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MULTI-AGENT NETWORK FLOWS THAT SOLVE LINEAR COMPLEMENTARITY PROBLEMS

Shu Liang and Xianlin Zeng

In this paper, we consider linear complementarity problems with positive definite matrices through a multi-agent network. We propose a distributed continuous-time algorithm and show its correctness and convergence. Moreover, with the help of Kalman–Yakubovich–Popov lemma and Lyapunov function, we prove its asymptotic convergence. We also present an alternative distributed algorithm in terms of an ordinary differential equation. Finally, we illustrate the effectiveness of our method by simulations.

Keywords: distributed algorithm, linear complementarity problem, multi-agent network, nonsmooth algorithm, continuous-time algorithm

Classification: 90C33, 68W15

1. INTRODUCTION

Recently, multi-agent networks have received much attention in various research fields such as distributed optimization and game [12, 13, 14, 29, 32], distributed machine learning [25] and distributed computation of equations [22, 31]. In contrast to centralized computations, network based distributed algorithms do not require overall information to accomplish a task and introduce an inherent robustness to communication or sensor failures, and environmental uncertainties. Moreover, distributed algorithms only require each agent to know a local part of the data, leading to a decomposition structure that is quite preferable for large scale problems.

This work focuses on another significant type of problems called *linear complementarity problems* (LCPs). The LCPs play a fundamental role in broad research areas such as game theory [28], geodetic network [27], contact problem [17], computer graphics [24], circuit modeling [21], energy market [6], and image restoration [3]. Rich theories and many conventional algorithms of LCPs were presented in the monograph [19]. Moreover, interesting research branches of LCPs include, just to mention a few, the robust version in the presence of uncertain data [26], which was motivated by robust optimization; properties of the solution map of a parametric LCP [7], which employed powerful tools from variational analysis; and various algorithms for solving LCPs [5, 10, 15]. Also, many works encountered or dealt with large scale LCPs, e. g., see [4, 16, 18, 23].

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Existing methods for designing distributed algorithms, in general, fail to solve LCPs. For example, saddle point conditions are the key technical foundation in many distributed optimization works. In a LCP, however, such conditions or tools do not hold any more. Some distributed computation methods for solving linear algebraic equations have just been proposed in recent works [11, 22, 30]. Comparing with linear algebraic equations, LCPs are quite different and difficult, due to the nonsmoothness caused by the complementarity.

In this paper, we develop a distributed continuous-time nonsmooth algorithm to solve the linear complementarity problem under reasonable assumptions. Note that continuous-time algorithms become more and more popular in distributed design [2, 9, 13, 22], which may be easily implemented by physical agents; moreover, continuous-time methods may provide effective approaches, possibly employing the powerful control theory. We design the algorithm in light of a differential inclusion with maximal monotone map, which guarantees the existence and uniqueness of its trajectory. Then we obtain the asymptotic convergence of the algorithm by virtue of Kalman–Yakubovich–Popov (KYP) lemma and a suitably constructed Lyapunov function. Furthermore, we present a modified algorithm in terms of a differential equation with discontinuous righthand side, which yields the same trajectory that solves the considered LCP by the original algorithm.

The rest of paper is organized as follows: Section 2 provides preliminaries and formulates the problem. Section 3 presents the main results, followed by simulations in Section 4. Finally, Section 5 gives some concluding remarks.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we introduce necessary preliminaries and formulate the problem.

2.1. Notations and preliminaries

$\mathbb{R}$ and $\mathbb{R}_+$ are the set of real numbers and nonnegative real numbers, respectively. $\mathbf{0}$ and $\mathbf{1}$ are vectors of proper dimension with all the elements as 0 and 1, respectively. $\boldsymbol{I}$ and $\boldsymbol{O}$ are the identity matrix and zero matrix, respectively. $col\{x_1, \ldots, x_n\} = (x_1^T, \ldots, x_n^T)^T$ is the column vector stacked with column vectors $x_1, \ldots, x_n$. $\mathrm{rge}(A)$ is the range space of matrix A. Given a vector $\boldsymbol{a}$ and a symmetric matrix P, $\boldsymbol{a} \geq \mathbf{0}$ means that each component of $\boldsymbol{a}$ is nonnegative, while $P \succ 0$ means that P is positive definite. Given a set S, the minimal selection operator $\mathfrak{m}(S)$ is any element of S with least norm.

For a convex set C and a point $x \in C$, the *tangent cone* and *normal cone* to C at x are

$$\mathcal{T}_C(x) \triangleq \Big\{ \lim_{k\to\infty} \frac{x_k - x}{t_k} \,|\, x_k \in C, t_k > 0, \text{ and } x_k \to x, t_k \to 0 \Big\}, \tag{1}$$

and

$$\mathcal{N}_C(x) \triangleq \{v \in \mathbb{R}^n \,|\, v^T(y - x) \leq 0, \text{ for all } y \in C\}, \tag{2}$$

respectively [20].

A network with its interaction topology described by a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, 2, \ldots N\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. $\mathcal{G}$ is said to be undirected if $\{i, j\} \in \mathcal{E} \Rightarrow \{j, i\} \in \mathcal{E}$. The adjacent matrix $\mathcal{A} = [a_{ij}]_{N\times N}$ satisfies $a_{ij} = 1$ if

$(j,i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. Let $d_i = \sum_{j=1}^{N} a_{ij}$ and $\mathcal{D} = \text{diag}\{d_1, \ldots, d_N\}$. Then the Laplacian matrix of $\mathcal{G}$ is defined as $L = \mathcal{D} - \mathcal{A}$.

Lemma 2.1. Graph $\mathcal{G}$ is connected and undirected if and only if $L = L^T$ is positive semidefinite with zero as its simple eigenvalue.

The positive realness of a linear dynamical system [8, page 237] is defined as follows.

Definition 2.2. A $p \times p$ proper rational transfer function matrix $G(s)$ is called positive real if

- poles of all element of $G(s)$ are in $\text{Re}(s) \leq 0$,
- for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite, and
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian.

The following lemma is about the positive realness, known as Kalman–Yakubovich–Popov (KYP) Lemma [8, page 240].

Lemma 2.3. (KYP Lemma) Let $G(s) = C(sI - A)^{-1}B$ be a $p \times p$ transfer function matrix where (A, B) is controllable and (A, C) is observable. Then $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T \succ 0, R$ such that

$$\begin{aligned} PA + A^T P &= -R^T R, \\ PB &= C^T. \end{aligned} \tag{3}$$

A differential inclusion can be expressed as:

$$\dot{x} \in \mathcal{S}(x), \quad x(0) = x_0, \tag{4}$$

where $\mathcal{S}$ is a set-valued map that associates any $w \in \mathbb{R}^n$ with a subset $\mathcal{S}(w)$ of $\mathbb{R}^n$ [1]. A trajectory $x(t) : [0, +\infty) \to \mathbb{R}^n$ is said to be a *solution* to (4) if it is absolutely continuous and satisfies the inclusion for almost all $t \in [0, +\infty)$. Moreover, $x(\cdot)$ is said to be a *viable* solution with respect to a convex set $K \subset \mathbb{R}^n$ if $x(t) \in K, \forall\, t \in [0, +\infty)$. It follows from the *viability theory* that (4) has a viable solution if it has a solution and

$$\forall\, w \in K, \quad \mathcal{S}(w) \cap \mathcal{T}_K(w) \neq \emptyset. \tag{5}$$

For a set-valued map $\mathcal{S}$, the domain of $\mathcal{S}$ is defined as $\text{dom}\, S \triangleq \{w \in \mathbb{R}^n \mid S(w) \neq \emptyset\}$, and the graph of $\mathcal{S}$ is defined as $\text{gph}\, \mathcal{S} \triangleq \{(w, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid v \in \mathcal{S}(w)\}$. $\mathcal{S}$ is said to be *monotone* if

$$(v - v')^T (w - w') \geq 0, \quad \forall\, (w, v), (w', v') \in \text{gph}(S). \tag{6}$$

Moreover, S is said to be *maximal monotone* if there is no other monotone set-valued map $\tilde{\mathcal{S}}$ with $\text{gph}(\tilde{\mathcal{S}}) \supset \text{gph}(\mathcal{S})$. We introduce the following lemma [1, Theorem 1, page 147], which lays a theoretic foundation for our nonsmooth algorithm design.

Lemma 2.4. Let $\mathcal{S}$ be a maximal monotone set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Consider the differential inclusion

$$\dot{x} \in -\mathcal{S}(x), \quad x(0) = x_0 \in \operatorname{dom} \mathcal{S}. \tag{7}$$

Then there exists a unique solution $x(\cdot)$ defined on $[0, +\infty)$, which is the slow solution as

$$\dot{x} = -\mathfrak{m}(\mathcal{S}(x)), \text{ for almost all } t > 0. \tag{8}$$

2.2. Problem formulation

Given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the *linear complementarity problem* [19], denoted by $\mathrm{LCP}(q, M)$, is to find a vector $z \in \mathbb{R}^n$ such that

$$z \geq \mathbf{0}, \tag{9a}$$

$$q + Mz \geq \mathbf{0}, \tag{9b}$$

$$z^T(q + Mz) = 0. \tag{9c}$$

The solution set of the $\mathrm{LCP}(q, M)$ is denoted by $\mathrm{SOL}(q, M)$.

Our goal is to solve $\mathrm{LCP}(q, M)$ in a distributed manner through a multi-agent network, described by a graph with nodes regarded as agents. The data matrix M and vector q is decomposed as

$$M = M_1 + M_2 + \cdots + M_N, \quad q = q_1 + q_2 + \cdots + q_N. \tag{10}$$

For each $i \in \mathcal{V}$, the ith agent updates its local variable $x_i \in \mathbb{R}^n$ to estimate the solution $z \in \mathrm{SOL}(q, M)$, based on private data q_i, M_i and information from its neighbors.

The LCPs do not share the "non-empty convex intersection" property, that is,

$$\mathrm{SOL}(q, M) \nsubseteq \bigcap_{i=1}^{N} \mathrm{SOL}(q_i, M_i).$$

This is easily seen from (9c) that $z \in \bigcap_{i=1}^{N} \mathrm{SOL}(q_i, M_i)$ implies $z^T(q_i + M_i z) = 0$, whereas the original problem corresponds to $\sum_{i=1}^{N} z^T(q_i + M_i z) = 0$. Therefore, the methods and techniques in [11, 22] and many distributed optimization works are inapplicable to our problem, even though each $\mathrm{SOL}(q_i, M_i)$ can be convex.

The decomposition (10) can be very flexible. In particular, each M_i can be quite sparse, though M may not be. Also, recall that the decomposition in [11, 22] takes the following form

$$M = \begin{pmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_N^T \end{pmatrix}, \quad q = \begin{pmatrix} z_1^T \\ z_2^T \\ \vdots \\ z_N^T \end{pmatrix}, \tag{11}$$

with h_i, z_i being assigned to the ith agent. Clearly, this is a special case of (10) with

$$M_i = \begin{pmatrix} \boldsymbol{O} \\ h_i^T \\ \boldsymbol{O} \end{pmatrix}, \quad q_i = \begin{pmatrix} \mathbf{0} \\ z_i \\ \mathbf{0} \end{pmatrix}. \tag{12}$$

LCP(q, M) can be equivalently represented as a nonsmooth equation problem $\mathbf{0} = \min\{z, q+Mz\}$ or a set-valued generalized equation problem $\mathbf{0} \in q+Mz+\mathcal{N}_{\mathbb{R}^n_+}(z)$, where the operator $\min\{\cdot,\cdot\}$ is in the componentwise sense. Such an inherit nonsmoothness makes our problem totally different from the linear algebraic equation problem $\mathbf{0} = q + Mz$ considered in [11, 22].

The following assumptions are adopted.

Assumption 1. M is positive definite (not necessarily symmetric). That is, $z^T Mz > 0$ for all nonzero $z \in \mathbb{R}^n$.

Assumption 2. The communication graph $\mathcal{G}$ is connected and undirected.

Assumption 1 is adopted mainly for two reasons. Firstly, the LCP with a positive definite matrix is encountered in many problems such as [3, 27]. Secondly, in general, a LCP can be very complicated and its solution set can be empty or set-valued (i. e., with multiple solutions). Thus, we need to impose some restriction for some well-posededness. Also, in order to solve the problem in a distributed manner, some stronger condition is often required than those in centralized algorithms. It follows from [19, Theorem 3.1.6] that LCP(q, M) has a unique solution for all $q \in \mathbb{R}^n$, if M is positive definite. Thus, we restrict our current attention on such a class of problems.

A special case of our problem includes LCP(q, M) with symmetric and positive definite matrix M, which corresponds to a quadratic programming with convex objective function as $f(z) = q^T z + \frac{1}{2} z^T Mz$ and the constraints $z \geq \mathbf{0}$. Important source problems of such type include the least square problems with inequality constraints. If each M_i is also symmetric, then $f(z)$ is separable with each $f_i(z) = q_i^T z + \frac{1}{2} z^T M_i z$. Note that $f_i(z)$ can be non-convex, since M_i is not necessarily positive semidefinite. Moreover, in our formulation, there is no restriction on the symmetry. These observations indicate that the considered problem differs from distributed convex optimizations.

3. MAIN RESULTS

In this section, we present the distributed algorithm design and give the convergence analysis.

3.1. Distributed algorithm

In this subsection, we present our distributed algorithm to solve LCP(q, M). For $i \in \mathcal{V}$, the ith agent has the following update rule:

$$\begin{cases} \dot{x}_i \in -(M_i x_i + q_i) - \gamma \sum_{j=1}^{N} a_{ij}(x_i - x_j) - \sum_{j=1}^{N} a_{ij}(\lambda_i - \lambda_j) - \mathcal{N}_{\mathbb{R}^n_+}(x_i), & x_i(0) \in \mathbb{R}^n_+ \\ \dot{\lambda}_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j), & \lambda_i(0) \in \mathbb{R}^n \end{cases} \tag{13}$$

Algorithm (13) is fully distributed since the ith agent only needs its local data q_i, M_i and exchanges the information of variables x, λ with its neighbors.

For convenience, we rewrite Algorithm (13) in a compact form as

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} \in -\mathcal{F}(\boldsymbol{x}, \boldsymbol{\lambda}) - \mathcal{N}_\Theta(\boldsymbol{x}, \boldsymbol{\lambda}), \quad (\boldsymbol{x}(0), \boldsymbol{\lambda}(0)) \in \Theta, \tag{14}$$

where $\boldsymbol{x} = col\{x_1, \ldots, x_N\}$, $\boldsymbol{\lambda} = col\{\lambda_1, \ldots, \lambda_N\}$, $\boldsymbol{q} = col\{q_1, \ldots, q_N\}$, $\Theta = \mathbb{R}_+^{nN} \times \mathbb{R}^{nN}$,

$$\mathcal{F}(\boldsymbol{x}, \boldsymbol{\lambda}) = \begin{bmatrix} \mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{x} + \boldsymbol{q} + \gamma(L \otimes I_n)\boldsymbol{x} + (L \otimes I_n)\boldsymbol{\lambda} \\ -(L \otimes I_n)\boldsymbol{x} \end{bmatrix},$$

and L is the Laplacian matrix of the communication topology. The parameter $\gamma > 0$ is chosen sufficiently large such that $\mathrm{diag}\{M_1, \ldots, M_N\} + \gamma(L \otimes I_n)$ is positive definite. Such a parameter γ always exists, as indicated by the following lemma.

Lemma 3.1. Under Assumption 1, there exists $\gamma^* > 0$ such that for any $\gamma > \gamma^*$, $\mathrm{diag}\{M_1, \ldots, M_N\} + \gamma(L \otimes I_n)$ is positive definite.

Proof. Define

$$\Gamma \triangleq (\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T) \otimes I_n, \quad \Gamma_\perp \triangleq \boldsymbol{I} - \Gamma. \tag{15}$$

For any $\boldsymbol{x} \in \mathbb{R}^{nN}$, define

$$\boldsymbol{v} = \Gamma\boldsymbol{x}, \quad \boldsymbol{w} = \Gamma_\perp\boldsymbol{x}, \quad \text{and} \quad z = (\frac{1}{N}\mathbf{1}_N^T \otimes I_n)\boldsymbol{x}. \tag{16}$$

Then $\boldsymbol{v} = col\{z, \ldots, z\}$, $\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$ and

$$\begin{aligned} &\boldsymbol{x}^T(\mathrm{diag}\{M_1, \ldots, M_N\} + \gamma(L \otimes I_n))\boldsymbol{x} \\ &= \boldsymbol{v}^T \mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{v} + \boldsymbol{v}^T \mathrm{diag}\{M_1 + M_1^T, \ldots, M_N + M_N^T\}\boldsymbol{w} \\ &\quad + \boldsymbol{w}^T \mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{w} + \gamma\boldsymbol{w}^T(L \otimes I_n)\boldsymbol{w} \\ &= z^T M z + z^T(\mathbf{1}_N^T \otimes I_n)\,\mathrm{diag}\{M_1 + M_1^T, \ldots, M_N + M_N^T\}\boldsymbol{w} \\ &\quad + \boldsymbol{w}^T \mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{w} + \gamma\boldsymbol{w}^T(L \otimes I_n)\boldsymbol{w} \\ &\geq k_1\|z\|^2 - k_2\|z\|\|\boldsymbol{w}\| - k_3\|\boldsymbol{w}\|^2 + \gamma k_4\|\boldsymbol{w}\|^2, \end{aligned}$$

where $k_1 > 0$ is the smallest eigenvalue of $\frac{1}{2}(M + M^T)$, $k_2 = \|(\mathbf{1}_N^T \otimes I_n)\,\mathrm{diag}\{M_1 + M_1^T, \ldots, M_N + M_N^T\}\|$, $k_3 = \|\mathrm{diag}\{M_1, \ldots, M_N\}\|$ and k_4 is the smallest nonzero eigenvalue of L. Then the conclusion holds with $\gamma^* = (\frac{k_2}{k_1} + k_3)\frac{1}{k_4}$, which completes the proof. □

We assume that an upper bound of γ^* is available to the algorithm designer.

3.2. Convergence analysis

We first present basic properties with respect to the trajectory of algorithm (14).

Theorem 3.2. Under Assumptions 1 − 2, (14) has a unique trajectory $(\boldsymbol{x}(t), \boldsymbol{\lambda}(t))$, which satisfies $(\boldsymbol{x}(t), \boldsymbol{\lambda}(t)) \in \Theta$.

Proof. Since

$$\begin{bmatrix} \boldsymbol{x}' - \boldsymbol{x} \\ \boldsymbol{\lambda}' - \boldsymbol{\lambda} \end{bmatrix}^T (\mathcal{F}(\boldsymbol{x}', \boldsymbol{\lambda}') - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\lambda})) = (\boldsymbol{x}' - \boldsymbol{x})^T (\mathrm{diag}\{M_1, \ldots, M_N\} + \gamma(L \otimes I_n))(\boldsymbol{x}' - \boldsymbol{x}) \geq 0,$$

the set-valued map $\mathcal{F}(\boldsymbol{x}, \boldsymbol{\lambda}) + \mathcal{N}_\Theta(\boldsymbol{x}, \boldsymbol{\lambda})$ is maximal monotone, according to [20, page 559]. Therefore, (14) is a differential inclusion with maximal monotone map. It follows from Lemma 2.4 that (14) has a unique solution $(\boldsymbol{x}(t), \boldsymbol{\lambda}(t))$.

Also, it is not difficult to verify that

$$\forall\, (\boldsymbol{x}, \boldsymbol{\lambda}) \in \Theta, \quad -(\mathcal{F}(\boldsymbol{x}, \boldsymbol{\lambda}) + \mathcal{N}_\Theta(\boldsymbol{x}, \boldsymbol{\lambda})) \cap \mathcal{T}_\Theta(\boldsymbol{x}, \boldsymbol{\lambda}) \neq \emptyset.$$

Thus, the unique trajectory $(\boldsymbol{x}(t), \boldsymbol{\lambda}(t))$ belongs to Θ, which completes the proof. □

Next, we check the equilibrium of (14), which corresponds to the solution of the considered LCP.

Theorem 3.3. Under Assumptions 1 − 2, a point $\boldsymbol{x}^* \in \mathbb{R}_+^{nN}$ together with some $\boldsymbol{\lambda}^* \in \mathbb{R}^{nN}$ is an equilibrium of (14) if and only if

$$x_1^* = \cdots = x_N^* = z^*, \quad z^* \in \mathrm{SOL}(q, M). \tag{17}$$

Proof. Necessity: Let $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ satisfy $\mathbf{0} \in \mathcal{A}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$. Then there exists some $z^* \in \mathbb{R}^n$ such that $x_1^* = \cdots = x_N^* = z^*$, and

$$\begin{aligned} \boldsymbol{x}^* &\geq \mathbf{0}, \\ \mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{x}^* + \boldsymbol{q} + \gamma(L \otimes I_n)\boldsymbol{x}^* + (L \otimes I_n)\boldsymbol{\lambda}^* &\geq \mathbf{0}, \\ \boldsymbol{x}^{*T}(\mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{x}^* + \boldsymbol{q} + \gamma(L \otimes I_n)\boldsymbol{x}^* + (L \otimes I_n)\boldsymbol{\lambda}^*) &= 0. \end{aligned} \tag{18}$$

Substituting the z^* into (18) yields

$$\begin{aligned} z^* &\geq \mathbf{0}, \\ Mz^* + q &\geq \mathbf{0}, \\ z^{*T}(Mz^* + q) &= 0, \end{aligned} \tag{19}$$

where the second condition in (19) is derived from the second condition in (18) by left multiplying $\mathbf{1}_N^T \otimes I_n$. Therefore, $z^* \in \mathrm{SOL}(q, M)$.

Sufficiency: Suppose that $\boldsymbol{x}$ satisfies (17). Then it suffices to verify the existence of some $\boldsymbol{\lambda}^*$ such that

$$\mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{x}^* + \boldsymbol{q} + (L \otimes I_n)\boldsymbol{\lambda}^* \geq \mathbf{0}, \tag{20}$$

or equivalently,

$$(\Gamma + \Gamma_\perp)(\mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{x}^* + \boldsymbol{q} + (L \otimes I_n)\boldsymbol{\lambda}^*) \geq \mathbf{0},$$

where Γ and $\Gamma_\perp$ are defined in (15). Note that

$$\Gamma(\mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{x}^* + \boldsymbol{q} + (L \otimes I_n)\boldsymbol{\lambda}^*) = \Gamma(\mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{x}^* + \boldsymbol{q}) \geq \mathbf{0}.$$

Then (20) holds if

$$\Gamma_\perp(\mathrm{diag}\{M_1, \ldots, M_N\}\boldsymbol{x}^* + \boldsymbol{q} + (L \otimes I_n)\boldsymbol{\lambda}^*) = \mathbf{0}.$$

Such a $\boldsymbol{\lambda}^*$ always exists because $\mathrm{rge}(\Gamma_\perp(L \otimes I_n)) = \mathrm{rge}(\Gamma_\perp)$.

Moreover, if $\boldsymbol{\lambda}^*$ is a solution to (20), then it is clear that any element of the following set

$$\Lambda = \{\boldsymbol{\lambda} \,|\, \boldsymbol{\lambda} = \boldsymbol{\lambda}^* + \mathbf{1}_N \otimes \nu, \nu \in \mathbb{R}^n\}, \tag{21}$$

is also a solution to (20). It completes the proof. □

Then we prove the convergence of the algorithm.

Theorem 3.4. Under Assumptions 1−2, algorithm (14) asymptotically converges to an equilibrium point $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$.

Proof. Since $\mathrm{diag}\{M_1, \ldots, M_N\} + \gamma(L \otimes I_n)$ is positive definite, there exists $\alpha > 0$ such that $\mathrm{diag}\{M_1, \ldots, M_N\} + \gamma(L \otimes I_n) - \alpha\boldsymbol{I}$ is positive semidefinite. Algorithm (14) can be rewritten as

$$\begin{cases} \dot{\boldsymbol{x}} = -\alpha\boldsymbol{x} - (L \otimes I_n)\boldsymbol{\lambda} + \boldsymbol{u} \\ \dot{\boldsymbol{\lambda}} = (L \otimes I_n)\boldsymbol{x} \\ \boldsymbol{u} \in -(\mathrm{diag}\{M_1, \ldots, M_N\} + \gamma(L \otimes I_n) - \alpha\boldsymbol{I})\boldsymbol{x} - \boldsymbol{q} - \mathcal{N}_{\mathbb{R}_+^{nN}}(\boldsymbol{x}) \end{cases} \tag{22}$$

Let $\boldsymbol{G}(s)$ be the transfer function matrix of the open-loop linear system from input $\boldsymbol{u}$ to output $\boldsymbol{x}$ in (22). That is,

$$\boldsymbol{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}, \tag{23}$$

where

$$\hat{A} = \begin{bmatrix} -\alpha\boldsymbol{I} & -(L \otimes I_n) \\ L \otimes I_n & \boldsymbol{O} \end{bmatrix}, \hat{B} = \begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix}, \hat{C} = \begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix}^T. \tag{24}$$

By some calculations, we have

$$\boldsymbol{G}(s) = s(s^2\boldsymbol{I} + \alpha s\boldsymbol{I} + (L \otimes I_n)^2)^{-1}. \tag{25}$$

Since L is symmetric, it can be transformed into a diagonal matrix $D = \mathrm{diag}\{0, \mu_2, \ldots, \mu_N\}$ via some orthogonal matrix Q as $L = QDQ^T$. Then $\boldsymbol{G}(s) = (Q\tilde{G}(s)Q^T) \otimes I_n$, where

$$\tilde{G}(s) = \begin{pmatrix} \frac{1}{s+\alpha} & & & \\ & \frac{s}{s^2+\alpha s+\mu_2^2} & & \\ & & \ddots & \\ & & & \frac{s}{s^2+\alpha s+\mu_N^2} \end{pmatrix}. \tag{26}$$

Consequently, $\boldsymbol{G}(s)$ is positive real according to Definition 2.2. Thus, for any minimal realization (A, B, C) of $\boldsymbol{G}(s)$, it follows from Lemma 2.3 that there exist $P = P^T \succ 0$ and R rendering (3).

Let $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ be an equilibrium of (14) that satisfies $(\mathbf{1}_N \otimes I_n)^T \boldsymbol{\lambda}^* = (\mathbf{1}_N \otimes I_n)^T \boldsymbol{\lambda}(0)$. Such an equilibrium exists, according to (21). Note that $(\hat{A}, \hat{B}, \hat{C})$ in (24) is not a minimal realization of $\boldsymbol{G}(s)$. The only internal subsystem that is not involved in $\boldsymbol{G}(s)$ is the dynamics of the variable $\boldsymbol{\lambda}_{\bar{c}\bar{o}}(t) \triangleq (\mathbf{1}_N \otimes I_n)^T \boldsymbol{\lambda}(t)$ with the system matrices as $(\boldsymbol{O}, \mathbf{0}, \mathbf{0})$. Clearly, $\boldsymbol{\lambda}_{\bar{c}\bar{o}}$ is neither controllable nor observable, but it is stable because $\dot{\boldsymbol{\lambda}}_{\bar{c}\bar{o}}(t) = \mathbf{0}$, $\forall\, t > 0$. As a result, $\boldsymbol{\lambda}_{\bar{c}\bar{o}}(t) \equiv (\mathbf{1}_N \otimes I_n)^T \boldsymbol{\lambda}^* \triangleq \boldsymbol{\lambda}^*_{\bar{c}\bar{o}}$. Since $\boldsymbol{\lambda}_{\bar{c}\bar{o}}(t) - \boldsymbol{\lambda}^*_{\bar{c}\bar{o}} \equiv \mathbf{0}$, we can choose some extended matrices $\hat{R}$ and $\hat{P} = \hat{P}^T \succ 0$ with

$$\boldsymbol{\theta}(t) \triangleq \begin{bmatrix} \boldsymbol{x}(t) - \boldsymbol{x}^* \\ \boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^* \end{bmatrix}, \quad V(t) \triangleq \boldsymbol{\theta}^T(t) \hat{P} \boldsymbol{\theta}(t), \tag{27}$$

such that for almost all $t > 0$,

$$\begin{aligned} \dot{V}(t) &= \boldsymbol{\theta}^T(t)(\hat{P}\bar{A} + \hat{A}^T\hat{P})\boldsymbol{\theta}(t) + \boldsymbol{\theta}^T(t)\hat{P}\hat{B}(\boldsymbol{u}(t) - \boldsymbol{u}^*) \\ &= \boldsymbol{\theta}^T(t)(-\hat{R}^T\hat{R})\boldsymbol{\theta}(t) + \boldsymbol{\theta}^T(t) \begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix} (\boldsymbol{u}(t) - \boldsymbol{u}^*) \\ &\le (\boldsymbol{x}(t) - \boldsymbol{x}^*)(\boldsymbol{u}(t) - \boldsymbol{u}^*) \\ &< 0, \end{aligned} \tag{28}$$

where $\boldsymbol{u}^* \in -(\mathrm{diag}\{M_1, \ldots, M_N\} + \gamma(L \otimes I_n) - \alpha \boldsymbol{I})\boldsymbol{x}^* - \boldsymbol{q} - \mathcal{N}_{\mathbb{R}^{nN}_+}(\boldsymbol{x}^*)$. The last inequality in (28) holds because of the monotonicity condition $-(\boldsymbol{x}(t) - \boldsymbol{x}^*)^T(\boldsymbol{u}(t) - \boldsymbol{u}^*) > 0$, $\forall\, t > 0$. Thus, the conclusion follows. □

Remark 3.5. Theorem 3.4 states that our distributed algorithm for the considered LCP achieves the exponential convergence. The key step in the proof is the construction of the Lyapunov function (27) and obtaining the inequality (28) for its first order derivative, which is fulfilled by taking advantage of KYP lemma, different from existing ones in [29]. Also, our nonsmooth analysis approach does not rely on the set-valued Lie derivative of the Lyapunov function, which is different from the work [32].

3.3. Alternative algorithm

In order to avoid the set-valued righthand side of (13) caused by the normal cone, we present an alternative distributed algorithm in the form of a differential equation with discontinuous term on the righthand side, which may be preferable for implementation due to its single-valued form.

For any $\boldsymbol{a} = col\{a_1, \ldots, a_n\} \in \mathbb{R}^n$ and $\boldsymbol{b} = col\{b_1, \ldots, b_n\} \in \mathbb{R}^n_+$, let us define a vector operator

$$\boldsymbol{\pi}(\boldsymbol{a}, \boldsymbol{b}) \triangleq col\{\pi(a_1, b_1), \ldots, \pi(a_n, b_n)\}, \tag{29}$$

where the scalar operator $\pi(a, b)$ is defined on $\mathbb{R} \times \mathbb{R}_+$ as

$$\pi(a, b) \triangleq \begin{cases} 0, & \text{if } b = 0 \text{ and } a \ge 0, \\ a, & \text{otherwise.} \end{cases} \tag{30}$$

Then the alternative distributed algorithm is

$$\begin{cases} \dot{x}_i = -\boldsymbol{\pi}(F_i, x_i) \\ \dot{\lambda}_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j) \end{cases} \tag{31}$$

where

$$F_i = M_i x_i + q_i + \gamma \sum_{j=1}^{N} a_{ij}(x_i - x_j) + \sum_{j=1}^{N} a_{ij}(\lambda_i - \lambda_j).$$

Clearly, algorithm (31) is fully distributed. Moreover, it can be rewritten in a compact form as

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\pi}(\boldsymbol{F}, \boldsymbol{x}) \\ (L \otimes I_n)\boldsymbol{x} \end{bmatrix}, \quad (\boldsymbol{x}(0), \boldsymbol{\lambda}(0)) \in \Theta, \tag{32}$$

where $\boldsymbol{F} = col\{F_1, \ldots, F_N\}$.

Then we have the following result.

Theorem 3.6. Under Assumptions 1 − 2, algorithm (14) and algorithm (32) yield the same trajectories. In particular, the following statements hold.

1. there is a unique trajectory $(\boldsymbol{x}(t), \boldsymbol{\lambda}(t))$ satisfying (32) for almost all $t \geq 0$. Moreover, $(\boldsymbol{x}(t), \boldsymbol{\lambda}(t)) \in \Theta$ for all $t > 0$;
2. the trajectory is exponentially convergent with

$$\lim_{t \to +\infty} (\boldsymbol{x}(t), \boldsymbol{\lambda}(t)) = (\boldsymbol{x}^*, \boldsymbol{\lambda}^*), \tag{33}$$

where $\boldsymbol{x}^*$ satisfies (17).

Proof. For any $(a, b) \in \mathbb{R} \times \mathbb{R}_+$, there holds

$$\mathfrak{m}(a + \mathcal{N}_{\mathbb{R}_+}(b)) = \begin{cases} \mathfrak{m}(\{a\}), & \text{if } b > 0 \\ \mathfrak{m}((-\infty, a]), & \text{if } b = 0 \end{cases} \tag{34}$$

From (34) and (30), one has

$$\pi(a, b) = \mathfrak{m}(a + \mathcal{N}_{\mathbb{R}_{\geq 0}}(b)), \quad \forall\, (a, b) \in \mathbb{R} \times \mathbb{R}_+.$$

Also, $\boldsymbol{\pi}(\boldsymbol{a}, \boldsymbol{b}) = \mathfrak{m}(\boldsymbol{a} + \mathcal{N}_{\mathbb{R}^n_{\geq 0}}(\boldsymbol{b}))$ holds for any $\boldsymbol{a} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^n_+$. Therefore, (32) can be equivalently written as

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = -\mathfrak{m}(\mathcal{F}(\boldsymbol{x}, \boldsymbol{\lambda}) + \mathcal{N}_\Theta(\boldsymbol{x}, \boldsymbol{\lambda})). \tag{35}$$

Thus, the conclusion follows from Lemma 2.4 and Theorems 3.2 − 3.4. □

Some discussions about our methods are summarized as follows.

- Theorems 3.2 – 3.6 provide a complete procedure to prove that the algorithm (13) or (31) solves LCP(q, M) in a distributed manner.

- Our techniques combine differential inclusions, viability theory, KYP lemma and Lyapunov method, as well as some elementary results of LCPs.

- Algorithm (31) is preferable for implementation since it has single-valued righthand side. However, since the term with operator $\boldsymbol{\pi}(\cdot,\cdot)$ in (31) is discontinuous, it is not easy to analyze the algorithm directly. Instead, we start from algorithm (13) in terms of a differential inclusion, which is much convenient for theoretical analysis.

4. NUMERICAL SIMULATIONS

In this section, we give numerical examples for illustration. Consider LCP(q, M) with

$$M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}, \quad q = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}.$$

To solve the problem, we employ a multi-agent system with 6 agents. Consider a decomposition as in (10), where

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$M_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

and

$$q_1 = q_2 = q_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad q_4 = q_5 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad q_6 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

The agents share information over a graph as shown in Figure 1. Our distributed algorithm with $\gamma = 3$ yields the solution $z^* = [3, 1, 0]^T$. The trajectories of the agents are shown in Figure 2.

Next, we consider LCP(q, M) with the decomposition as in (11) and (12) for different network sizes as $N = 5, 6, \ldots, 20$. The data and the communication graph are randomly generated such that Assumption 1 holds. The solution of LCP(q, M), denoted by z^*, is calculated and confirmed by both the conventional (centralized) pivoting algorithm and Newton type iterative algorithm [19]. Then we use our distributed algorithm to solve the problem and take the relative error $e(t) = \frac{\max_{i=1,\ldots,N} \|x_i(t) - z^*\|}{\|z^*\|}$ for $t = 20, 40, 60, 80, 100$. The results of our numerical experiment are shown in Table 1, which verifies the performance of our distributed algorithm for LCPs with different network sizes.

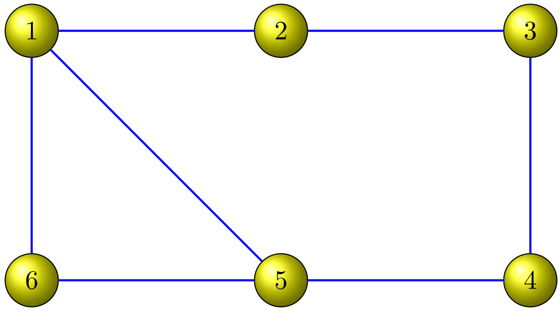

Fig. 1. The communication graph of six agents.

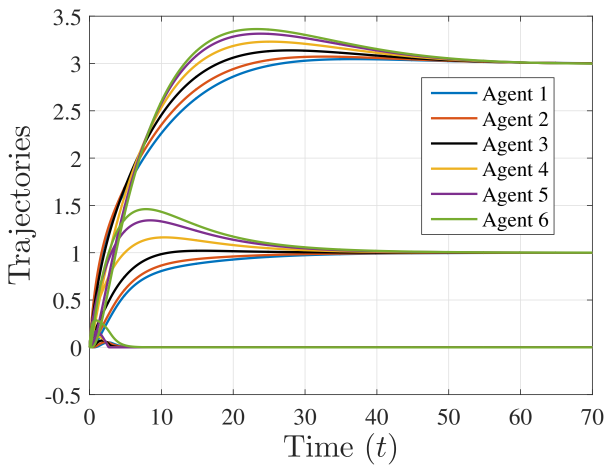

Fig. 2. The trajectories of the agents.

5. CONCLUSIONS

In the paper, distributed linear complementarity problems with positive definite matrices have been studied, and a continuous-time nonsmooth algorithm in terms of a differential inclusion with maximal monotone map has been proposed to solve the problem in a distributed manner. The asymptotic convergence of the algorithm has been proved by virtue of the KYP lemma and Lyapunov method. In addition, an algorithm described by an ordinary differential equation has also been presented. Finally, simulations have illustrated the effectiveness of our algorithm.

	$t = 20$	$t = 40$	$t = 60$	$t = 80$	$t = 100$
$N = 5$	0.0015	0.0000	0.0000	0.0000	0.0000
$N = 6$	0.0058	0.0000	0.0000	0.0000	0.0000
$N = 7$	0.0072	0.0001	0.0000	0.0000	0.0000
$N = 8$	0.0343	0.0012	0.0000	0.0000	0.0000
$N = 9$	0.0420	0.0018	0.0001	0.0000	0.0000
$N = 10$	0.0560	0.0044	0.0004	0.0000	0.0000
$N = 11$	0.0981	0.0099	0.0010	0.0001	0.0000
$N = 12$	0.1021	0.0118	0.0015	0.0002	0.0000
$N = 13$	0.1323	0.0192	0.0030	0.0005	0.0001
$N = 14$	0.1370	0.0218	0.0038	0.0007	0.0001
$N = 15$	0.1217	0.0176	0.0031	0.0006	0.0001
$N = 16$	0.1860	0.0362	0.0073	0.0015	0.0003
$N = 17$	0.2114	0.0458	0.0102	0.0023	0.0005
$N = 18$	0.1905	0.0395	0.0087	0.0020	0.0005
$N = 19$	0.2247	0.0522	0.0124	0.0030	0.0007
$N = 20$	0.2414	0.0601	0.0154	0.0040	0.0011

Tab. 1. relative error vs. problem size.

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