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ARITHMETIC GENUS OF INTEGRAL SPACE CURVES

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Abstract. We give an estimation for the arithmetic genus of an integral space curve which is not contained in a surface of degree $k-1$. Our main technique is the Bogomolov-Gieseker type inequality for $\mathbb{P}^3$ proved by Macrì.

Keywords: space curve; arithmetic genus; Bridgeland stability; Bogomolov-Gieseker inequality

MSC 2010: 14H50, 14F05

1. Introduction

A classical problem, which goes back to Halphen in [7], is to determine, for given integers d and k, the maximal genus $G(d,k)$ of a smooth projective space curve of degree d not contained in a surface of degree less than k. This problem is actually very natural, and has been investigated by many people (see [5], [6], [9], [11], [8]).

In this paper, we consider the same problem for an integral space curve. Our main result is:

Theorem 1.1. *Let C be an integral complex projective curve in $\mathbb{P}^3$ of degree d. Let $p_a(C)$ be its arithmetic genus. If C is not contained in a surface of degree less than k, then*

$$p_a(C) \leqslant \begin{cases} \dfrac{2}{3}\dfrac{d^2}{k} + \dfrac{1}{3}d(k-6)+1 & \text{if } k^2 < d, \\ d\bigl(\sqrt{d}-2\bigr)+1 & \text{if } k^2 \geqslant d. \end{cases}$$

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For the case of $k \leqslant 2$, this inequality has also been obtained by Macrì in [13], Corollary 4.1. When $k^2 < d$, our bound is weaker than that of Castelnuovo, Harris and Gruson-Peskine for smooth space curves, but still nontrivial. Our bound can be reached in some cases, when $k^2 \geqslant d$. For example, the arithmetic genus of a complete intersection of two surfaces of degree k is

$$k^2(k-2)+1 = d\big(\sqrt{d}-2\big)+1.$$

The idea of the proof of Theorem 1.1 is to establish the tilt-stability of $\mathcal{I}_C$ via computing its walls; then the Bogomolov-Gieseker type inequality for $\mathbb{P}^3$ proved by Macrì in [13] implies Theorem 1.1. This Bogomolov-Gieseker type inequality naturally appears in the construction of Bridgeland stability conditions on threefolds (cf. [4], [3], [2]). There are also some other interesting applications of the Bogomolov-Gieseker type inequality in [1] and [14].

Our tilt-stability of $\mathcal{I}_C$ also gives a version of the Halphen speciality theorem:

Theorem 1.2. *Let $C \subset \mathbb{P}^3$ be an integral complex projective degree d curve not contained in any surface of degree $< k$. Then $h^2(\mathcal{I}_C(l)) = h^1(\mathcal{O}_C(l)) = 0$ if $l > 2d/k - 4$ when $k^2 < d$, or $l > 2\sqrt{d} - 4$ when $k^2 \geqslant d$.*

Our paper is organized as follows. In Section 2, we review basic properties of tilt-stability, the conjectural inequality proposed in [3], [2] and variants of the classical Bogomolov-Gieseker inequality satisfied by tilt-stable objects. Then in Section 3 the tilt-stability of $\mathcal{I}_C$ is established via computing its walls. Finally, we show the proof of Theorems 1.1 and 1.2 in Section 4.

Notation. In this paper, we will always denote by C an integral projective curve in the three dimensional complex projective space $\mathbb{P}^3$ and by $\mathcal{I}_C$ its ideal sheaf in $\mathbb{P}^3$. We let $p_a(C) := h^1(C, \mathcal{O}_C)$ be the arithmetic genus of C. By X we denote a complex smooth projective threefold and by $\mathrm{D}^b(X)$ its bounded derived category of coherent sheaves.

2. Preliminaries

In this section, we review the notion of tilt-stability for threefolds introduced in [3], [2]. Then we recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed there.

Let X be a smooth projective threefold over $\mathbb{C}$, and let H be an ample divisor on X. Let $\alpha > 0$ and β be two real numbers. We write $\mathrm{ch}^{\beta}(E) = \mathrm{e}^{-\beta H}\,\mathrm{ch}(E)$ to denote the Chern character twisted by βH. More explicitly, we have

$$\begin{aligned} \mathrm{ch}_0^\beta &= \mathrm{ch}_0 = \mathrm{rank}, & \mathrm{ch}_2^\beta &= \mathrm{ch}_2 - \beta H \,\mathrm{ch}_1 + \frac{\beta^2}{2} H^2 \,\mathrm{ch}_0, \\ \mathrm{ch}_1^\beta &= \mathrm{ch}_1 - \beta H \,\mathrm{ch}_0, & \mathrm{ch}_3^\beta &= \mathrm{ch}_3 - \beta H \,\mathrm{ch}_2 + \frac{\beta^2}{2} H^2 \,\mathrm{ch}_1 - \frac{\beta^3}{6} H^3 \,\mathrm{ch}_0 . \end{aligned}$$

Slope-stability. We define the slope μ_β of a coherent sheaf $E \in \mathrm{Coh}(X)$ by

$$\mu_\beta(E) = \begin{cases} \infty & \text{if } \mathrm{ch}_0^\beta(E) = 0, \\ \dfrac{H^2 \,\mathrm{ch}_1^\beta(E)}{H^3 \,\mathrm{ch}_0^\beta(E)} & \text{otherwise.} \end{cases}$$

Definition 2.1. A coherent sheaf E on X is slope-(semi)stable (μ_β-(semi)stable) if, for all nonzero subsheaves $F \hookrightarrow E$, we have

$$\mu_\beta(F) < \mu_\beta(E/F) \quad \big(\mu_\beta(F) \leqslant \mu_\beta(E/F)\big).$$

Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to slope-stability exist in $\mathrm{Coh}(X)$: given a nonzero sheaf $E \in \mathrm{Coh}(X)$, there is a filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

such that $G_i := E_i / E_{i-1}$ is slope-semistable, and $\mu_\beta(G_1) > \ldots > \mu_\beta(G_n)$. We set $\mu_\beta^+(E) := \mu_\beta(G_1)$ and $\mu_\beta^-(E) := \mu_\beta(G_n)$.

Tilt-stability. There exists a *torsion pair* $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ in $\mathrm{Coh}(X)$ defined as follows:

$$\begin{aligned} \mathcal{T}_\beta &= \{E \in \mathrm{Coh}(X) \colon \ \mu_\beta^-(E) > 0\}, \\ \mathcal{F}_\beta &= \{E \in \mathrm{Coh}(X) \colon \ \mu_\beta^+(E) \leqslant 0\}. \end{aligned}$$

Equivalently, $\mathcal{T}_\beta$ and $\mathcal{F}_\beta$ are the extension-closed subcategories of $\mathrm{Coh}(X)$ generated by slope-stable sheaves of positive and nonpositive slope, respectively.

Definition 2.2. We let $\mathrm{Coh}^\beta(X) \subset \mathrm{D}^b(X)$ be the extension-closure

$$\mathrm{Coh}^\beta(X) = \langle \mathcal{T}_\beta, \mathcal{F}_\beta[1] \rangle.$$

By the general theory of torsion pairs and tilting [10], $\mathrm{Coh}^\beta(X)$ is the heart of a bounded t-structure on $\mathrm{D}^b(X)$; in particular, it is an abelian category.

Now we can define the following slope function on $\mathrm{Coh}^\beta(X)$: for an object $E \in \mathrm{Coh}^\beta(X)$, we set

$$\nu_{\alpha,\beta}(E) = \begin{cases} \infty & \text{if } H^2 \operatorname{ch}_1^\beta(E) = 0, \\ \dfrac{H \operatorname{ch}_2^\beta(E) - \frac{1}{2}\alpha^2 H^3 \operatorname{ch}_0^\beta(E)}{H^2 \operatorname{ch}_1^\beta(E)} & \text{otherwise.} \end{cases}$$

Definition 2.3. An object $E \in \mathrm{Coh}^\beta(X)$ is *tilt-(semi)stable* ($\nu_{\alpha,\beta}$*-(semi)stable*) if for all nontrivial subobjects $F \hookrightarrow E$ we have

$$\nu_{\alpha,\beta}(F) < \nu_{\alpha,\beta}(E/F) \quad \big(\nu_{\alpha,\beta}(F) \leqslant \nu_{\alpha,\beta}(E/F)\big).$$

Lemma 3.2.4 in [3] shows that the Harder-Narasimhan property holds with respect to $\nu_{\alpha,\beta}$-stability, i.e., for any $\mathcal{E} \in \mathrm{Coh}^\beta(X)$ there is a filtration in $\mathrm{Coh}^\beta(X)$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_n = \mathcal{E}$$

such that $\mathcal{F}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is $\nu_{\alpha,\beta}$-semistable with $\nu_{\alpha,\beta}(\mathcal{F}_1) > \ldots > \nu_{\alpha,\beta}(\mathcal{F}_n)$.

Definition 2.4. In the above filtration, we call $\mathcal{E}_1$ the $\nu_{\alpha,\beta}$-maximal subobject of $\mathcal{E} \in \mathrm{Coh}^\beta(X)$. If $\mathcal{E}$ is $\nu_{\alpha,\beta}$-semistable, we say $\mathcal{E}$ itself is its $\nu_{\alpha,\beta}$-maximal subobject.

Bogomolov-Gieseker type inequality. We now recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed in [3], [2].

Definition 2.5. We define the generalized discriminant

$$\overline{\Delta}_H^\beta := (H^2 \operatorname{ch}_1^\beta)^2 - 2H^3 \operatorname{ch}_0^\beta (H \operatorname{ch}_2^\beta).$$

A short calculation shows $\overline{\Delta}_H^\beta = (H^2 \operatorname{ch}_1)^2 - 2H^3 \operatorname{ch}_0 (H \operatorname{ch}_2)$. Hence the generalized discriminant is independent of β.

Theorem 2.6 ([3], Theorem 7.3.1)**.** *Assume $E \in \mathrm{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$-semistable. Then*

$$\overline{\Delta}_H^\beta(E) \geqslant 0. \tag{2.1}$$

Conjecture 2.7 ([2], Conjecture 4.1)**.** *Assume $E \in \mathrm{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$-semistable. Then*

$$\alpha^2 \overline{\Delta}_H^\beta(E) + 4(H \operatorname{ch}_2^\beta(E))^2 - 6H^2 \operatorname{ch}_1^\beta(E) \operatorname{ch}_3^\beta(E) \geqslant 0. \tag{2.2}$$

Such an inequality was proved by Macrì in [13] in the case of the projective space $\mathbb{P}^3$:

Theorem 2.8. *The inequality* (2.2) *holds for* $\nu_{\alpha,\beta}$*-semistable objects in* $\mathrm{D}^b(\mathbb{P}^3)$.

3. Tilt-stability of ideal sheaves of space curves

In this section, we establish the tilt-stability of ideal sheaves of space curves via computing their walls. Then from Theorem 2.8 we can deduce a Castelnuovo type inequality for integral curves in $\mathbb{P}^3$.

Throughout this section, let C be an integral projective curve in $\mathbb{P}^3$ of degree d not contained in a surface of degree $< k$, and let $\mathcal{I}_C$ be the ideal sheaf of C in $\mathbb{P}^3$. We keep the same notation as that in the previous section for $X = \mathbb{P}^3$ and $H =$ a plane of $\mathbb{P}^3$. To simplify, we directly identify $H^{3-i}\operatorname{ch}_i^\beta(E) = \operatorname{ch}_i^\beta(E)$ for $E \in \mathrm{D}^b(\mathbb{P}^3)$. The tilted slope becomes:

$$\nu_{\alpha,\beta} = \frac{\operatorname{ch}_2^\beta - \frac{1}{2}\alpha^2 \operatorname{ch}_0^\beta}{\operatorname{ch}_1^\beta} = \frac{\operatorname{ch}_2 - \beta \operatorname{ch}_1 + \frac{1}{2}(\beta^2 - \alpha^2)\operatorname{ch}_0}{\operatorname{ch}_1 - \beta \operatorname{ch}_0}.$$

The following lemma is a key observation for us to establish the tilt-stability of $\mathcal{I}_C$.

Lemma 3.1. *Let E be the $\nu_{\alpha,\beta}$-maximal subobject of $\mathcal{I}_C \in \operatorname{Coh}^\beta(\mathbb{P}^3)$ for some $(\alpha,\beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. If $2\alpha^2 + \beta^2 \geqslant 4d$, then* $\operatorname{ch}_0(E) = 1$.

Proof. By the long exact sequence of cohomology sheaves induced by the short exact sequence

$$0 \to E \to \mathcal{I}_C \to Q \to 0$$

in $\operatorname{Coh}^\beta(\mathbb{P}^3)$, one sees that E is a torsion free sheaf with $\operatorname{ch}_0(E) \geqslant 1$. If $\mathcal{I}_C$ is $\nu_{\alpha,\beta}$-semistable, then $E = \mathcal{I}_C$ by our definition. Hence $\operatorname{ch}_0(E) = 1$.

Now we assume that $\mathcal{I}_C$ is not $\nu_{\alpha,\beta}$-semistable. One deduces

$$\nu_{\alpha,\beta}(E) = \frac{\operatorname{ch}_2^\beta(E) - \frac{1}{2}\alpha^2 \operatorname{ch}_0(E)}{\operatorname{ch}_1^\beta(E)} > \nu_{\alpha,\beta}(\mathcal{I}_C) = \frac{\frac{1}{2}(\beta^2 - \alpha^2) - d}{-\beta},$$

i.e.,

$$\operatorname{ch}_2^\beta(E) > \frac{\frac{1}{2}(\beta^2 - \alpha^2) - d}{-\beta} \operatorname{ch}_1^\beta(E) + \frac{1}{2}\alpha^2 \operatorname{ch}_0(E). \tag{3.1}$$

By Theorem 2.6, we obtain

$$\frac{(\mathrm{ch}_1^{\beta}(E))^2}{2\,\mathrm{ch}_0(E)} \geqslant \mathrm{ch}_2^{\beta}(E). \tag{3.2}$$

Combining (3.1) and (3.2), one sees that

$$\alpha^2(\mathrm{ch}_0(E))^2 + \frac{\beta^2-\alpha^2-2d}{-\beta}\,\mathrm{ch}_1^{\beta}(E)\,\mathrm{ch}_0(E) - (\mathrm{ch}_1^{\beta}(E))^2 < 0.$$

This implies

$$\mathrm{ch}_0(E) < \left(\frac{\beta^2-\alpha^2-2d}{\beta} + \sqrt{\left(\frac{\beta^2-\alpha^2-2d}{\beta}\right)^2 + 4\alpha^2}\right)\frac{\mathrm{ch}_1^{\beta}(E)}{2\alpha^2}. \tag{3.3}$$

Since E is a subobject of $\mathcal{I}_C$ in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$, by the definition of $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ we deduce that

$$0 < \mathrm{ch}_1^{\beta}(E) \leqslant \mathrm{ch}_1^{\beta}(\mathcal{I}_C) = -\beta.$$

From (3.3) it follows that

$$\mathrm{ch}_0(E) < \frac{(\alpha^2-\beta^2+2d) + \sqrt{(\beta^2-\alpha^2-2d)^2+4\alpha^2\beta^2}}{2\alpha^2}. \tag{3.4}$$

On the other hand, since $2\alpha^2+\beta^2 \geqslant 4d$, a direct computation shows

$$\frac{(\alpha^2-\beta^2+2d) + \sqrt{(\beta^2-\alpha^2-2d)^2+4\alpha^2\beta^2}}{2\alpha^2} \leqslant 2.$$

Therefore, by (3.4) we conclude that $\mathrm{ch}_0(E) < 2$, i.e., $\mathrm{ch}_0(E) = 1$. □

We now compute the walls of $\mathcal{I}_C$. See [12] for the surface case.

Lemma 3.2. *Let E be a subobject of $\mathcal{I}_C$ in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ with*

$$(\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E)) = (r, \theta, c).$$

Then $\nu_{\alpha,\beta}(E)\left\{\begin{matrix}\leqslant\\<\end{matrix}\right\}\nu_{\alpha,\beta}(\mathcal{I}_C)$ if and only if

$$\frac{\theta}{2}(\alpha^2+\beta^2) - (c+rd)\beta + \theta d\left\{\begin{matrix}\leqslant\\<\end{matrix}\right\}0.$$

Proof. Since E is a subobject of $\mathcal{I}_C$ in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$, one has

$$0 < \mathrm{ch}_1^{\beta}(E) = \theta - r\beta \leqslant \mathrm{ch}_1^{\beta}(\mathcal{I}_C) = -\beta,$$

i.e., $r\beta < \theta \leqslant (r-1)\beta \leqslant 0$.

Hence

$$\nu_{\alpha,\beta}(E) = \frac{\frac{r}{2}(\beta^2-\alpha^2) - \beta\theta + c}{\theta - r\beta} \left\{ \begin{matrix} \leqslant \\ < \end{matrix} \right\} \nu_{\alpha,\beta}(\mathcal{I}_C) = \frac{\frac{1}{2}(\beta^2-\alpha^2) - d}{-\beta}$$

is equivalent to

$$-\beta\Big(\frac{r}{2}(\beta^2-\alpha^2) - \beta\theta + c\Big) \left\{ \begin{matrix} \leqslant \\ < \end{matrix} \right\} (\theta - r\beta)\Big(\frac{1}{2}(\beta^2-\alpha^2) - d\Big),$$

i.e.,

$$\frac{\theta}{2}(\alpha^2+\beta^2) - (c+rd)\beta + \theta d \left\{ \begin{matrix} \leqslant \\ < \end{matrix} \right\} 0.$$

$\square$

Proposition 3.3. *If $k^2 < d$, then $\mathcal{I}_C$ is $\nu_{\alpha,\beta}$-semistable for any $\alpha > 0$ and $\beta = -2d/k$.*

Proof. We let α_0 be an arbitrary positive real number, $\beta_0 = -2d/k$, and let E be the ν_{α_0,β_0}-maximal subobject of $\mathcal{I}_C \in \mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$.

Since $k^2 < d$, one sees that $2\alpha_0^2 + \beta_0^2 > \beta_0^2 > 4d$. Hence, by Lemma 3.1, one has $\mathrm{ch}_0(E) = 1$, and E is a subsheaf of $\mathcal{I}_C$. We can write $E = \mathcal{I}_W(-l)$, where $W \subset \mathbb{P}^3$ is a scheme of dimension $\leqslant 1$ and $l \geqslant 0$. The Chern characters of $\mathcal{I}_W(-l)$ are

$$(\mathrm{ch}_0(\mathcal{I}_W(-l)), \mathrm{ch}_1(\mathcal{I}_W(-l)), \mathrm{ch}_2(\mathcal{I}_W(-l))) = \Big(1, -l, \frac{1}{2}l^2 + \mathrm{ch}_2(\mathcal{I}_W)\Big).$$

Since $\mathcal{I}_W(-l)$ is a subobject of $\mathcal{I}_C$ in $\mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$, one deduces

$$0 < \mathrm{ch}_1^{\beta_0}(\mathcal{I}_W(-l)) = -l - \beta_0 \leqslant \mathrm{ch}_1^{\beta_0}(\mathcal{I}_C) = -\beta_0,$$

i.e.,

$$0 \leqslant l < -\beta_0. \tag{3.5}$$

If $C \subseteq W$, then $\mathrm{ch}_2(\mathcal{I}_W) \leqslant \mathrm{ch}_2(\mathcal{I}_C) = -d$. Thus one sees that

$$\begin{aligned} -\frac{l}{2}(\alpha_0^2+\beta_0^2) - \Big(\frac{1}{2}l^2 + \mathrm{ch}_2(I_W) + d\Big)\beta_0 - ld &\leqslant -\frac{l}{2}\beta_0^2 - \Big(\frac{1}{2}l^2 - d + d\Big)\beta_0 \\ &= -\frac{\beta_0 l}{2}(l+\beta_0) \leqslant 0. \end{aligned}$$

By Lemma 3.2, we conclude that $\nu_{\alpha_0,\beta_0}(\mathcal{I}_W(-l)) \leqslant \nu_{\alpha_0,\beta_0}(\mathcal{I}_C)$. Therefore the ν_{α_0,β_0}-maximal subobject of $\mathcal{I}_C$ in $\mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$ is $\mathcal{I}_C$ itself. Namely, $\mathcal{I}_C$ is ν_{α_0,β_0}-semistable.

If $C \nsubseteq W$, then $\mathcal{I}_W(-l) \subset \mathcal{I}_C$ implies $\mathcal{O}_{\mathbb{P}^3}(-l) \subset \mathcal{I}_C$. Thus $l \geqslant k$. One deduces by (3.5) that

$$
\begin{aligned}
(3.6) \qquad -\frac{l}{2}(\alpha_0^2+\beta_0^2) - \Big(\frac{1}{2}l^2 + \mathrm{ch}_2(I_W) + d\Big)\beta_0 - ld& \\
&< -\frac{l}{2}\beta_0^2 - \Big(\frac{1}{2}l^2 + d\Big)\beta_0 - ld \\
&= -\frac{l}{2}\Big(\beta_0^2 + \Big(l + \frac{2d}{l}\Big)\beta_0 + 2d\Big) \\
&= -\frac{l}{2}(\beta_0 + l)\Big(\beta_0 + \frac{2d}{l}\Big) \\
&= -\frac{l}{2}(\beta_0 + l)\Big(\frac{2d}{l} - \frac{2d}{k}\Big) \leqslant 0.
\end{aligned}
$$

Lemma 3.2 yields that $\mathcal{I}_C$ is also ν_{α_0,β_0}-semistable in this case. □

Proposition 3.4. *If $k^2 \geqslant d$, then $\mathcal{I}_C$ is $\nu_{\alpha,\beta}$-semistable for any $\alpha > 0$ and $\beta = -2\sqrt{d}$.*

Proof. The proof is almost the same as that of Proposition 3.3. We let α_0 be an arbitrary positive real number, $\beta_0 = -2\sqrt{d}$, and let E be the ν_{α_0,β_0}-maximal subobject of $\mathcal{I}_C \in \mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$.

By Lemma 3.1, the assumption $\beta_0 = -2\sqrt{d}$ makes sure that $\mathrm{ch}_0(E) = 1$. We can still write $E = \mathcal{I}_W(-l)$ as in the proof of Proposition 3.3. When $C \subseteq W$, the same proof of Proposition 3.3 shows that $\mathcal{I}_C$ is ν_{α_0,β_0}-semistable.

In the case of $C \nsubseteq W$, one sees that $l \geqslant k$. Thus it follows from (3.6) and (3.5) that

$$
\begin{aligned}
-\frac{l}{2}(\alpha_0^2+\beta_0^2) - \Big(\frac{1}{2}l^2 + \mathrm{ch}_2(I_W) + d\Big)\beta_0 - ld &< -\frac{l}{2}(\beta_0 + l)\Big(\beta_0 + \frac{2d}{l}\Big) \\
&\leqslant -\frac{l}{2}(\beta_0 + l)\Big(\frac{2d}{k} - 2\sqrt{d}\Big).
\end{aligned}
$$

The assumption $k^2 \geqslant d$ guarantees that the left hand side of the above inequality is negative. Therefore we are done by Lemma 3.2. □

4. The proof of the main theorems

Now we can prove Theorems 1.1 and 1.2 easily.

Proof of Theorem 1.1. Since C is an integral curve, one sees that

$$\mathrm{ch}_3^{\beta}(\mathcal{I}_C) = -\frac{1}{6}\beta^3 + d\beta + 2d - \chi(\mathcal{O}_C).$$

If $\mathcal{I}_C$ is $\nu_{\alpha,\beta}$-semistable, then Theorem 2.8 implies that

$$\begin{aligned}
&\alpha^2\overline{\Delta}_H^{\beta}(\mathcal{I}_C) + 4(H\,\mathrm{ch}_2^{\beta}(\mathcal{I}_C))^2 - 6H^2\,\mathrm{ch}_1^{\beta}(\mathcal{I}_C)\,\mathrm{ch}_3^{\beta}(\mathcal{I}_C)\\
&\qquad = 2\alpha^2 d + 4d^2 + \beta^4 - 4\beta^2 d - 6(-\beta)\Big(-\frac{1}{6}\beta^3 + d\beta + 2d - \chi(\mathcal{O}_C)\Big)\\
&\qquad = 2\alpha^2 d + 4d^2 + 2\beta^2 d + 6\beta(2d - \chi(\mathcal{O}_C)) \geqslant 0,
\end{aligned}$$

i.e.,

$$h^1(\mathcal{O}_C) - 1 = -\chi(\mathcal{O}_C) \leqslant \frac{2d^2 + (\alpha^2+\beta^2)d}{3(-\beta)} - 2d. \tag{4.1}$$

By Propositions 3.3 and 3.4, one can substitute $(\alpha,\beta) = (0,-2d/k)$ and $(\alpha,\beta) = \big(0,-2\sqrt{d}\big)$ into (4.1) respectively to obtain our desired conclusion. $\square$

Proof of Theorem 1.2. The short exact sequence

$$0 \to \mathcal{I}_C(m) \to \mathcal{O}_{\mathbb{P}^3}(m) \to \mathcal{O}_C(m) \to 0$$

induces a long exact sequence

$$H^1(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^1(\mathcal{O}_C(m)) \to H^2(\mathcal{I}_C(m)) \to H^2(\mathcal{O}_{\mathbb{P}^3}(m)).$$

Since $H^1(\mathcal{O}_{\mathbb{P}^3}(m)) = H^2(\mathcal{O}_{\mathbb{P}^3}(m)) = 0$, we deduce $h^2(\mathcal{I}_C(m)) = h^1(\mathcal{O}_C(m))$.

Now we assume

Assumption 4.1. *$m > 2d/k$, $k^2 < d$ and $\beta_0 = -2d/k$.*

One sees that

$$\mathrm{ch}_1^{\beta_0}(\mathcal{O}_{\mathbb{P}^3}(-m)) = -m + \frac{2d}{k} < 0.$$

Thus $\mathcal{O}_{\mathbb{P}^3}(-m)[1] \in \mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$. It turns out that

$$\nu_{\alpha_0,\beta_0}(\mathcal{O}_{\mathbb{P}^3}(-m)[1]) = \frac{-\frac{1}{2}(m+\beta_0)^2 + \frac{1}{2}\alpha_0^2}{m+\beta_0} < \nu_{\alpha_0,\beta_0}(\mathcal{I}_C) = \frac{\frac{1}{2}(\beta_0^2 - \alpha_0^2) - d}{-\beta_0}$$

is equivalent to

$$-\beta_0\Big(-\frac{1}{2}(m+\beta_0)^2+\frac{1}{2}\alpha_0^2\Big) < (m+\beta_0)\Big(\frac{1}{2}(\beta_0^2-\alpha_0^2)-d\Big),$$

i.e.,

$$\alpha_0^2+\beta_0^2+\Big(m+\frac{2d}{m}\Big)\beta_0+2d<0.$$

Assumption 4.1 implies

$$\begin{aligned}\beta_0^2+\Big(m+\frac{2d}{m}\Big)\beta_0+2d &= (\beta_0+m)\Big(\beta_0+\frac{2d}{m}\Big)\\ &= (\beta_0+m)\Big(\frac{2d}{m}-\frac{2d}{k}\Big)\\ &< (\beta_0+m)\Big(k-\frac{2d}{k}\Big)<0.\end{aligned}$$

Thus we can find an $\alpha_0>0$ such that $\nu_{\alpha_0,\beta_0}(\mathcal{O}_{\mathbb{P}^3}(-m)[1]) < \nu_{\alpha_0,\beta_0}(\mathcal{I}_C)$. On the other hand, by [3], Proposition 7.4.1, and Proposition 3.3, one deduces that both $\mathcal{O}_{\mathbb{P}^3}(-m)[1]$ and $\mathcal{I}_C$ are ν_{α_0,β_0}-semistable. We conclude that

$$\mathrm{Hom}_{\mathrm{D}^b(\mathbb{P}^3)}(\mathcal{I}_C,\mathcal{O}_{\mathbb{P}^3}(-m)[1])=0.$$

By the Serre duality theorem, one obtains $h^2(\mathcal{I}_C(m-4))=0$. Therefore we conclude that $h^2(\mathcal{I}_C(l))=h^1(\mathcal{O}_C(l))=0$ if $l>2d/k-4$ and $k^2<d$.

Similarly, one can show $h^2(\mathcal{I}_C(l))=h^1(\mathcal{O}_C(l))=0$ if $l>2\sqrt{d}-4$ and $k^2\geqslant d$. $\square$

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