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ON WEAKLY-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS

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The paper is dedicated to Professor John Cossey for his 75th birthday

Abstract. A subgroup H of a finite group G is weakly-supplemented in G if there exists a proper subgroup K of G such that $G = HK$. In this paper, some interesting results with weakly-supplemented minimal subgroups to a smaller subgroup of G are obtained.

Keywords: weakly-supplemented subgroup; complemented subgroup; supersolvable group

MSC 2010: 20D10, 20D20

1. Introduction

Throughout this article, all groups are finite. A subgroup H of G is complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. In 1937, Hall in [2] proved that a finite group is solvable if and only if every Sylow subgroup of G is complemented. Arad and Ward in [1] proved that a finite group is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented. In particular, Hall in [3] proved that a finite group G is supersolvable with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G. In a recent paper, the author and Liu studied finite groups for which every minimal subgroup is weakly-supplemented (see [5]). A subgroup H of G is weakly-supplemented in G if there exists a proper subgroup K of G such that $G = HK$. They proved that every minimal subgroup of G is weakly-supplemented in G if and only if G is a supersolvable group and all Sylow subgroups of G are elementary abelian. In addition, we also proved the following result:

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Theorem A. *Let $\mathscr{R}$ be a formation containing $\mathscr{F}$, the class of supersolvable groups. Let H be a normal subgroup of a solvable group G such that $G/H \in \mathscr{R}$. If every minimal subgroup of the Fitting subgroup $F(G' \cap H)$ of $G' \cap H$ is weakly-supplemented in G, then G belongs to $\mathscr{R}$.*

Recently, Pan in [7] weakened the hypothesis that the group G is solvable in Theorem A and proved the following:

Theorem B. *Let $\mathscr{R}$ be a formation containing $\mathscr{F}$, the class of supersolvable groups. Let H be a solvable normal subgroup of a group G such that $G/H \in \mathscr{R}$. If every minimal subgroup of the Fitting subgroup $F(G' \cap H)$ of $G' \cap H$ is weakly-supplemented in G, then G belongs to $\mathscr{R}$.*

In this paper, we further investigate the influence of weakly-supplemented subgroups on the structure of finite groups along the above direction. It is significant to remove the hypothesis that the group G is solvable in Theorem A. However, if a group G is not solvable, then its Fitting subgroup $F(G)$ will sometimes be a trivial subgroup, and therefore we could not expect a detailed structure if we only give the conditions on the minimal subgroups of $F(G)$. However, if we replace the Fitting subgroup $F(G)$ by the generalized Fitting subgroup $F^*(G)$, then we are able to get some interesting results. If the hypothesis that the group G is solvable in Theorem A is removed, then we can have the following result:

Theorem C. *Let $\mathscr{R}$ be a saturated formation containing $\mathscr{F}$, the class of supersolvable groups. Let H be a normal subgroup of a group G such that $G/H \in \mathscr{R}$. If every minimal subgroup of $F^*(H) \cap G'$ is weakly-supplemented in G, then G belongs to $\mathscr{R}$.*

In order to prove Theorem C, we need the following result:

Theorem D. *Let G be a group with a normal subgroup H such that G/H is supersolvable. If every minimal subgroup of $F^*(H) \cap G'$ is weakly-supplemented in G, then G is supersolvable.*

The following Corollary E follows immediately from Theorem D.

Corollary E. *Let G be a group. If every minimal subgroup of $F^*(G) \cap G'$ is weakly-supplemented in G, then G is supersolvable.*

2. Preliminary results

In this section, we give some results that are needed in this paper.

Lemma 2.1 ([5], Lemma 2.2). *Let G be a group and N be a normal subgroup of G.*

(1) *If $H \leqslant K \leqslant G$ and H is weakly-supplemented in G, then H is weakly-supplemented in K.*

(2) *If N is contained in H and H is weakly-supplemented in G, then H/N is weakly-supplemented in G/N.*

(3) *Let π be a set of primes. Let N be a π'-subgroup and A be a π-subgroup of G. If A is weakly-supplemented in G, then AN/N is weakly-supplemented in G/N.*

Lemma 2.2 ([4], Chapter X, 13). *Let G be a group and M a subgroup of G.*

(i) *If M is normal in G, then $F^*(M) \leqslant F^*(G)$;*

(ii) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G))$;*

(iii) *$F^*(F^*(G)) = F^*(G) \geqslant F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

Lemma 2.3 ([5], Lemma 2.5). *Every minimal subgroup of G is weakly-supplemented in G if and only if G is a supersolvable group and all Sylow subgroups of G are elementary abelian.*

Lemma 2.4 ([6], Lemma 2.6). *Let N, $N \neq 1$ be a solvable normal subgroup of G. If every minimal normal subgroup of G which is contained in N is not contained in $\Phi(G)$ (the Frattini subgroup of G), then the Fitting subgroup $F(N)$ of N is the direct product of the minimal normal subgroup of G which is contained in N.*

3. The proof of the main result

P r o o f of Theorem D. Suppose that the theorem is false and let G be a counterexample of the smallest order. Then we prove the theorem by the following steps.

Step 1. Every proper normal subgroup of G is supersolvable.

Let L be a maximal normal subgroup of G. Since the class of supersolvable groups is subgroup-closed and $L/(H \cap L) \simeq HL/H \leqslant G/H$, we see that $L/(H \cap L)$ is supersolvable. By Lemma 2.2, $F^*(H \cap L)$ is contained in $F^*(H)$. Thus, L with normal subgroup $L \cap H$ satisfies the hypotheses of the theorem. The minimal choice of G implies that L is supersolvable.

Step 2. $H = G$, $G' = G$ and $F^*(G) = F(G) < G$.

If $H < G$, then H is supersolvable by Step 1, and therefore, H is solvable, and $F^*(H) = F(H)$ by Lemma 2.2. It follows from Theorem B that G is supersolvable, a contradiction. Thus, $H = G$.

Since G/G' is abelian and so supersolvable, it follows that so $G/(H \cap G')$ is supersolvable. If $G' < G$, then $G' \cap H = G'$ is supersolvable by Step 1. It is clear that every minimal subgroup of $F(G') \cap G' \leqslant F^*(G) \cap G' = F^*(H) \cap G'$ is weakly-supplemented in G. By Theorem B, G is supersolvable, a contradiction.

If $F^*(G) = G$, then every minimal subgroup of G is weakly-supplemented in G by the hypotheses. Now by Lemma 2.3, G is supersolvable, a contradiction. Hence, $F^*(G) < G$. By Step 1, we see that $F^*(G)$ is supersolvable, and therefore by Lemma 2.2, $F^*(G) = F(G)$.

Step 3. $\Phi(G) = 1$.

In fact, if $\Phi(G) \neq 1$, then there is a minimal subgroup A of G such that $A \leqslant \Phi(G)$. Noticing that $\Phi(G) \leqslant F(G) = F^*(G)$ and by Step 2 together with our hypotheses, we see that we have a subgroup K of G such that $G = AK$ and $K < G$. It follows from $G = AK$ and $A \leqslant \Phi(G)$ that $G = K$, in contradiction to $K < G$. Thus, $\Phi(G) = 1$.

Step 4. Final contradiction. In fact, by applying Lemma 2.4, we have

$$F^*(G) = F(G) = N_1 \times N_2 \times \ldots \times N_t,$$

where N_j, $j = 1, 2, \ldots, t$ are minimal normal subgroups of G.

Since every minimal subgroup of N_j is weakly-supplemented in G, N_j is a cyclic group of prime order ($j = 1, 2, \ldots, t$). It follows that $G/C_G(N_j)$ is an abelian group and therefore $G' \leqslant C_G(N_j)$. Hence, $F^*(G) = F(G) \leqslant Z(G)$. Now by the hypotheses there is a subgroup K_j of G such that $G = N_j K_j$ and $K_j < G$ for every N_j. Of course, we may choose $N_2, N_3, \ldots, N_t$ such that

$$N_1 \times \ldots \times N_{j-1} \times N_{j+1} \times \ldots \times N_t \leqslant K_j$$

for $j = 1, 2, \ldots, t$. Let $K = \bigcap_{j=1}^{t} K_j$. Then $G = F(G)K$ and $F(G) \cap K = 1$. Since $F(G) \leqslant Z(G)$, K is a proper normal subgroup of G. By Step 1, we see that K is supersolvable. Of course, $F(G)$ itself is supersolvable. It follows that $G \simeq G/(F(G) \cap K)$ is supersolvable, this arrives at a final contradiction. The proof of the theorem is complete. □

P r o o f of Theorem C. By the hypotheses and Lemma 2.1, every minimal subgroup of $F^*(H) \cap H' \leqslant F(H) \cap G'$ is weakly-supplemented in H. Corollary E implies that H is supersolvable, and therefore, $F^*(H) = F(H)$. Thus, $G \in \mathscr{R}$ by Theorem B, and therefore, the proof is complete. □

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