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A NOTE ON WEAKLY-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS

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The paper is dedicated to Professor John Cossey for his 75th birthday

Abstract. A subgroup H of a finite group G is weakly-supplemented in G if there exists a proper subgroup K of G such that $G = HK$. In the paper, we extend one main result of Kong and Liu (2014).

Keywords: weakly-supplemented subgroup; p-nilpotent group; supersolvable group

MSC 2010: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite groups. Let G be a group. A subgroup H of G is complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Such a subgroup K of G is called a complement to H in G. It is quite clear that the existence of complements for certain subgroups of a finite group gives a lot of useful information about its structure. For instance, Hall in 1937 proved that a finite group is solvable if and only if every Sylow subgroup of G is complemented, see [2]. Arad and Ward in [1] proved that a finite group is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented. In particular, Hall in [3] proved that a finite G is supersolvable with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G. In a recent paper, Kong and Liu in [4] studied finite groups for which every minimal subgroup is weakly-supplemented. A subgroup H of G is weakly-supplemented in G

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if there exists a proper subgroup K of G such that $G = HK$. The authors proved that every minimal subgroup of G is weakly-supplemented in G if and only if G is a supersolvable group and all Sylow subgroups of G are elementary abelian. In addition, they also proved the following result:

Theorem 1.1. *Let $\mathscr{R}$ be a formation containing $\mathscr{F}$, the class of supersolvable groups. Let H be a normal subgroup of a solvable group G such that $G/H \in \mathscr{R}$. If every minimal subgroup of the Fitting subgroup $F(G' \cap H)$ of $G' \cap H$ is weakly-supplemented in G, then G belongs to $\mathscr{R}$.*

In this note, we further investigate the influence of weakly-supplemented subgroups on the structure of finite groups along the above direction. It is significant to weaken the hypothesis that the G is solvable in Theorem 1.1. Our main result is the following:

Theorem 1.2. *Let $\mathscr{R}$ be a formation containing $\mathscr{F}$, the class of supersolvable groups. Let H be a solvable normal subgroup of a group G such that $G/H \in \mathscr{R}$. If every minimal subgroup of the Fitting subgroup $F(G' \cap H)$ of $G' \cap H$ is weakly-supplemented in G, then G belongs to $\mathscr{R}$.*

Remark 1.3. Since $F(G' \cap H) = G' \cap F(H) = (G' \cap P_1) \times (G' \cap P_2) \times \ldots \times (G' \cap P_k)$, we know that every minimal subgroup of $F(G' \cap H)$ in [4], Theorem 3.7, is still a minimal subgroup of some $G' \cap P_i$, where P_i is the Sylow p_i-subgroup of $F(H)$ for some prime p_i.

2. Preliminary results

In this section, we give some results that are needed in this paper.

Lemma 2.1 ([5], Lemma 2.6)**.** *Let N, $N \neq 1$, be a solvable normal subgroup of G. If every minimal normal subgroup of G which is contained in N is not contained in $\Phi(G)$ (the Frattini subgroup of G), then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroup of G which is contained in N.*

Lemma 2.2 ([4], Lemma 2.6)**.** *Let N be a minimal normal subgroup of G. If every minimal subgroup of N is weakly-supplemented in G, then N is cyclic of prime order.*

3. The proof of main result

P r o o f of Theorem 1.2. Assume that the theorem is false and let G be a counterexample of the smallest order. Since G/G' is abelian and $\mathscr{F} \subseteq \mathscr{R}$, we have $G/G' \in \mathscr{R}$, and so $G/(H \cap G') \in \mathscr{R}$. Thus, we can prove our theorem by replacing $G' \cap H$ by H and assume that $H \leqslant G'$.

We first claim that $\Phi(G) \cap H = 1$. In fact, if $\Phi(G) \neq 1$, then there is a minimal subgroup A of G such that $A \leqslant \Phi(G) \cap H$. Noticing that $\Phi(G) \cap H \leqslant F(H)$, then by the hypothesis of the theorem there exists a subgroup K of G such that $G = AK$ and $K < G$. It follows from $G = AK$ and $A \leqslant \Phi(G)$ that $G = K$, in contradiction to $K < G$. Thus $\Phi(G) \cap H = 1$ and our claim is established.

Next, by applying Lemma 2.1, we deduce that

$$F(H) = N_1 \times \ldots \times N_t$$

where each N_j is a minimal normal subgroup of G, $j = 1, 2, \ldots, t$. Since every minimal subgroup of N_j is weakly-supplemented in G, by Lemma 2.2 N_j is a cyclic group of prime order $j = 1, 2, \ldots, t$. Then it follows that $G/C_G(N_j)$ is an abelian group and therefore $G' \leqslant C_G(N_j)$. Hence $H \leqslant G' \leqslant C_G(F(H))$. The solvability of H implies that $H \leqslant C_H(F(H)) \leqslant F(H)$. Hence, H is an abelian group and $H = F(H)$.

Now consider the quotient group G/N_j. Since $H = F(H)$, we may prove that $G/N_j \in \mathscr{R}$, $j = 1, 2, \ldots, t$ by using arguments similar to the ones in the proof of Theorem 3.7 in [4]. Hence we may assume that $H = N_1$ is a minimal subgroup.

Finally, by the hypothesis, there is a subgroup K of G such that $G = HK$ and $K < G$. By the above we know $H \cap K = 1$. Since $C_G(H) = C_G(N_1) \geqslant H = N_1$, we have $C_G(H) = H(C_G(H) \cap K)$. It is easy to see that $C_G(H) \cap K$ is normal in G. If $C_G(H) \cap K = 1$, then $C_G(H) = H$, and therefore, $K \simeq G/H = G/C_G(H)$ is a cyclic group. This shows that G is supersolvable, and therefore, $G \in \mathscr{R}$, a contradiction. If $C_G(H) \cap K \neq 1$, then we consider the quotient group $G/(C_G(H) \cap K)$. Since $G/C_G(H)$ is a cyclic group and $C_G(H)/(C_G(H) \cap K) \simeq H$ is also a cyclic group, we see that $G/(C_G(H) \cap K)$ is supersolvable, and therefore, $G/(C_G(H) \cap K) \in \mathscr{R}$. It follows that

$$G \simeq G/(H \cap (C_G(H) \cap K)) \in \mathscr{R},$$

the final contradiction. Thus, the proof of the theorem is complete. □

References

[1] *Z. Arad, M. B. Ward*: New criteria for the solvability of finite groups. J. Algebra *77* (1982), 234–246. zbl MR doi

[2] *P. Hall*: A characteristic property of soluble groups. J. Lond. Math. Soc. *12* (1937), 198–200. zbl MR doi

[3] *P. Hall*: Complemented groups. J. Lond. Math. Soc. *12* (1937), 201–204. zbl MR doi

[4] *Q. Kong, Q. Liu*: The influence of weakly-supplemented subgroups on the structure of finite groups. Czech. Math. J. *64* (2014), 173–182. zbl MR doi

[5] *D. Li, X. Guo*: The influence of c-normality of subgroups on the structure of finite groups II. Commun. Algebra *26* (1998), 1913–1922. zbl MR doi

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