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ON THE NEGATIVE DEPENDENCE IN HILBERT SPACES WITH APPLICATIONS

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Abstract. This paper introduces the notion of pairwise and coordinatewise negative dependence for random vectors in Hilbert spaces. Besides giving some classical inequalities, almost sure convergence and complete convergence theorems are established. Some limit theorems are extended to pairwise and coordinatewise negatively dependent random vectors taking values in Hilbert spaces. An illustrative example is also provided.

Keywords: negative dependence; pairwise negative dependence; Hilbert space; law of large numbers

MSC 2010: 60B11, 60B12, 60F15

1. Introduction

The concept of negative dependence was introduced by Lehmann [15] and further investigated by Ebrahimi and Ghosh [7] and Block et al. [1]. A collection of random variables $\{X_1, \ldots, X_n\}$ is said to be negatively dependent (ND) if for all $x_1, \ldots, x_n \in \mathbb{R}$,

$$P(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n) \leqslant P(X_1 \leqslant x_1) \ldots P(X_n \leqslant x_n),$$

and

$$P(X_1 > x_1, \ldots, X_n > x_n) \leqslant P(X_1 > x_1) \ldots P(X_n > x_n).$$

A sequence of random variables $\{X_i,\ i \geqslant 1\}$ is said to be ND if for any $n \geqslant 1$, the collection $\{X_1, \ldots, X_n\}$ is ND. A sequence of random variables $\{X_i,\ i \geqslant 1\}$ is said

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to be pairwise negatively dependent (PND) if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$(1.1) \qquad P(X_i \leqslant x, X_j \leqslant y) \leqslant P(X_i \leqslant x)P(X_j \leqslant y).$$

It is well known and easy to prove that $\{X_i,\ i \geqslant 1\}$ is PND if and only if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$P(X_i > x, X_j > y) \leqslant P(X_i > x)P(X_j > y).$$

A simple consequence of (1.1) is that if $\{X_1, \ldots, X_n\}$ are PND random variables with finite variances, then for all $i \neq j$, $E(X_i X_j) \leqslant EX_i EX_j$, and so

$$(1.2) \qquad \operatorname{Var}\Big(\sum_{i=1}^{n} X_i\Big) \leqslant \sum_{i=1}^{n} \operatorname{Var}(X_i).$$

Matuła [18] proved the Kolmogorov strong law of large numbers for sequences of PND identically distributed random variables, and extended the Etemadi strong law of large numbers [8] to the PND case. Besides the work of Matuła [18], there is a long and interesting literature on the laws of large numbers and the complete convergence for sequences of ND or PND random variables, see Chen and Sung [4], Li et al. [16], Li and Yang [17], Patterson and Taylor [23], Wu and Rosalsky [28] and references therein.

In this paper, we introduce the notion of coordinatewise negative dependence (CND), and pairwise and coordinatewise negative dependence (PCND) for random vectors in Hilbert spaces. We prove the Rademacher-Menshov type inequality, the Hájek-Rényi type inequality, and establish some strong laws of large numbers and a complete convergence theorem for PCND random vectors in Hilbert spaces.

Let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A sequence $\{X_n,\ n \geqslant 1\}$ of random vectors taking values in H is said to be CND or PCND if for some orthonormal basis $\{e_j,\ j \geqslant 1\}$ and for each $j \geqslant 1$, the sequence of random variables $\{\langle X_i, e_j \rangle,\ i \geqslant 1\}$ is ND or PND, respectively. We would like to note that the CND (or PCND) structure is not preserved if we change the basis, as was so kindly pointed out to us by an anonymous referee.

ND and PND random variables became popular because of their applications. Many useful distributions enjoy the ND properties, including multinomial distribution, multivariate hypergeometric distribution, Dirichlet distribution, strongly Rayleigh distribution, and distribution of random sampling without replacement. We refer to Sections 3 and 4 in [2] for negative dependence in mathematics and physics, and to Pemantle [24] for a major survey on these dependence structures. It was shown by Li et al. [16], Proposition 2.1 that for every sequence of continuous distribution functions $\{F_n, n \geqslant 1\}$, there exists a sequence of PND random variables $\{X_n, n \geqslant 1\}$ such that for each $n \geqslant 1$, F_n is the distribution function of X_n and $\{X_n, n \geqslant k\}$ is not a sequence of pairwise independent random variables for every $k \geqslant 1$.

In the notions of CND and PCND random vectors in Hilbert spaces, there are no ND (or PND) requirements between two different coordinates of each vector; even repetitions are permitted. An obvious example of a sequence of CND (resp., PCND) random vectors is as follows. Let ℓ_2 denote the real separable Hilbert space of all square summable real sequences with inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i \quad \text{for } x, y \in \ell_2.$$

The standard orthonormal basis of ℓ_2 is $\{e_n, n \geqslant 1\}$ where e_n denote the element of ℓ_2 having 1 in its nth position and 0 elsewhere. Let $\{Y_i, i \geqslant 1\}$ be a sequence of ND (resp., PND) real-valued random variables and let $a = (a_1, \ldots, a_n \ldots) \in \ell_2$. For $n \geqslant 1$, set

$$X_n = (a_1 Y_n, a_2 Y_n, , \ldots, a_n Y_n, \ldots).$$

Then $\{X_n, n \geqslant 1\}$ is a sequence of CND (resp., PCND) random vectors in ℓ_2. More generally, let $\{X_n^{(j)}, n \geqslant 1, j \geqslant 1\}$ be a double array of column-wise ND (resp., PND) real-valued random variables, i.e., for each $j \geqslant 1$ fixed, $\{X_n^{(j)}, n \geqslant 1\}$ is a sequence of ND (resp., PND) real-valued random variables. Assume further that for each $n \geqslant 1$, $\sum\limits_{j=1}^{\infty} (X_n^{(j)})^2 < \infty$ a.s. Let

$$X_n = \sum_{j=1}^{\infty} X_n^{(j)} e_j, \quad n \geqslant 1.$$

Then $\{X_n,\ n \geqslant 1\}$ is a sequence of CND (resp., CPND) random vectors in ℓ_2.

We would like to note that the concept of negative association, a dependence structure stronger than PND, was first extended to Hilbert space valued random vectors by Burton et al. [3]. After the appearance of the paper by Burton et al. [3], a literature of investigation concerning the limit theorems of negatively associated random vectors in Hilbert spaces has emerged. We refer the reader to Dabrowski and Dehling [6], Hien and Thanh [10], Huan et al. [12], Ko et al. [14], Miao [19], Thanh [27], Zhang [29] among others. This stronger dependence notion will not be discussed here.

Throughout this paper, H denotes a real separable Hilbert space with orthonormal basis $\{e_j, j \geqslant 1\}$, inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. The symbol K denotes a generic positive constant whose value may be different for each appearance, and log denotes the logarithm with base 2. By saying $\{X_n, n \geqslant 1\}$ is a sequence of PCND random vectors, we mean that the random vectors are PCND with respect to the orthonormal basis $\{e_j, j \geqslant 1\}$.

The plan of the paper is as follows. In Section 2 of the paper, we prove some inequalities for PCND random vectors in H. As applications, we establish some strong laws of large numbers and a complete convergence theorem for sums of PCND random vectors in Section 3.

2. Hájek-Rényi type inequality for sums of PCND random vectors

In this section, we prove Rademacher-Menshov type inequality and the Hájek-Rényi type inequality for sums of PCND random vectors in H. The proof is similar to that of Theorem 3.2 in [13] and Lemma 3.2 in [14]. Inequality (2.2) in Theorem 2.1 below is the Rademacher-Menshov type inequality. It is a consequence of (2.1) and Theorem 2.3 in Móricz et al. [21]. Theorem 2.3 in Móricz et al. [21] is stated for Banach space-valued random vectors. In our special case, it implies that if $\{X_n, n \geqslant 1\}$ is a sequence of random vectors in H satisfying

$$E\left\|\sum_{i=b+1}^{b+n} X_i\right\|^2 \leqslant \sum_{i=b+1}^{b+n} E\|X_i\|^2 \quad \forall\, b \geqslant 0,\ n \geqslant 1,$$

then

$$E\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{k=b+1}^{b+k} X_i\right\|^2\right) \leqslant \log^2(2n)\sum_{i=b+1}^{b+n} E\|X_i\|^2.$$

Theorem 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of PCND mean 0 random vectors in H satisfying $E\|X_n\|^2 < \infty$ for all $n \geqslant 1$. Then for any $n \geqslant 1$, we have*

$$E\left\|\sum_{i=1}^{n} X_i\right\|^2 \leqslant \sum_{i=1}^{n} E\|X_i\|^2, \tag{2.1}$$

and

$$E\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{i=1}^{k} X_i\right\|^2\right) \leqslant \log^2(2n)\sum_{i=1}^{n} E\|X_i\|^2. \tag{2.2}$$

P r o o f. By applying (1.2), we have

$$E\Big\|\sum_{i=1}^n X_i\Big\|^2 = E\bigg(\sum_{j\geqslant 1}\Big\langle\sum_{i=1}^n X_i, e_j\Big\rangle^2\bigg) = \sum_{j\geqslant 1} E\bigg(\sum_{i=1}^n \langle X_i, e_j\rangle\bigg)^2$$
$$\leqslant \sum_{j\geqslant 1}\bigg(\sum_{i=1}^n E\langle X_i, e_j\rangle^2\bigg) = \sum_{i=1}^n E\bigg(\sum_{j\geqslant 1}\langle X_i, e_j\rangle^2\bigg) = \sum_{i=1}^n E\|X_i\|^2.$$

The proof of (2.1) is completed. From (2.1), by applying Theorem 2.3 in Móricz et al. [21] mentioned above, we obtain (2.2). □

Theorem 2.2. *Let $\{X_n, n\geqslant 1\}$ be a sequence of PCND mean 0 random vectors in H satisfying $E\|X_n\|^2 < \infty$ for all $n\geqslant 1$, and let $\{b_n, n\geqslant 1\}$ be a nondecreasing sequence of positive constants. Then for any $n\geqslant 1$,*

$$E\bigg(\max_{1\leqslant k\leqslant n}\Big\|\frac{\sum_{i=1}^k X_i}{b_k}\Big\|^2\bigg) \leqslant 4\log^2(2n)\sum_{i=1}^n \frac{E\|X_i\|^2}{b_i^2}. \tag{2.3}$$

Moreover, for any $1\leqslant m\leqslant n$,

$$E\bigg(\max_{m\leqslant k\leqslant n}\Big\|\frac{\sum_{i=1}^k X_i}{b_k}\Big\|^2\bigg) \leqslant \frac{2\sum_{i=1}^m E\|X_i\|^2}{b_m^2} + 8\log^2(2(n-m))\sum_{i=m+1}^n \frac{E\|X_i\|^2}{b_i^2}. \tag{2.4}$$

P r o o f. Set $b_0 = 0$, by using the argument of Hájek and Rényi [9], we have

$$\sum_{j=1}^k X_j = \sum_{j=1}^k\bigg(\sum_{i=1}^j (b_i - b_{i-1})\frac{X_j}{b_j}\bigg) = \sum_{i=1}^k (b_i - b_{i-1})\sum_{j=i}^k \frac{X_j}{b_j}.$$

It follows that

$$\max_{1\leqslant k\leqslant n}\Big\|\frac{\sum_{i=1}^k X_i}{b_k}\Big\| = \max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^k \frac{b_i - b_{i-1}}{b_k}\sum_{j=i}^k \frac{X_j}{b_j}\Big\|$$
$$\leqslant \max_{1\leqslant k\leqslant n}\sum_{i=1}^k \frac{b_i - b_{i-1}}{b_k}\bigg(\max_{1\leqslant i\leqslant k}\Big\|\sum_{j=i}^k \frac{X_j}{b_j}\Big\|\bigg)$$
$$= \max_{1\leqslant k\leqslant n}\bigg(\max_{1\leqslant i\leqslant k}\Big\|\sum_{j=i}^k \frac{X_j}{b_j}\Big\|\bigg)\sum_{i=1}^k \frac{b_i - b_{i-1}}{b_k}$$
$$= \max_{1\leqslant k\leqslant n}\bigg(\max_{1\leqslant i\leqslant k}\Big\|\sum_{j=i}^k \frac{X_j}{b_j}\Big\|\bigg) = \max_{1\leqslant i\leqslant k\leqslant n}\Big\|\sum_{j=1}^k \frac{X_j}{b_j} - \sum_{j=1}^{i-1}\frac{X_j}{b_j}\Big\|$$
$$\leqslant 2\max_{1\leqslant k\leqslant n}\Big\|\sum_{j=1}^k \frac{X_j}{b_j}\Big\|.$$

Therefore, by applying (2.2), we have

$$E\left(\max_{1\leqslant k\leqslant n}\left\|\frac{\sum_{i=1}^{k}X_i}{b_k}\right\|^2\right)\leqslant 4E\left(\max_{1\leqslant k\leqslant n}\left\|\sum_{j=1}^{k}\frac{X_j}{b_j}\right\|^2\right)\leqslant 4\log^2(2n)\sum_{j=1}^{n}\frac{E\|X_j\|^2}{b_j^2}.$$

The proof of (2.3) is completed. For the proof of (2.4), we note that

$$\begin{aligned}\max_{m\leqslant k\leqslant n}\left\|\frac{\sum_{i=1}^{k}X_i}{b_k}\right\| &\leqslant \max_{m\leqslant k\leqslant n}\left\|\frac{\sum_{i=1}^{m}X_i}{b_k}\right\|+\max_{m<k\leqslant n}\left\|\frac{\sum_{i=m+1}^{k}X_i}{b_k}\right\| \\ &= \left\|\frac{\sum_{i=1}^{m}X_i}{b_m}\right\|+\max_{1\leqslant k\leqslant n-m}\left\|\frac{\sum_{i=1}^{k}X_{m+i}}{b_{m+k}}\right\|.\end{aligned}\tag{2.5}$$

Combining (2.1), (2.3), and (2.5), we have

$$\begin{aligned}E\left(\max_{m\leqslant k\leqslant n}\left\|\frac{\sum_{i=1}^{k}X_i}{b_k}\right\|\right)^2 &\leqslant 2\left(E\left\|\frac{\sum_{i=1}^{m}X_i}{b_m}\right\|^2+E\left(\max_{1\leqslant k\leqslant n-m}\left\|\frac{\sum_{i=1}^{k}X_{m+i}}{b_{m+k}}\right\|\right)^2\right) \\ &\leqslant 2\left(\frac{\sum_{i=1}^{m}E\|X_i\|^2}{b_m^2}+4\log^2(2(n-m))\sum_{i=m+1}^{n}\frac{E\|X_i\|^2}{b_i^2}\right).\end{aligned}$$

This ends the proof of (2.4). □

3. Almost sure and complete convergence

In this section, we prove some theorems on almost sure convergence and complete convergence for sequences of PCND vectors in Hilbert spaces. An illustrative example is also provided.

Following the idea of Móricz [20] and applying a recent result of Hu et al. [11], we have the following theorem. It is an extension of the Rademacher-Menshov strong law of large numbers to the blockwise PCND random vectors in Hilbert spaces.

Theorem 3.1. *Let $\{X_n, n\geqslant 1\}$ be a sequence of mean 0 random vectors in H such that for any $k\geqslant 0$, the random vectors $\{X_i,\ 2^k\leqslant i<2^{k+1}\}$ are PCND. Let $\{b_n, n\geqslant 1\}$ be a nondecreasing sequence of positive constants satisfying*

$$\inf_{n\geqslant 0}\frac{b_{2^{n+1}}}{b_{2^n}}>1 \quad \textit{and} \quad \sup_{n\geqslant 0}\frac{b_{2^{n+1}}}{b_{2^n}}<\infty.\tag{3.1}$$

If

$$\sum_{n=1}^{\infty}\frac{E\|X_n\|^2\log^2 n}{b_n^2}<\infty,\tag{3.2}$$

then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{b_n} = 0 \quad a.s. \tag{3.3}$$

P r o o f. For $k \geqslant 0$, let

$$T_k = \frac{1}{b_{2^{k+1}}} \max_{2^k \leqslant m < 2^{k+1}} \left\| \sum_{n=2^k}^{m} X_n \right\|.$$

By (2.2), we have

$$ET_k^2 \leqslant \frac{(\log 2^{k+1})^2}{b_{2^{k+1}}^2} \sum_{n=2^k}^{2^{k+1}-1} E\|X_n\|^2 \leqslant K \sum_{n=2^k}^{2^{k+1}-1} \frac{E\|X_n\|^2 \log^2 n}{b_n^2}, \tag{3.4}$$

since $\{b_n, n \geqslant 1\}$ is nondecreasing. Combining (3.2) and (3.4), we have

$$\sum_{k=0}^{\infty} ET_k^2 \leqslant K \sum_{n=1}^{\infty} \frac{E\|X_n\|^2 \log^2 n}{b_n^2} < \infty.$$

By Markov's inequality and the Borel-Cantelli lemma, this ensures that

$$\lim_{k\to\infty} T_k = 0 \quad \text{a.s.} \tag{3.5}$$

From (3.5) and Theorem 2.1 of Hu et al. [11], we obtain the strong law of large numbers (3.3). □

R e m a r k 3.2. (i) Csörgő, Tandori, and Totik [5] proved that even for mean 0 pairwise independent random variables, the condition

$$\sum_{n=1}^{\infty} \frac{EX_n^2}{n^2} < \infty \tag{3.6}$$

does not imply the strong law of large numbers (3.3) with $b_n = n$.

(ii) In Theorem 3.1, the random vectors with indices in each block $[2^k, 2^{k+1})$ are PCND but there are no PCND requirements for the random vectors with indices in different blocks; even repetitions are permitted. This weaker assumption may prevent some limit theorems from being valid. For example, there exists a sequence of mean zero random variables $\{X_n, n \geqslant 1\}$ which is independent within each block $[2^k, 2^{k+1})$ and satisfies

$$\sum_{n=1}^{\infty} EX_n^2 < \infty$$

but the series $\sum\limits_{n=1}^{\infty} X_n$ diverges a.s. It means that the Kolmogorov-Khintchin theorem fails to hold for this kind of dependence (see examples in Móricz [20], Rosalsky and Thanh [25], [26]).

(iii) We see that the inequalities (2.1) and (2.2) play a key role in many strong limit theorems. By the same method as in Móricz [20], Theorem 2, we can prove that if $\{X_n, n \geqslant 1\}$ is a sequence of PCND vectors in H such that

$$\sum_{n=1}^{\infty} E\|X_n - EX_n\|^2 \log^2 n < \infty,$$

then $\sum\limits_{i=n}^{\infty} (X_n - EX_n)$ converges a.s.

The last theorem in this section establishes the complete convergence for sequences of PCND vectors in Hilbert spaces. Let X be a random vector in H. Here and thereafter, we denote the jth coordinate of X by $X^{(j)}$, i.e.,

$$X^{(j)} = \langle X, e_j \rangle, \quad j \geqslant 1.$$

Then, we can write

$$X = \sum_{j \geqslant 1} X^{(j)} e_j.$$

Theorem 3.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of PCND identically distributed mean 0 random vectors in H, and let $1 \leqslant p < 2$. If*

$$\sum_{j \geqslant 1} E|X_1^{(j)}|^p < \infty, \tag{3.7}$$

then for all $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1 \leqslant k \leqslant n} \Big\| \sum_{i=1}^{k} X_i \Big\| > \varepsilon (n \log^2 n)^{1/p} \Big) < \infty. \tag{3.8}$$

P r o o f. For $n \geqslant 1$, $k \geqslant 1$, $j \geqslant 1$, set

$$\begin{aligned} Y_{nk}^{(j)} &= -(n\log^2 n)^{1/p} I(X_k^{(j)} < -(n\log^2 n)^{1/p}) + X_k^{(j)} I(|X_k^{(j)}| \leqslant (n\log^2 n)^{1/p}) \\ &\quad + (n\log^2 n)^{1/p} I(X_k^{(j)} > (n\log^2 n)^{1/p}), \\ Y_{nk} &= \sum_{j \geqslant 1} Y_{nk}^{(j)} e_j, \end{aligned}$$

and

$$S_{nk} = \sum_{i=1}^{k} (Y_{ni} - EY_{ni}).$$

For any $\varepsilon > 0$, we have

(3.9)

$$\begin{aligned}
&P\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} X_i\Big\| > \varepsilon(n\log^2 n)^{1/p}\Big)\\
&\leqslant P\Big(\bigcup_{1\leqslant i\leqslant n}\bigcup_{j\geqslant 1}(|X_i^{(j)}| > (n\log^2 n)^{1/p})\Big) + P\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} Y_{ni}\Big\| > \varepsilon(n\log^2 n)^{1/p}\Big)\\
&\leqslant P\Big(\bigcup_{1\leqslant i\leqslant n}\bigcup_{j\geqslant 1}(|X_i^{(j)}| > (n\log^2 n)^{1/p})\Big) + P\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} EY_{ni}\Big\| > \frac{\varepsilon}{2}(n\log^2 n)^{1/p}\Big)\\
&\quad + P\Big(\max_{1\leqslant k\leqslant n}\|S_{nk}\| > \frac{\varepsilon}{2}(n\log^2 n)^{1/p}\Big).
\end{aligned}$$

By (3.7), we have

$$\begin{aligned}
(3.10)\qquad &\sum_{n=1}^{\infty} n^{-1} P\Big(\bigcup_{1\leqslant i\leqslant n}\bigcup_{j\geqslant 1}(|X_i^{(j)}| > (n\log^2 n)^{1/p})\Big)\\
&\leqslant \sum_{n=1}^{\infty} n^{-1}\sum_{j\geqslant 1}\sum_{i=1}^{n} P(|X_i^{(j)}| > (n\log^2 n)^{1/p})\\
&\leqslant \sum_{j\geqslant 1}\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P(|X_i^{(j)}| > n^{1/p})\\
&= \sum_{j\geqslant 1}\sum_{n=1}^{\infty} P(|X_1^{(j)}| > n^{1/p}) \leqslant K\sum_{j\geqslant 1} E|X_1^{(j)}|^p < \infty,
\end{aligned}$$

and

$$\begin{aligned}
(3.11)\quad &(n\log^2 n)^{-1/p}\max_{1\leqslant k\leqslant n}\Big\|\sum_{i=1}^{k} EY_{ni}\Big\| \leqslant (n\log^2 n)^{-1/p}\sum_{i=1}^{n}\|EY_{ni}\|\\
&\leqslant (n\log^2 n)^{-1/p}\sum_{i=1}^{n}\sum_{j\geqslant 1}|EY_{ni}^{(j)}|\\
&\leqslant (n\log^2 n)^{-1/p}\sum_{i=1}^{n}\sum_{j\geqslant 1}|EX_i^{(j)} I(|X_i^{(j)}| \leqslant (n\log^2 n)^{1/p})|\\
&\quad + (n\log^2 n)^{-1/p}\sum_{i=1}^{n}\sum_{j\geqslant 1}(n\log^2 n)^{1/p} P(|X_i^{(j)}| > (n\log^2 n)^{1/p})
\end{aligned}$$

$$
\begin{aligned}
&= (n\log^2 n)^{-1/p}\sum_{i=1}^{n}\sum_{j\geqslant 1}|EX_i^{(j)}I(|X_i^{(j)}|>(n\log^2 n)^{1/p})|\\
&\quad+\sum_{i=1}^{n}\sum_{j\geqslant 1}P(|X_i^{(j)}|>(n\log^2 n)^{1/p})\\
&\leqslant n^{1-1/p}(\log^{-2/p} n)\sum_{j\geqslant 1}E(|X_1^{(j)}|I(|X_1^{(j)}|>(n\log^2 n)^{1/p}))\\
&\quad+\sum_{j\geqslant 1}nP(|X_1^{(j)}|>(n\log^2 n)^{1/p})\\
&\leqslant 2n^{1-1/p}(\log^{-2/p} n)\sum_{j\geqslant 1}E(|X_1^{(j)}|I(|X_1^{(j)}|>(n\log^2 n)^{1/p}))\\
&\leqslant 2(\log^{-2} n)\sum_{j\geqslant 1}E(|X_1^{(j)}|^pI(|X_1^{(j)}|>(n\log^2 n)^{1/p}))\\
&\leqslant 2(\log^{-2} n)\sum_{j\geqslant 1}E(|X_1^{(j)}|^p)\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

From (3.9)–(3.11) to prove (3.8), it suffices to show that

$$
\sum_{n=1}^{\infty} n^{-1}P\Big(\max_{1\leqslant k\leqslant n}\|S_{nk}\|>\varepsilon(n\log^2 n)^{1/p}/2\Big)<\infty. \tag{3.12}
$$

It is well known that for all $j\geqslant 1$ and for all $n\geqslant 1$, $\{Y_{nk}^{(j)}-EY_{nk}^{(j)},\ k\geqslant 1\}$ is a sequence of PND random variables. So $\{Y_{nk}-EY_{nk},\ k\geqslant 1\}$ is a sequence of PCND random vectors in H. By applying Markov's inequality and Theorem 2.1, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1}P\Big(\max_{1\leqslant k\leqslant n}\|S_{nk}\|>\frac{\varepsilon}{2}(n\log^2 n)^{1/p}\Big) \qquad (3.13)\\
&\quad\leqslant 1+\sum_{n=2}^{\infty}\frac{4}{\varepsilon^2(n\log^2 n)^{2/p}}n^{-1}\log^2(2n)\sum_{i=1}^{n}E\|Y_{ni}-EY_{ni}\|^2\\
&\quad\leqslant 1+K\sum_{n=2}^{\infty}\frac{1}{n^{1+2/p}\log^{4/p-2} n}\sum_{i=1}^{n}E\|Y_{ni}\|^2\\
&\quad= 1+K\sum_{j\geqslant 1}\sum_{n=2}^{\infty}\frac{1}{n^{2/p}\log^{4/p-2} n}(E(|X_1^{(j)}|^2I(|X_1^{(j)}|^p\leqslant n\log^2 n))\\
&\qquad+(n\log^2 n)^{2/p}P(|X_1^{(j)}|^p>n\log^2 n)) =: 1+J_1+J_2.
\end{aligned}
$$

We estimate J_1 as follows:

$$(3.14)\qquad \begin{aligned} J_1 &\leqslant K\sum_{j\geqslant 1}\sum_{n=2}^{\infty}\frac{1}{n^{2/p}\log^{4/p-2} n} \\ &\qquad \times \sum_{i=1}^{n-1} E(|X_1^{(j)}|^2 I(i\log^2 i < |X_1^{(j)}|^p \leqslant (i+1)\log^2(i+1))) \\ &\leqslant K\sum_{j\geqslant 1}\sum_{i=1}^{\infty}\Bigl(\sum_{n=i+1}^{\infty}\frac{1}{n^{2/p}\log^{4/p-2} n}\Bigr) \\ &\qquad \times E(|X_1^{(j)}|^2 I(i\log^2 i < |X_1^{(j)}|^p \leqslant (i+1)\log^2(i+1))) \\ &\leqslant K\sum_{j\geqslant 1}\sum_{i=1}^{\infty}\frac{1}{(i+1)^{2/p-1}\log^{4/p-2}(i+1)} \\ &\qquad \times E(|X_1^{(j)}|^2 I(i\log^2 i < |X_1^{(j)}|^p \leqslant (i+1)\log^2(i+1))) \\ &\leqslant K\sum_{j\geqslant 1}\sum_{i=1}^{\infty} E(|X_1^{(j)}|^p I(i\log^2 i < |X_1^{(j)}|^p \leqslant (i+1)\log^2(i+1))) \\ &\leqslant K\sum_{j\geqslant 1} E|X_1^{(j)}|^p < \infty. \end{aligned}$$

For J_2, we have

$$(3.15)\qquad \begin{aligned} J_2 &\leqslant K\sum_{j\geqslant 1}\sum_{n=2}^{\infty}(\log^2 n)P(|X_1^{(j)}|^p > n\log^2 n) \\ &\leqslant K\sum_{j\geqslant 1}\sum_{n=2}^{\infty}\sum_{i=n}^{\infty}(\log^2 n)P(i\log^2 i < |X_1^{(j)}|^p \leqslant (i+1)\log^2(i+1)) \\ &\leqslant K\sum_{j\geqslant 1}\sum_{i=2}^{\infty}\Bigl(\sum_{n=2}^{i}\log^2 n\Bigr)P(i\log^2 i < |X_1^{(j)}|^p \leqslant (i+1)\log^2(i+1)) \\ &\leqslant K\sum_{j\geqslant 1}\sum_{i=2}^{\infty} i(\log^2 i)P(i\log^2 i < |X_1^{(j)}|^p \leqslant (i+1)\log^2(i+1)) \\ &\leqslant K\sum_{j\geqslant 1} E|X_1^{(j)}|^p < \infty. \end{aligned}$$

Combining (3.13)–(3.15), we obtain (3.12). The proof is completed. □

R e m a r k 3.4. If H is a finite dimensional Hilbert space, then the condition (3.7) is equivalent to $E\|X_1\|^p < \infty$. So Theorem 3.3 can be viewed as a Baum-Katz type theorem for PCND random vectors in Hilbert spaces.

The following corollary is the Marcinkiewicz-Zygmund type strong law of large numbers for PCND random vectors in Hilbert spaces. The proof is standard.

Corollary 3.5. *Let* $\{X_n,\ n \geqslant 1\}$ *be a sequence of PCND identically distributed mean* 0 *random vectors in* H, *and let* $1 \leqslant p < 2$. *If*

$$\sum_{j\geqslant 1} E|X_1^{(j)}|^p < \infty,$$

then

$$\frac{\sum_{i=1}^n X_i}{(n\log^2 n)^{1/p}} \to 0 \quad \text{a.s. as } n \to \infty.$$

Proof. For any $\varepsilon > 0$, by applying Theorem 3.3, we have

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{j=1}^k X_j\Big\| > \varepsilon(n\log^2 n)^{1/p}\Big) \\
&= \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{j=1}^k X_j\Big\| > \varepsilon(n\log^2 n)^{1/p}\Big) \\
&\geqslant \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} 2^{-i-1} P\Big(\max_{1\leqslant k\leqslant 2^i}\Big\|\sum_{j=1}^k X_j\Big\| > 4\varepsilon(2^i i^2)^{1/p}\Big) \\
&= \frac{1}{2}\sum_{i=0}^{\infty} P\Big(\max_{1\leqslant k\leqslant 2^i}\Big\|\sum_{j=1}^k X_j\Big\| > 4\varepsilon(2^i i^2)^{1/p}\Big).
\end{aligned} \tag{3.16}$$

By the Borel-Cantelli lemma, (3.16) ensures that

$$\frac{1}{(2^i i^2)^{1/p}} \max_{1\leqslant k\leqslant 2^i}\Big\|\sum_{j=1}^k X_j\Big\| \to 0 \quad \text{a.s. as } i \to \infty. \tag{3.17}$$

For $2^{i-1} \leqslant n < 2^i$, we have

$$0 \leqslant \frac{1}{(n\log^2 n)^{1/p}}\Big\|\sum_{j=1}^n X_j\Big\| \leqslant \frac{4}{(2^i i^2)^{1/p}} \max_{1\leqslant k\leqslant 2^i}\Big\|\sum_{j=1}^k X_j\Big\|. \tag{3.18}$$

The conclusion of the corollary follows from (3.17) and (3.18). □

It is well known that the complete convergence implies the a.s. convergence but the converse is not true. Móricz and Taylor [22], Theorem 1A provided conditions for complete convergence for sequences of quasi-orthogonal random vectors in Hilbert spaces. The following example presents a sequence $\{X_n, n \geqslant 1\}$ of PCND random vectors in ℓ_2 which is not a pairwise independent sequence. We show that this sequence satisfies the conditions of Theorem 3.1 so that the strong law of large numbers (3.3) holds. However, it does not satisfy the conditions in Móricz and Taylor [22], Theorem 1A so that we cannot apply Theorem 1A in Móricz and Taylor [22] to obtain the corresponding complete convergence.

Example 3.6. Let F_n, $n \geqslant 1$, be the distribution function of probability density function

$$p_n(x) = \begin{cases} \dfrac{1}{(2\log 2)|x|} & \text{if } \dfrac{1}{\log^2(n+1)} \leqslant |x| \leqslant \dfrac{2}{\log^2(n+1)}, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.1 of Li et al. [16], there exists a sequence of PND random variables $\{X^{(n)}, n \geqslant 1\}$ such that for each $n \geqslant 1$, F_n is the distribution function of $X^{(n)}$ and $\{X^{(n)}, n \geqslant m\}$ is not a sequence of pairwise independent random variables for every $m \geqslant 1$. Now let

$$X_n = X^{(n)} \sum_{k=1}^{n} e_k, \quad n \geqslant 1.$$

It follows that $\{X_n, n \geqslant 1\}$ is a sequence of PCND mean 0 random vectors in ℓ_2 but $\{X_n, n \geqslant m\}$ is not a pairwise independent sequence for all $m \geqslant 1$. It is easy to see that

$$E(X^{(n)}) = 0,\ E(X^{(n)})^2 = \frac{3}{(2\log 2)\log^4(n+1)}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{E\|X_n\|^2 \log^2 n}{n^2} = \sum_{n=1}^{\infty} \frac{nE(X^{(n)})^2 \log^2 n}{n^2} \leqslant \sum_{n=2}^{\infty} \frac{3}{(2\log 2) n \log^2 n} < \infty$$

so that (3.2) holds with $b_n \equiv n$. By Theorem 3.1, we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = 0 \quad \text{a.s.}$$

Theorem 1A in Móricz and Taylor [22] states that if $\{X_n, n \geqslant 1\}$ is a sequence of mean 0 random vectors in a real separable H satisfying $E\|X_n\|^2 < \infty$ and $E(X_m X_n) \leqslant 0$ for all $m \neq n$, then the condition

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{i=1}^{n} E\|X_i\|^2 < \infty \tag{3.19}$$

implies

$$\sum_{n=1}^{\infty} P\Big(\frac{\|\sum_{i=1}^{n} X_i\|}{n} > \varepsilon\Big) < \infty \quad \forall\, \varepsilon > 0. \tag{3.20}$$

In our example, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{i=1}^{n}E\|X_i\|^2 &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{i=1}^{n} iE(X^{(i)})^2\\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{i=1}^{n}\frac{3i}{(2\log 2)\log^4(i+1)}\\ &\geqslant \frac{3}{4\log 2}\sum_{n=1}^{\infty}\frac{1}{\log^4(n+1)} = \infty\end{aligned}$$

so that (3.19) fails. Therefore, we cannot apply Theorem 1A of Móricz and Taylor [22] to obtain (3.20).

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