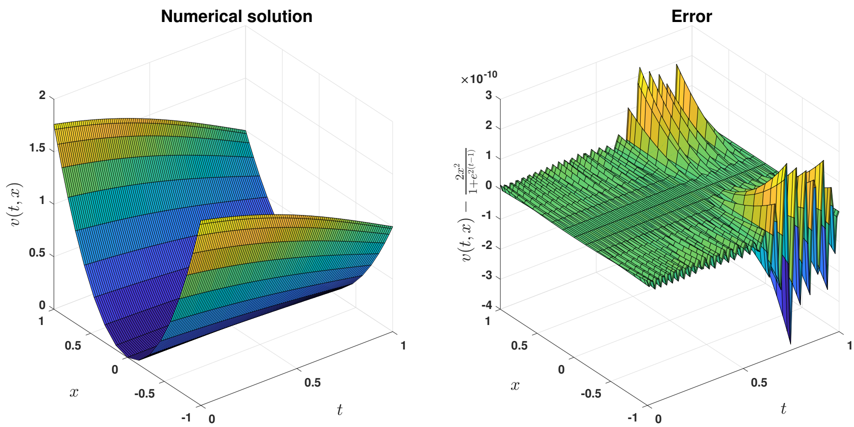

Fig. 1. (Example 1) (Left) The value function $v(t,x)$ computed for $n=20$, (Right) The absolute error.

n	2	3	4	5
E_n	1.2617×10^{-9}	9.4614×10^{-10}	7.5690×10^{-10}	6.3094×10^{-10}

Tab. 1. The average absolute error computed by $E_n = \frac{1}{n+1}\|v_{Exact}(t,x) - v(t,x)\|_\infty$ for Example 1.

5.2. Example 2 (Two dimensional, discontinuous derivative and nonconvex)

Let us consider problem (1) with two dimensional case [39]

$$\min \quad -y_1(t_f) - y_2(t_f) \tag{43a}$$

$$\tag{43b}$$

$$\text{s.t.} \quad \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ -y_2(t) \end{bmatrix} u, \tag{43c}$$

$$y_1(s) = x_1, y_2(s) = x_2, \tag{43d}$$

$$u : [0, t_f] \to [0,1]. \tag{43e}$$

In this example we have $L(t, y(t), u(t)) = 0$, $h(x_1, x_2) = -x_1 - x_2$, $h(y(t_f)) = -y_1(t_f) - y_2(t_f)$ and $f_1(t, y(t), u(t)) = y_1(t)u(t)$, $f_2(t, y(t), u(t)) = -y_2(t)u(t)$ for $x_1, x_2 \in [-1,1]$ and $t \in [s, t_f]$, in which $s = 0, t_f = 1$. Thus, the HJB problem is

$$-v_t + \sup_{0 \le u \le 1} (-v_{x_1} x_1 u + v_{x_2} x_2 u) = 0, \tag{44a}$$

$$v(1, x_1, x_2) = -x_1 - x_2. \tag{44b}$$

method in Ref. [16]		The proposed method here	
n	Averaged absolute error	n	Averaged absolute error
8	8.31×10^{-4}	3	9.5001×10^{-6}
13	1.87×10^{-4}	5	8.4445×10^{-6}
23	3.6×10^{-5}	7	6.3750×10^{-7}

Tab. 2. The computed errors for Example 2.

The exact value function for (44a) – (44b) is [39]

$$v_{Exact}(t, x_1, x_2) = \begin{cases} -(x_1 + x_2)e^{1-t} & x_1 + x_2 > 0, \\ -(x_1 + x_2) & x_1 + x_2 \leq 0. \end{cases} \tag{45}$$

Applying the Chebyshev pseudospectral method for $i, j = 0, \ldots, n$, yields

$$-\dot{\alpha}_{in+j}(t) + \sup_{0 \leq u \leq 1} \Big\{ -\boldsymbol{\alpha}(t)^T \big[\big(\mathbf{D}^T \mathbf{e}_i \otimes \mathbf{e}_j\big)\, \xi_i + \big(\mathbf{e}_i \otimes \mathbf{D}^T \mathbf{e}_j\big)\, \xi_j \big] u \Big\} = 0, \tag{46a}$$

$$\alpha_{in+j}(1) = -\xi_i - \xi_j. \tag{46b}$$

The supremum $u(t)$ for the linear term $-\boldsymbol{\alpha}(t)^T \big[\big(\mathbf{D}^T \mathbf{e}_i \otimes \mathbf{e}_j\big)\, \xi_i + \big(\mathbf{e}_i \otimes \mathbf{D}^T \mathbf{e}_j\big)\, \xi_j \big] u$ is either 0 or 1. Consequently, we conclude that the maximizer is

$$u(t) = \begin{cases} 0, & \textbf{if } -\boldsymbol{\alpha}(t)^T \big[\big(\mathbf{D}^T \mathbf{e}_i \otimes \mathbf{e}_j\big)\, \xi_i + \big(\mathbf{e}_i \otimes \mathbf{D}^T \mathbf{e}_j\big)\, \xi_j \big] < 0, \\ 1, & \textbf{if } -\boldsymbol{\alpha}(t)^T \big[\big(\mathbf{D}^T \mathbf{e}_i \otimes \mathbf{e}_j\big)\, \xi_i + \big(\mathbf{e}_i \otimes \mathbf{D}^T \mathbf{e}_j\big)\, \xi_j \big] > 0. \end{cases} \tag{47}$$

By using (47), the 'sup' function in problem, (46a) – (46b) is resolved and the problem is converted to a classical system of differential equations with terminal conditions. We note that function u in (47) is not continuous in $t = 0$. Thus, to get a better accuracy, we consider the domain decomposition strategy with $m = 1$ and $[-1, +1] = [-1, 0] \cup [0, +1]$.

The results of hybrid method for various values of n are reported in Table 2. Moreover, to make a comparison, the results for method in Ref. [16] which is based on radial basis functions are given. Comparison is very helpful, because using the pseudospectral method we reach the global minimum of the cost functional, even if $v(t, x)$ is not continuously differentiable and the problem is not convex.

5.3. Example 3 (Discontinuous derivative and nonconvex)

Consider the following optimal control problem [16]

$$\min \quad -y(t_f) \tag{48a}$$

$$\tag{48b}$$

$$\text{s.t.} \quad \dot{y}(t) = y(t)u(t), \tag{48c}$$

$$y(s) = x, \tag{48d}$$

$$u : [s, t_f] \to [0, 1]. \tag{48e}$$

The associated HJB problem is as follows

$$-v_t + \sup_{0 \le u \le 1} (-v_x x u) = 0, \tag{49a}$$

$$v(1, x) = -x, \tag{49b}$$

with the following exact solution

$$v_{Exact}(t, x) = \begin{cases} -xe^{1-t} & x > 0, \\ -x & x \le 0. \end{cases} \tag{50}$$

By applying the hybrid method, we get

$$\begin{aligned} &-\dot{\alpha}_i(t) + \sup_{0 \le u \le 1} \{-[\mathbf{D}\boldsymbol{\alpha}(t)]_i \xi_i u\} = 0, \\ &\alpha_i(1) = -\xi_i, \\ &\qquad\qquad i = 0, \ldots, n. \end{aligned}$$

Similar to Example 2, we find that

$$u(t) = \begin{cases} 0, & -[\mathbf{D}\boldsymbol{\alpha}(t)]_i \, \xi_i < 0, \\ 1, & -[\mathbf{D}\boldsymbol{\alpha}(t)]_i \, \xi_i > 0. \end{cases}$$

Since $u(t)$ is discontinuous in $t = 0$, we apply the multidomain strategy with $m = 1$ and $[-1, +1] = [-1, 0] \cup [0, +1]$.

In Figure 2 (Left), the value function with $m = 1$ and $n = 20$ collocation points is depicted, where the absolute error is shown in the right hand side.

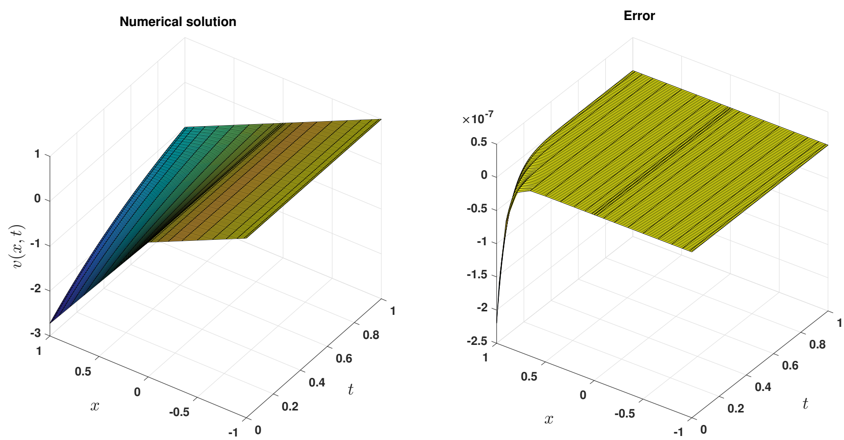

Fig. 2. (Example 3) (Left) The value function $v(t, x)$ computed for $s = 0, t_f = 1$ and $n = 20$, (Right) The absolute error.

method in Ref. [16]		The proposed method here	
n	Averaged absolute error	n	Averaged absolute error
5	1.575×10^{-2}	3	3.7898×10^{-10}
9	5.34×10^{-3}	5	2.2742×10^{-10}
14	2.18×10^{-3}	7	1.6239×10^{-10}
26	6.59×10^{-4}	9	1.2623×10^{-10}

Tab. 3. Computed errors for Example 3.

The results for the hybrid method and the results in Ref. [16] are demonstrated in Table 3. From this table, one can conclude that only a few number of grids is required to achieve much better accurate solution. Here, it is shown that the current method combined with the domain decomposition technique produces solutions accurate up to the accuracy of the machine, even if the problem is not convex.

5.4. Example 4 (Discontinuous Derivative and Nonconvex)

In this example, we consider the following optimal control problem [16]

$$\min \quad -y^2(t_f) \tag{51a}$$

$$\tag{51b}$$

$$\text{s.t.} \quad \dot{y}(t) = u(t), \tag{51c}$$

$$y(s) = x, \tag{51d}$$

$$u : [s, t_f] \to [-1, 1], \tag{51e}$$

with the following HJB problem

$$-v_t + \sup_{-1 \le u \le 1} (-v_x u) = 0, \tag{52a}$$

$$v(1, x) = -x^2. \tag{52b}$$

The exact value function for (49) is [16]

$$v_{Exact}(t, x) = -[|x| + (1 - t)]^2. \tag{53}$$

The results for the hybrid method with $m = 1$ and various values of n, together with the results of [16] are compared in Table 4.

method in Ref. [16]		The proposed method here	
n	Averaged absolute error	n	Averaged absolute error
5	1.933×10^{-2}	5	2.5802×10^{-14}
10	6.127×10^{-3}	7	2.1094×10^{-14}
17	1.933×10^{-3}		
32	6.39×10^{-4}		

Tab. 4. Computed errors for Example 4.

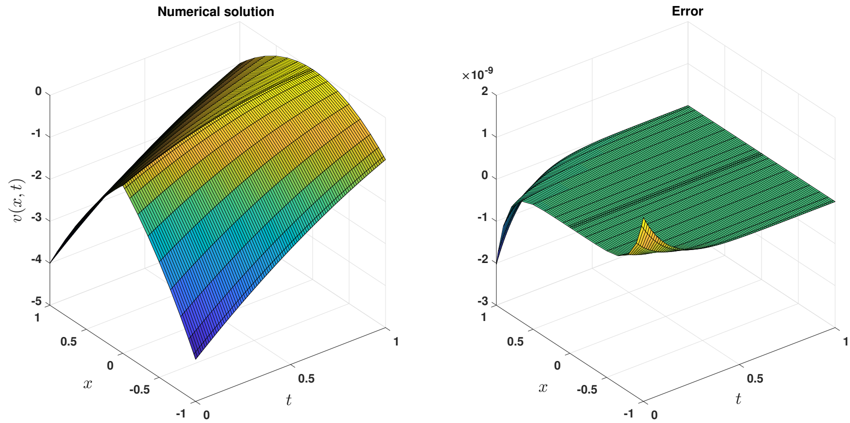

Fig. 3. (Example 4) (Left) The value function $v(t, x)$ computed for $s = 0, t_f = 1$ and $n = 20$, (Right) The absolute error.

Figure 3 shows the value function and absolute error for $n = 10$.

Example 4, works with the simple supremum term (see (52a)). This is very helpful in efficient solving the problem altogether, (see Table 4 and compare it with Tables 1, 2 and 3).

5.5. Example 5 (Highly Nonlinear)

Here, we consider the problem [23]

$$\min \quad \int_0^1 \Big(y^4(t) + u^4(t)\Big)\,\mathrm{d}t \tag{54a}$$

$$\tag{54b}$$

$$\text{s.t.} \quad \dot{y}(t) = u(t), \tag{54c}$$

$$y(s) = x. \tag{54d}$$

Writing the HJB problem for it, yields

$$-v_t + \sup_u \left(-uv_x - x^4 - u^4\right) = 0, \tag{55a}$$

$$v(1, x) = 0. \tag{55b}$$

By the help of the Chebyshev pseudospectral discretization for $i = 0, \ldots, n$, one can write

$$-\dot{\alpha}_i(t) + \sup_u \left\{-u[\mathbf{D}\boldsymbol{\alpha}(t)]_i - {\xi_i}^4 - u^4\right\} = 0, \tag{56a}$$

$$\alpha_i(1) = 0. \tag{56b}$$

Supremum $u(t)$ for the $forth$ degree term $-u[\mathbf{D}\boldsymbol{\alpha}(t)]_i - {\xi_i}^4 - u^4$ happens if

$$u(t) = -\left(\frac{1}{4}[\mathbf{D}\boldsymbol{\alpha}(t)]_i\right)^{\frac{1}{3}}. \tag{57}$$

By substituting the above control function in problem (56a) − (56b) for $i = 0, \dots, n$, we get

$$\begin{aligned} -\dot{\alpha}_i(t) &= -3\left(\frac{1}{4}[\mathbf{D}\boldsymbol{\alpha}(t)]_i\right)^{\frac{4}{3}} + {\xi_i}^4, \\ \alpha_i(1) &= 0. \end{aligned}$$

This problem can be expressed in the following vector form

$$\begin{aligned} -\dot{\boldsymbol{\alpha}}(t) &= -3\left(\frac{1}{4}\mathbf{D}\boldsymbol{\alpha}(t)\right)^{\frac{4}{3}} + \boldsymbol{\xi}^4, \\ \boldsymbol{\alpha}(1) &= 0. \end{aligned}$$

Now from the above problem, one can calculate coefficients $\alpha_0, \dots, \alpha_n$. Figure 4 shows the value function for $n = 30$.

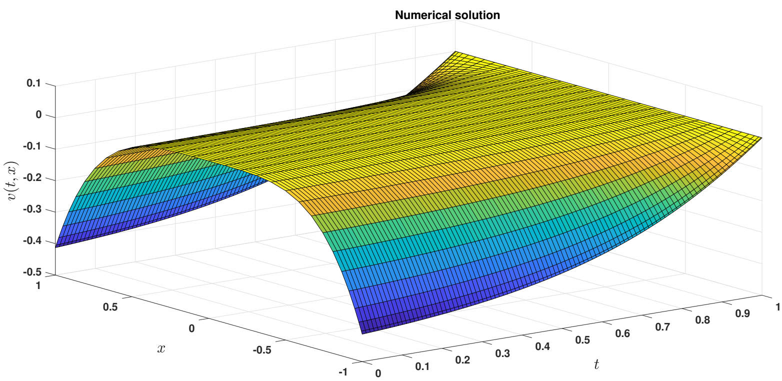

Fig. 4. (Example 5) The value function $v(t, x)$ computed for $s = 0, t_f = 1$ and $n = 30$.

6. CONCLUSIONS

In this paper, we present a hybrid pseudospectral scheme for solving a class of linear and nonlinear optimal control problems. This method is very helpful, because using it we reach the global minimum of the cost functional, even if the problem is not convex. Using these techniques, we approximate functions as a weighted sum of smooth basis functions which are Lagrange polynomials over Chebyshev–Gauss–Lobatto nodes. We begin by introducing the HJB problem and develop a fully coherent numerical method. Hybrid method here, consists of collocation points in the form of Chebyshev–Gauss–Lobatto grids, Lagrange interpolators, efficient ordinary differential equation solvers, matrix differentiation and domain decomposition technique to solve linear and nonlinear convex and nonconvex HJB problems. Numerical results show that hybrid method is a reliable scheme and it can be utilized as a powerful tool to solve HJB problems. Moreover, the proposed method has a very good performance, even when a few amounts of grids are used. The main advantages of this approach lie in its good accuracy, very low numerical complexity, easy implementation and finding a feedback solution.

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